Supersymmetric solutions with G_2 and Spin(7) structure

Kazuki Hinoue(D1)
Yukinori Yasui
Osaka City University
Research Group for Mathematical Physics

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Contents

Generating solutions of type II supergravity

1. We find first order equations which are consistent to equations of motion for SUGRA by imposing some conditions. This equations generate SUSY solutions.

2. We solve the first order equations explicitly.

There is a way to lift the solution from 7-dim or 8-dim abelian heterotic SUGRA to 10-dim type II SUGRA.

D.Martelli and J.Sparks (arXiv:1010.4031v2[hep-th])

So, in order to generate type II solutions we will study 7 and 8 dimensional abelian heterotic supergravity.

$$g_{10} = (dz + A_{(1)})^2 + dx^2 + dy^2 + g_7$$

$$g_{10} = (dz + A_{(1)})^2 + dx^2 + g_8$$

The reason why we select 7 dimension and 8 dimension is that 7-dim manifolds may have G_2 –structure and 8-dim manifolds may have Spin(7)-structure.

Then, G_2 and Spin(7) structures give strong restrictions so that we may obtain the solution explicitly.

So, we will study 7-dim case and 8-dim case.

7-dim or 8-dim abelian heterotic SUGRA

Lagrangian

$$\mathcal{L} = e^{-\Phi}(R * 1 + *d\Phi \wedge d\Phi - *F_{(2)} \wedge F_{(2)} - \frac{1}{2} * H_{(3)} \wedge H_{(3)})$$

Where, $F_{(2)} = dA_{(1)}$ **and** $H_{(3)} = dB_{(2)} + A_{(1)} \wedge F_{(2)}$.

Equations of motion

$$\begin{aligned}
R_{\mu\nu} &= -\nabla_\mu \nabla_\nu \Phi + F_\mu{}^\rho F_{\nu\rho} + \frac{1}{4} H_\mu{}^{\rho\sigma} H_{\nu\rho\sigma} \\
\nabla^2 e^{-\Phi} &= \frac{1}{2} e^{-\Phi} F_{\mu\nu} F^{\mu\nu} + \frac{1}{6} e^{-\Phi} H_{\mu\nu\rho} H^{\mu\nu\rho} \\
d(e^{-\Phi} * F_{(2)}) &= -e^{-\Phi} * H_{(3)} \wedge F_{(2)} \\
d(e^{-\Phi} * H_{(3)}) &= 0
\end{aligned}$$

Killing spinor equations

$$\left(\nabla_\mu + \frac{1}{4\cdot 2!} H_{\mu\nu\rho}\gamma^{\nu\rho}\right)\chi = 0$$

$$(\partial_\mu \Phi)\gamma^\mu \chi + \frac{1}{3!} H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\chi = 0$$

$$F_{\mu\nu}\gamma^{\mu\nu}\chi = 0$$

Bianchi identity in abelian heterotic supergravity

$$dH_{(3)} = F_{(2)} \wedge F_{(2)}$$

J.P.Gauntlett, D.Martelli, etal, CMP vol.247,issue2,pp421-445

- **Defining equations of the G-structure**

- **Bianchi identity**

Contents

$$\wedge_{-}^{2}\boldsymbol{R}^4 \cong \boldsymbol{R}^7$$

Define fundamental 3-form on **R⁷** $\Omega_0^{(3)} = \mathrm{d}a_{123} - \sum da_i \wedge \beta^i$

Where, (a_1, a_2, a_3) is fiber coordinate on $\wedge_{-}{}^{2}\,\boldsymbol{R}^4$

$$\beta^i \in \Gamma(\wedge_{-}^{2}\,\boldsymbol{R}^4) = \left\{\alpha \in \Gamma(\wedge^2\,\boldsymbol{R}^4)\,\middle|\, *\alpha = -\alpha\right\}$$

Explicitly β^i are given by

$$\beta^1 = dy_{12} - dy_{34}, \beta^2 = dy_{42} - dy_{13}, \beta^3 = dy_{14} - dy_{23}$$

Thus $\Omega_0{}^{(3)}$ is written by choosing basis on **R⁷**

$$\Omega_0^{(3)} = dx_{123} + dx_{147} + dx_{165} + dx_{257} + dx_{246} + dx_{354} + dx_{367}$$

