2013 / March 23 , 日露 mini workshop (大阪)

"Noncommutative Instanton Revisited,"

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Based on work with M. Hamanaka.

Moyal $\star$-積 (非可換結合代数)

$\theta^{\mu\nu}$ real const. anti-symm. tensor

$$f(x) \star g(x) \equiv \exp\left(\frac{\sqrt{-1}}{2}\theta^{\mu\nu}\partial_\mu^{(x')}\partial_\nu^{(x'')}\right) f(x') g(x'') \Big|_{x'=x''=x}$$

- Canonical form $\quad (\theta^{\mu\nu}) = \begin{pmatrix} 0 & -\theta_1 & & \\ \theta_1 & 0 & & \\ & & 0 & -\theta_2 \\ & & \theta_2 & 0 \end{pmatrix}$

- $\bar{f} = f$ の cpx. conj. $\quad \overline{f \star g} = \bar{g} \star \bar{f}$

- $\displaystyle\int_{\mathbb{R}^4} d^4x\, f \star g = \int_{\mathbb{R}^4} d^4x\, g \star f = \int_{\mathbb{R}^4} d^4x\, f g$

- Derived Lie algebra $\quad [f, g]_\star \equiv f \star g - g \star f$

$$[x^\mu, x^\nu]_\star = \sqrt{-1}\,\theta^{\mu\nu}$$

非可換 $\mathbb{R}^4$ 上のゲージ理論 (古典論)

$$A(x) \in u(N) \qquad \text{gauge potential}$$

gauge変換 $$A_\mu^g = g * \partial_\mu g^\dagger + g * A_\mu * g^\dagger$$

共変微分 $$D_\mu \phi = \partial_\mu \phi + A_\mu * \phi$$

曲率テンソル $$F_{\mu\nu} \equiv [D_\mu, D_\nu]_*$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu * A_\nu - A_\nu * A_\mu$$

作用

$$I_{YM} = -\frac{1}{4g^2} \int_{\mathbb{R}^4} d^4x \, \mathrm{Tr}\left(F_{\mu\nu} * F^{\mu\nu} \right)$$

GOAL OF TALK

一般の配位の非可換 $\mathbb{R}^4$ 上の ASD Instanton と ADHM data の間の One-to-One 対応の実際を示す

$\boxed{\text{ASD Instanton}}$ $\quad d_A = d + A$

- $F_{\mu\nu}^{(+)} = 0$

- $A_\mu(x) = g_\infty \partial_\mu g_\infty^\dagger + O(|x|^{-3}) \qquad |x| \to \infty$

$$g_\infty \in U(N), \quad g_\infty = O(1)$$

- $\text{"}C_2\text{"} \equiv \frac{1}{16\pi^2} \int_{\mathbb{R}^4} d^4x \, \mathrm{Tr}_{\mathbb{C}^N} F_{\mu\nu} * \tilde{F}^{\mu\nu} = k \quad (>0)$

<u>Dirac operators</u>

$$\begin{cases} \bar{\mathcal{D}} = \bar{e}^\mu \otimes (d_A)_\mu & \mathcal{S}_+ \longrightarrow \mathcal{S}_- \\ \mathcal{D} = \bar{\mathcal{D}}^\dagger & \mathcal{S}_- \longrightarrow \mathcal{S}_+ \end{cases}$$

$$\bar{\mathcal{D}} * \mathcal{D} = \mathbb{1}_2 \otimes \Box_A$$

Index Th. (+ ASD)

$$\mathrm{Ker}\,\bar{\mathcal{D}} \simeq \mathbb{C}^k \qquad \mathrm{Ker}\,\mathcal{D} = \phi$$

$\mathrm{Ker}\,\overline{\mathcal{D}}$ の正規直交基底行列

$$\Psi(x) = \underbrace{\left(\psi^{(1)}, \quad \cdots, \psi^{(k)} \right)}_{k} \updownarrow 2N$$

$$\left\{ \psi^{(i)} = \left(\psi^{(i)\,a}_{\ \alpha} \right)_{\substack{a=1\sim N \\ \alpha=1,2}} \right\}_{i=1}^{N} \qquad \text{正規直交基底}$$

$$\overline{\mathcal{D}}\,\Psi(x) = 0$$

$$\int_{\mathbb{R}^4} d^4x \; \Psi(x)^\dagger \star \Psi(x) = \mathbb{1}_k$$

$\tilde{\psi}^{\alpha a} = \varepsilon^{\alpha\beta}\psi^{a}_{\beta}$ 　$(\varepsilon^{12} = 1)$ として Ψ を $\tilde{\Psi}$ に読み替える

$$\tilde{\Psi}(x) = \underbrace{\left(\tilde{\psi}^{(1)\alpha=1}, \cdots, \tilde{\psi}^{(k)1}, \tilde{\psi}^{(1)\alpha=2}, \cdots, \tilde{\psi}^{(k)2} \right)}_{2k} \updownarrow N$$