Exceptional Lie group G_2 is defined as subgroup of GL(7;**R**)

$$G_2 = \{A\,;\ A^*\Omega_0^{(3)} = \Omega_0^{(3)}\ ,\ A \in GL(7;\boldsymbol{R})\}$$

M_7 ; 7-dim manifold

Frame bundle over M_7 ;

$$\mathcal{F}(M_7) := \coprod_{p \in M_7} \{(p,u) \; ; \; u \text{ is a base of } T_pM_7\}$$

Definition of G_2-structure :

G_2-structure on M_7 is principal subbundle $\mathcal{R}$ of frame bundle $\mathcal{F}(M_7)$ with fibre G_2 .
Here,frame bundle $\mathcal{F}(M_7)$ is principal fiber bundle over M_7 with fibre $GL(7;R)$.

By choosing a suitable base it takes the following form:

$$\Omega^{(3)} = e^{123} + e^{147} + e^{165} + e^{246} + e^{257} + e^{354} + e^{367}$$

e^{μ} **; orthonarmal base on tangent space on M_7 .**

$$e^{\mu\nu\rho} \equiv e^{\mu} \wedge e^{\nu} \wedge e^{\rho}$$

More explicitly, G_2 invariant 3-form $\Omega^{(3)}$ exists on M_7 which has one to one correspondence with G_2 –structure.

It is enough to study G_2 invariant $\Omega^{(3)}$.

Our purpose :

1. **find first order equations of abelian heterotic theory.**
2. **find SUSY solutions.**

We start with number 1.

When we impose $\Omega^{(3)}$ on conditions ;

$$\begin{cases} \nabla^T \Omega^{(3)} = 0 \\ \qquad (\ \nabla^T \ \text{; affine connection with 3-form torsion}\ ) \\ d\Omega^{(3)} \wedge \Omega^{(3)} = 0 \end{cases}$$

we have

$$d * \Omega^{(3)} + \frac{1}{3}\Theta \wedge *\Omega^{(3)} = 0 \qquad T^{(3)} = - * \left(d\Omega^{(3)} + \frac{1}{3}\Theta \wedge \Omega^{(3)} \right)$$

Here,Lee form Θ is defined by $\Theta = *(*d\Omega^{(3)} \wedge \Omega^{(3)})$

When there is no torsion and Lee-form, the condition represents G_2 holonomy and the correspondence metric is Ricci-flat. So the above condition is a generalization by torsion.

Such a generalization is natural because there is a candidate for the torsion, 3-form flux.

We identify 3-form flux with 3-form torsion ; $H_{(3)} = T^{(3)}$

and exterior derivative of dilaton with Lee form ; $d\Phi = -\frac{1}{3}\Theta$

First order equations

$$e^{\Phi} d(e^{-\Phi} * \Omega^{(3)}) = 0$$

$$d\Omega^{(3)} \wedge \Omega^{(3)} = 0$$

$$*(\Omega^{(3)} \wedge F_{(2)}) = F_{(2)}$$

$$H_{(3)} = -e^{\Phi} * d(e^{-\Phi}\Omega^{(3)})$$

$$d\Phi = -\frac{1}{3} * (*d\Omega^{(3)} \wedge \Omega^{(3)})$$

$$dH_{(3)} = F_{(2)} \wedge F_{(2)}$$

Generalized self dual equation

Consistency condition between flux and Maxwell field and this is obtained from definition of $H_{(3)}$;