逆構成

定数行列 $S = \underbrace{(S_{\dot\alpha i}{}^{a})}_{2k} \updownarrow N$, $T^\mu = \underbrace{(T^\mu_{ij})}_{k} \updownarrow k$ を

次で定める

$$
\begin{cases}
\tilde{\Psi}(x) = \dfrac{1}{\pi |x|^4} g_\infty S(\bar{e}\cdot x \otimes \mathbb{1}_k) + O(|x|^{-4}) \\
\qquad\qquad \text{as } |x| \longrightarrow +\infty \\
T^\mu = \displaystyle\int_{\mathbb{R}^4} d^4z \; x^\mu \star \Psi(x)^\dagger \star \Psi(x) \\
\qquad\qquad (\Rightarrow \text{エルミート行列})
\end{cases}
$$

このとき

$$
\nabla(x) \equiv \begin{pmatrix} S \\ e_\mu \otimes (x^\mu - T^\mu) \end{pmatrix} \qquad \mathbb{C}^{2k} = \mathbb{C}^2 \otimes \mathbb{C}^k \longrightarrow \mathbb{C}^{N+2k}
$$

は次を満たす

$$
\mathrm{Tr}_{\mathbb{C}^2}\left(\sigma^i \nabla(x)^\dagger \star \nabla(x)\right) = 0 \qquad i = 1, 2, 3
$$

$(\Longleftrightarrow$ ADHM 方程式 $)$

$$\nabla(x)^\dagger \star \nabla(x)$$

$$= \left(s^\dagger ,\ \overline{e_\mu} \otimes (x^\mu - T^\mu) \right) \star \begin{pmatrix} s \\ e_\nu \otimes (x^\nu - T^\nu) \end{pmatrix}$$

$$= \mathbb{1}_2 \otimes (x - T)^2$$

$$+ \frac{\sqrt{-1}}{2} \eta^{(+)}_{\mu\nu} \otimes [x^\mu - T^\mu , x^\nu - T^\nu]_\star + s^\dagger s$$

$\boxed{\text{ADHM 方程式}}$

$$T^\mu \in \mathrm{End}(\mathbb{C}^k) \qquad (T^\mu)^\dagger = T^\mu$$

$$S = (I^\dagger, J) \in \mathrm{Hom}(\mathbb{C}^2 \otimes \mathbb{C}^k, \mathbb{C}^N)$$

cpx. str.

$$z_1 = x^2 + \sqrt{-1}\,x^1, \qquad z_2 = x^4 + \sqrt{-1}\,x^3$$

$$B_1 = T^2 + \sqrt{-1}\,T^1, \qquad B_2 = T^4 + \sqrt{-1}\,T^3$$

$$\left([z_1, \bar{z}_1]_* = 2\theta_1, \quad [z_2, \bar{z}_2]_* = 2\theta_2 \right)$$

$$\begin{tikzcd} \mathbb{C}^k \arrow[loop right, "B_{1,2}"] \arrow[d, shift left, "J"] \\ \mathbb{C}^N \arrow[u, shift left, "I"] \end{tikzcd}$$

$$\mathrm{Tr}_{\mathbb{C}^2}\left(\sigma^i \, \nabla(x)^\dagger * \nabla(x)\right) = 0 \qquad i = 1, 2, 3$$

$$\Leftrightarrow \begin{cases} [z_1 - B_1, z_2 - B_2]_* + IJ = 0 \\ [z_1 - B_1, \bar{z}_1 - B_1^\dagger]_* + [z_2 - B_2, \bar{z}_2 - B_2^\dagger]_* + II^\dagger - J^\dagger J = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} [B_1, B_2] + IJ = 0 \\ [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = -2(\theta_1 + \theta_2)\mathbb{1}_k \end{cases}$$

（ADHM 方程式）

順構成（ADHM 構成）

(S, T^μ) ADHM data

$$\nabla(x)^\dagger = \left(S^\dagger, \ \bar{e}_\mu \otimes (x^\mu - T^\mu) \right) \quad \mathbb{C}^{N+2k} \longrightarrow \mathbb{C}^{2k}$$

全射 $(\zeta \equiv -2(\theta_1 + \theta_2) \neq 0)$

$\mathrm{Ker}\, \nabla(x)^\dagger$ の正規直交基底行列

$$V(x) = \underbrace{\left(v_1(x), \ \cdots, v_N(x) \right)}_{N} \updownarrow N+2k$$