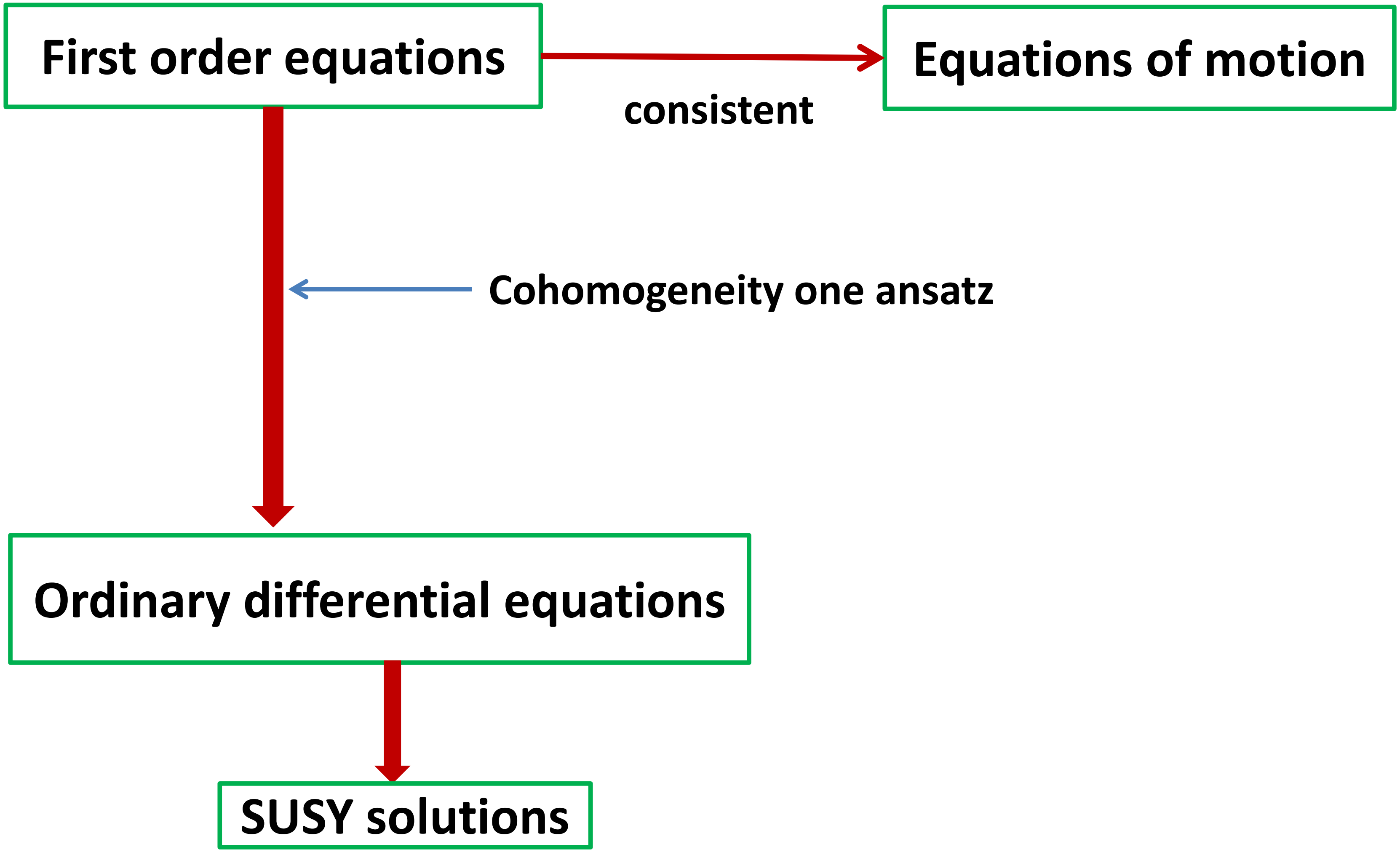
$$H_{(3)} = dB_{(2)} + A_{(1)} \wedge F_{(2)}$$

If we solve first order equations under this identification, we can obtain SUSY solutions.

In order to solve them, we do cohomogeneity one ansatz .

$$M_7 = \boldsymbol{R} \times \text{homogeneous space}$$

first order equations result in ODEs.


First order equations
Equations of motion
consistent
Cohomogeneity one ansatz
Ordinary differential equations
SUSY solutions

Cohomogeneity one manifold admitting G_2 -structure

$$\left\{ \begin{array}{l} \boldsymbol{R} \times SU(2) \times SU(2) \\ \boldsymbol{R} \times Sp(2)/(Sp(1) \times U(1)) \\ \boldsymbol{R} \times SU(3)/(U(1) \times U(1)) \end{array} \right.$$