$\{ v_i(x) \}_{i=1}^{N}$ 正規直交基底（exists?）

$$\nabla(x)^\dagger * V(x) = 0$$

$$V(x)^\dagger * V(x) = \mathbb{1}_N$$

(順構成のつづき)

$$\begin{tikzcd}
\mathbb{C}^{N+2k} \arrow[r, "P"] \arrow[d, "d"'] & \operatorname{Ker}\nabla^{\dagger} \arrow[r, "\sim", "V^{\dagger}\star"'] & \mathbb{C}^{N} \arrow[d, "d_A = d + A"] \\
\mathbb{C}^{N+2k} \arrow[r, "P"'] & \operatorname{Ker}\nabla^{\dagger} \arrow[r, "\sim", "V^{\dagger}\star"'] & \mathbb{C}^{N}
\end{tikzcd}$$

$$V^{\dagger} \star P \star d = d_A \star V^{\dagger} \star P$$

$$\left.\begin{aligned} V^{\dagger} \star P &= V^{\dagger} \star (V \star V^{\dagger}) \\ &= V^{\dagger} \end{aligned}\right\} \text{より} \quad d_A \star V^{\dagger} = V^{\dagger} \star d$$

$$d_A = V^{\dagger} \star d V$$

$(d_A = d + A)$

$$\underline{A_\mu(x) = V^{\dagger}(x) \star \partial_\mu V(x)} \quad (\text{unitary})$$

(順構成のつづき)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]_\star$$ は次の形

$$F = \frac{1}{2} F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

$$= \sqrt{-1}\, V^\dagger \star \begin{pmatrix} 0 & 0 \\ 0 & \sigma^i \otimes \frac{1}{\Box} \end{pmatrix} \star V$$

$$\times\ \varepsilon_{ijk} \left(dx^j \wedge dx^k - \underset{\substack{\uparrow \\ \text{Hodge } *}}{*} dx^j \wedge dx^k \right)$$

where

$$\begin{cases} \dfrac{1}{\Box} \star \Box = \Box \star \dfrac{1}{\Box} = \mathbb{1}_k \\ \Box \equiv (x - T)^2 + \dfrac{1}{2}(II^\dagger + J^\dagger J) \end{cases}$$

Explictly
$$\frac{1}{\Box} = \lim_{R \to \infty} \int_0^R ds\ \exp_\star(-s\Box)$$

$$\left(\exp_\star(f) \equiv \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{f \star \cdots \star f}_{n} \right)$$

$$\Longrightarrow \quad \underline{F^{(+)} = 0} \qquad (ASD)$$

(順構成のつづき)

$$"C_2" = \frac{1}{16\pi^2}\int_{\mathbb{R}^4} d^4x \, \mathrm{Tr}_{\mathbb{C}^N} F_{\mu\nu} \star \tilde{F}^{\mu\nu}$$

$$= \frac{1}{16\pi^2}\int_{\mathbb{R}^4} d^4x \, \partial^2 \left[\mathrm{Tr}_{\mathbb{C}^k} \left\{ \partial^\mu \left(\square \star \partial_\mu \left(\frac{1}{\square} \right) \right) \right\} \right]$$

$$\frac{1}{16\pi^2}\int_{\mathbb{R}^4} d^4x \, \partial^2 \left[\quad \right] = |x| \to \infty \text{ での境界積分}$$

$$(\star \longrightarrow \cdot)$$

$$= \underline{k}$$

$\therefore$

$\boxed{\text{One-to-One 対応}}$

$$
\begin{array}{ccc}
\text{ADHM data} & \underset{\text{逆構成}}{\overset{\text{順構成}}{\rightleftarrows}} & \text{ASD Instanton} \\
(S,\ T^{\mu}) & & A \\
 & & \\
V & & \Psi
\end{array}
$$

次の公式を用いて、1対1対応を示せる

<u>Prop. 1</u>

$$
\tilde{\Psi} = \frac{(-1)}{\pi} V^{\dagger} * \begin{array}{c} \begin{pmatrix} 0 \\ \mathbb{1}_2 \otimes \dfrac{1}{\square} \end{pmatrix} \\ \xleftrightarrow{\quad 2k \quad} \end{array} \begin{array}{l} \updownarrow N \\ \updownarrow 2k \end{array}
$$

<u>Prop. 2</u>

$$
\square_A V^{\dagger} = \Big(\underset{\xleftrightarrow{N}}{0}\ ,\ \underset{\xleftrightarrow{\quad 2k \quad}}{4\pi\tilde{\Psi}} \Big) \updownarrow N
$$