A.Dancer and M.Y.Wang

We select case of $R \times SU(2) \times SU(2)$ because the other cases are similar to this one. So we consider the manifold ;

$$\mathrm{M}_7 = \boldsymbol{R} \times SU(2) \times SU(2)$$

We consider the follwing metric ansatz ;

$$g = dt^2 + \sum_{i=1}^{3} a_i(t)^2(\sigma_i - \Sigma_i)^2 + \sum_{i=1}^{3} b_i(t)^2(\sigma_i + \Sigma_i)^2$$

t : coordinate on R

$$\begin{aligned}
\sigma_1 &= \sin\psi d\theta - \sin\theta\cos\psi d\phi \\
\sigma_2 &= \cos\psi d\theta + \sin\theta\sin\psi d\phi \\
\sigma_3 &= d\psi + \cos\theta d\phi \\
\Sigma_1 &= \sin\tilde{\psi} d\tilde{\theta} - \sin\tilde{\theta}\cos\tilde{\psi} d\tilde{\phi} \\
\Sigma_2 &= \cos\tilde{\psi} d\tilde{\theta} + \sin\tilde{\theta}\sin\tilde{\psi} d\tilde{\phi} \\
\Sigma_3 &= d\tilde{\psi} + \cos\tilde{\theta} d\tilde{\phi}
\end{aligned}$$

We assume that $F_{(2)}$ is invariant under $SU(2)\times SU(2)$ isometry .
Thus components of $F_{(2)}$ are dependent only t.

$F_{(2)}$ and $H_{(3)}$ are restricted by $dH_{(3)} = F_{(2)} \wedge F_{(2)}$.

$$\Downarrow$$

$$F_{(2)} = F_{12}(t)e^{12} + F_{45}(t)e^{45} + F_{67}(t)e^{67}$$

and

$$H_{(3)} = H_{126}(t)e^{126} + H_{456}(t)e^{456}$$

Here, the orthonormal base $\{e^{\mu}\}$ is defined by

$$e^1 \equiv a_1(t)(\sigma_1 - \Sigma_1)\ ,\ e^2 \equiv a_2(t)(\sigma_2 - \Sigma_2)\ ,\ e^3 \equiv a_3(t)(\sigma_3 - \Sigma_3)$$
$$e^4 \equiv b_1(t)(\sigma_1 + \Sigma_1)\ ,\ e^5 \equiv b_2(t)(\sigma_2 + \Sigma_2)\ ,\ e^6 \equiv b_3(t)(\sigma_3 + \Sigma_3)$$

Thus first order equations consist of ODEs and algebraic equations.

First order equations reduce to ordinary differential equations

$$\frac{da_1}{dt} = -\frac{{a_1}^2}{4a_2b_3} - \frac{{a_1}^2}{4a_3b_2} + \frac{a_2}{4b_3} + \frac{a_3}{4b_2} + \frac{b_2}{4a_3} + \frac{b_3}{4a_2} - \frac{1}{2}a_1H_{126}$$

$$\frac{da_2}{dt} = -\frac{{a_2}^2}{4a_1b_3} - \frac{{a_2}^2}{4a_3b_1} + \frac{a_1}{4b_3} + \frac{a_3}{4b_1} + \frac{b_1}{4a_3} + \frac{b_3}{4a_1} - \frac{1}{2}a_2H_{126}$$

$$\frac{da_3}{dt} = -\frac{{a_3}^2}{4a_2b_1} - \frac{{a_3}^2}{4a_1b_2} + \frac{a_1}{4b_2} + \frac{a_2}{4b_1} + \frac{b_1}{4a_2} + \frac{b_2}{4a_1}$$

$$\frac{db_1}{dt} = -\frac{{b_1}^2}{4a_2a_3} + \frac{{b_1}^2}{4b_2b_3} + \frac{a_2}{4a_3} + \frac{a_3}{4a_2} - \frac{b_2}{4b_3} - \frac{b_3}{4b_2} + \frac{a_1a_2}{2b_2}H_{126}$$

$$\frac{db_2}{dt} = -\frac{{b_2}^2}{4a_1a_3} + \frac{{b_2}^2}{4b_1b_3} + \frac{a_1}{4a_3} + \frac{a_3}{4a_1} - \frac{b_1}{4b_3} - \frac{b_3}{4b_1} + \frac{a_1a_2}{2b_1}H_{126}$$

$$\frac{db_3}{dt} = -\frac{{b_3}^2}{4a_1a_2} + \frac{{b_3}^2}{4b_1b_2} + \frac{a_1}{4a_2} + \frac{a_2}{4a_1}$$

$$\frac{dH_{126}}{dt} = -H_{126}\left(\frac{1}{a_1}\frac{da_1}{dt} + \frac{1}{a_2}\frac{da_2}{dt} + \frac{1}{b_3}\frac{db_3}{dt}\right) + \left(\frac{a_1a_2}{b_1b_2} - 1\right)\frac{b_3}{a_1a_2}H_{126}$$

$$\frac{d\Phi}{dt} = \left(\frac{a_1a_2}{b_1b_2} - 1\right)H_{126}$$

algebraic equations

$$H_{456} = \frac{a_1 a_2}{b_1 b_2} H_{126}$$

$$F_{12} = \sqrt{-\frac{1}{2}\left(\frac{b_3}{a^2} H_{126}\right)} \qquad F_{45} = \frac{a_1 a_2}{b_1 b_2} F_{12}$$

$$F_{67} = F_{45} - F_{12}$$

It is shown numerically that regular solutions doesn't exist , so we consider ansatz, $a_1 = a_2$ **and** $b_1 = b_2$ **.**

$$\begin{cases} a(t) \equiv a_1(t) = a_2(t) \\ b(t) \equiv b_1(t) = b_2(t) \end{cases}$$

After all, the equations reduce to

$$\left\{\begin{aligned}
\frac{da}{dt} &= \frac{b}{4a_3} + \frac{a_3}{4b} - \frac{a^2}{4a_3 b} + \frac{b_3}{4a} - \frac{1}{2} a H_{126} \\
\frac{db}{dt} &= -\frac{b^2}{4aa_3} + \frac{a_3}{4a} + \frac{a}{4a_3} - \frac{b_3}{4b} + \frac{1}{2}\frac{a^2}{b} H_{126} \\
\frac{da_3}{dt} &= \frac{b}{2a} - \frac{{a_3}^2}{2ab} + \frac{a}{2b} \\
\frac{db_3}{dt} &= -\frac{{b_3}^2}{4a^2} + \frac{{b_3}^2}{4b^2} + \frac{1}{2} b_3 \left(\frac{a^2}{b^2} - 1\right) H_{126} \\
\frac{dH_{126}}{dt} &= -H_{126}\left(\frac{2}{a}\frac{da}{dt} + \frac{1}{b_3}\frac{db_3}{dt}\right) + \left(\frac{a^2}{b^2} - 1\right)\frac{b_3}{a^2} H_{126} \\
\frac{d\Phi}{dt} &= H_{126}\left(\frac{a^2}{b^2} - 1\right)
\end{aligned}\right.$$

$$\left\{\begin{aligned}
H_{456} &= \frac{a^2}{b^2} H_{126} \qquad & F_{12} &= \sqrt{-\frac{1}{2}\left(\frac{b_3}{a^2} H_{126}\right)} \\
F_{45} &= \frac{a^2}{b^2} F_{12} \qquad & F_{67} &= F_{45} - F_{12}
\end{aligned}\right.$$

Solutions of these equations automatically solves equations of motion

$$
\begin{aligned}
R_{\mu\nu} &= -\nabla_\mu \nabla_\nu \Phi + F_\mu{}^\rho F_{\nu\rho} + \frac{1}{4} H_\mu{}^{\rho\sigma} H_{\nu\rho\sigma} \\
\nabla^2 e^{-\Phi} &= \frac{1}{2} e^{-\Phi} F_{\mu\nu} F^{\mu\nu} + \frac{1}{6} e^{-\Phi} H_{\mu\nu\rho} H^{\mu\nu\rho} \\
d(e^{-\Phi} * F_{(2)}) &= -e^{-\Phi} * H_{(3)} \wedge F_{(2)} \\
d(e^{-\Phi} * H_{(3)}) &= 0
\end{aligned}
$$

and Killing spinor exists on the back ground.

$$\left(\nabla_\mu + \tfrac{1}{4\cdot 2!} H_{\mu\nu\rho} \gamma^{\nu\rho}\right) \chi = 0$$

$$(\partial_\mu \Phi) \gamma^\mu \chi + \tfrac{1}{3!} H_{\mu\nu\rho} \gamma^{\mu\nu\rho} \chi = 0$$

$$F_{\mu\nu} \gamma^{\mu\nu} \chi = 0$$

We impose boundary conditions at t=0 ;

$$a(0) = 1 \qquad b(0) = 0 \qquad a_3(0) = -1 \qquad b_3(0) = 0$$

$$H_{126}(0) = 0$$

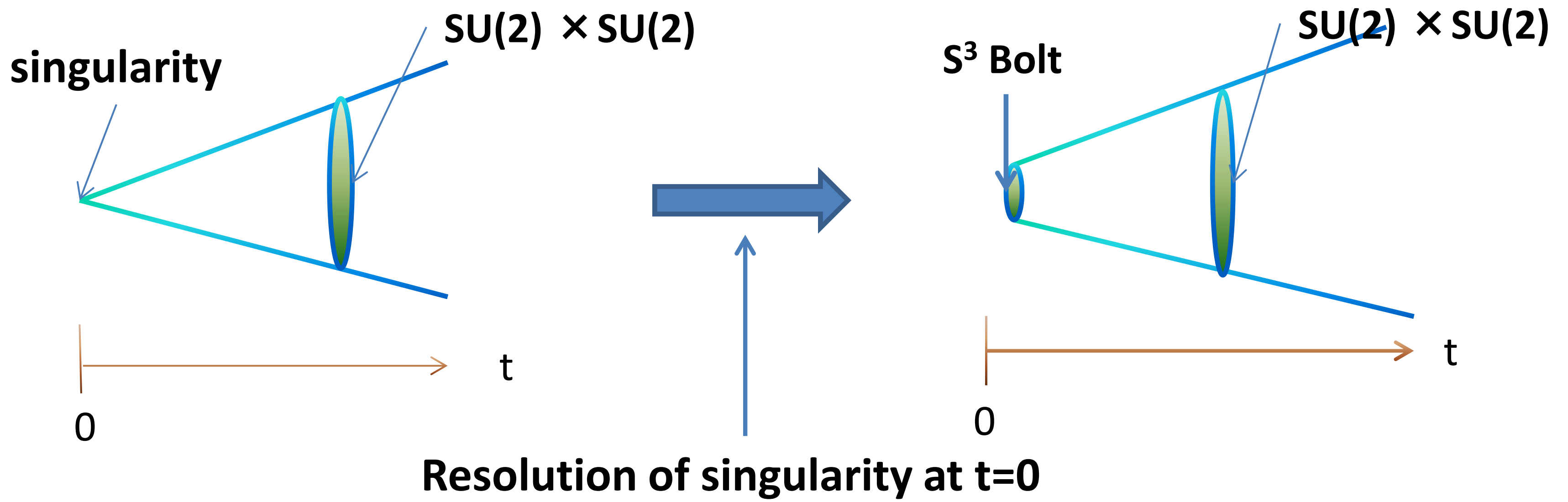
Then we have the following expansions near t=0 ;

$$
\begin{cases}
a(t) = 1 + \dfrac{1}{16}t^2 + \dfrac{64p-7}{2560}t^4 + \mathcal{O}(t^6) \\[2ex]
b(t) = -\dfrac{1}{4}t + pt^3 + \dfrac{98304p^2 - 1344p - 1}{10240}t^5 + \mathcal{O}(t^7) \\[2ex]
a_3(t) = -1 - \dfrac{1}{16}t^2 + \dfrac{64p+3}{1280}t^4 + \mathcal{O}(t^6) \\[2ex]
b_3(t) = \dfrac{1}{4}t + \dfrac{128p+128q-1}{64}t^3 + \left(\dfrac{1}{640} - \dfrac{27}{80}p + \dfrac{132}{5}p^2 - \dfrac{3}{8}q + 48pq + 16q^2\right)t^5 + \mathcal{O}(t^7) \\[2ex]
H_{126}(t) = qt^3 + \dfrac{384B_3h_3 + 128q^2 - 5q}{16}t^5 + \mathcal{O}(t^7)
\end{cases}
$$

Each coefficients are determined by method of series expansion.

The series includes two free parameters, p and q.

If the expansion has analytic continuation up to large t region, the expansion gives a regular metric.



The left figure illustrates conical metrics and it has a singularity at t=0.

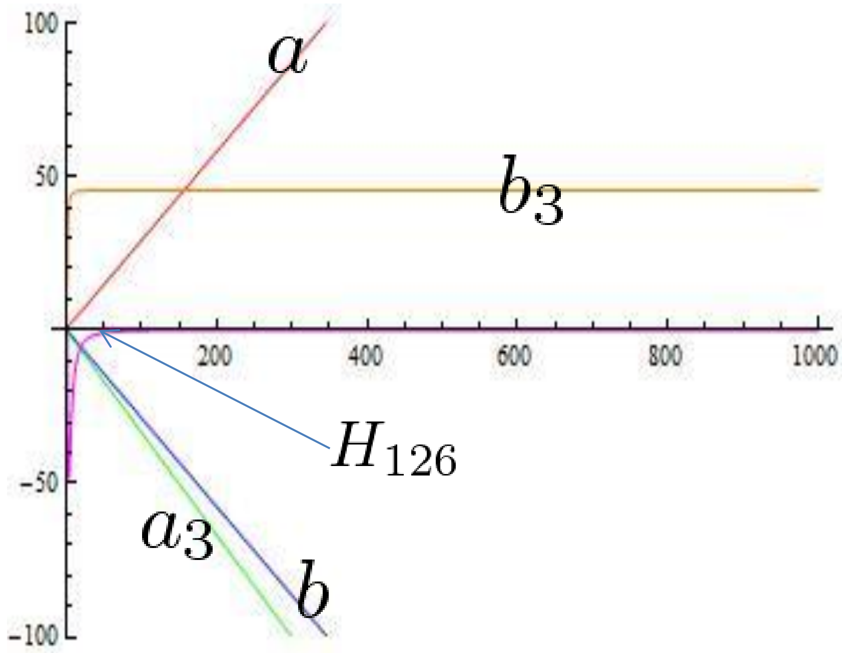
We resolve the singularity of the conical metric;

Figure illustrates regular metrics corresponding to the above expansion; the metric collapses into S^3 metric at t=0.

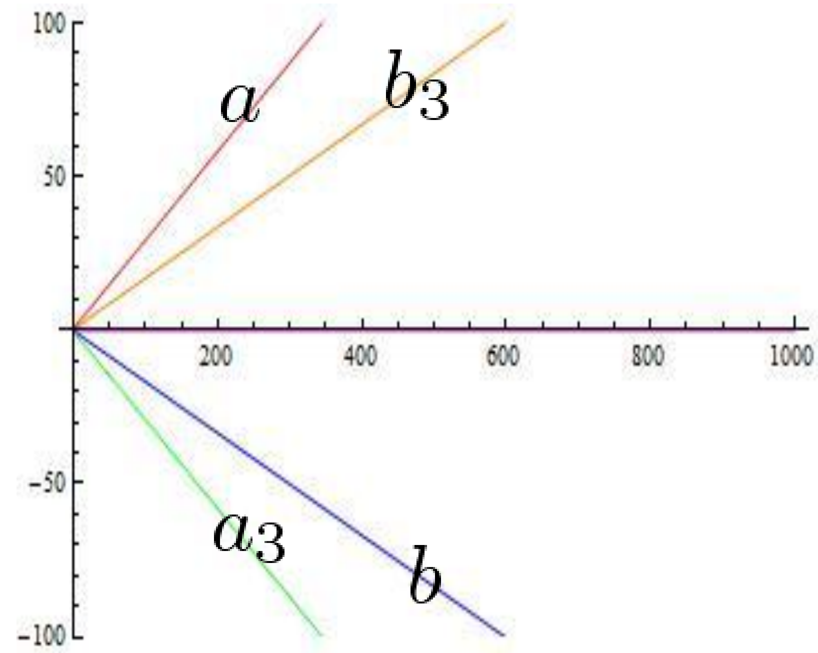
This is the reason that we imposed the boundary condition.

$$a(0) = 1 \quad b(0) = 0 \quad a_3(0) = -1 \quad b_3(0) = 0 \quad H_{126}(0) = 0$$

We give two examples by numerical calculation.



ALC(Asymptotically Locally Conical)



AC(Asymptotically Conical)

AC metric : $a(t) \to k_1 t \quad b(t) \to k_2 t \quad a_3(t) \to k_3 t \quad b_3(t) \to k_4 t$ **for** $t \to \infty$

ALC metric : $a(t) \to k_1 t \quad b(t) \to k_2 t \quad a_3(t) \to k_3 t \quad b_3(t) \to k_4$ **for** $t \to \infty$

When 3-form flux, Maxwell filed and dilaton exist, AC solution doesn't exist.

AC metric is a Ricci-flat metric with G_2 holonomy.

We find the following exact solutions ;

$$g = \frac{9}{4}\frac{r^2-l^2}{r^2-9l^2}dr^2 + \frac{3}{16}(r-l)(r+3l)\{(\sigma_1-\Sigma_1)^2+(\sigma_2-\Sigma_2)^2\} + \frac{1}{4}r^2(\sigma_3-\Sigma_3)^2$$
$$+\frac{3}{16}(r+l)(r-3l)\{(\sigma_1+\Sigma_1)^2+(\sigma_2+\Sigma_2)^2\} + \frac{l^2(r^2-9l^2)}{Cl^2(r^2-9l^2)+r^2-l^2}(\sigma_3+\Sigma_3)^2$$

$$H_{(3)} = -\frac{8C}{3\left(1+C\frac{r^2-9l^2}{r^2-l^2}\right)}\sqrt{\frac{r^2-9l^2}{r^2-l^2}}\left[\frac{r-3l}{3(r-l)^2(r+l)}e^{126} + \frac{r+3l}{(r+l)^2(r-l)}e^{456}\right]$$

$$\Phi = -\log\left[1+C\frac{r^2-9l^2}{r^2-l^2}\right]$$

$$F_{(2)} = -\frac{8\sqrt{C}}{3\left(1+C\frac{r^2-9l^2}{r^2-l^2}\right)}\left[\frac{r-3l}{3(r-l)^2(r+l)}e^{12} + \frac{r+3l}{(r+l)^2(r-l)}e^{45} + \frac{2(r^2-3l^2)}{(r^2-l^2)^2}e^{67}\right]$$

$r > 3l$ $\frac{3}{2}l$ **is a constant which represents radius of S^3.**

C is a positive constant .

General solution

We investigate general solutions by using 'flow method'. Because it is difficult to see the flows going to infinity, it is convenience to compactify configuration space ;

$$\boldsymbol{R}^5 = \{\vec{R} = (a(t), b(t), a_3(t), b_3(t), \tilde{F}_{12}(t))\}$$

So, we will study flows on S^4 (4-dimensional sphere).

In this purpose, we rewrite first ODEs.

$$\left\{\begin{array}{ll} \dfrac{da}{dt} = \dfrac{b}{4a_3} + \dfrac{a_3}{4b} - \dfrac{a^2}{4a_3 b} + \dfrac{b_3}{4a} + \dfrac{\tilde{F}_{12}{}^2}{ab_3} & \dfrac{db}{dt} = -\dfrac{b^2}{4aa_3} + \dfrac{a_3}{4a} + \dfrac{a}{4a_3} - \dfrac{b_3}{4b} - \dfrac{\tilde{F}_{12}{}^2}{bb_3} \\ \dfrac{da_3}{dt} = \dfrac{b}{2a} - \dfrac{a_3{}^2}{2ab} + \dfrac{a}{2b} & \dfrac{db_3}{dt} = -\dfrac{b_3{}^2}{4a^2} + \dfrac{b_3{}^2}{4b^2} - \dfrac{a^2 - b^2}{a^2 b^2}\tilde{F}_{12}{}^2 \\ \dfrac{d\tilde{F}_{12}}{dt} = \dfrac{1}{2}\dfrac{a^2 - b^2}{a^2 b^2} b_3 \tilde{F}_{12} & \end{array}\right.$$

For simplicity, we rewrite these equations the following ;

$$\frac{d\vec{R}}{dt} = \vec{X}(\vec{R}) \qquad \vec{R} = (a(t), b(t), a_3(t), b_3(t), \tilde{F}_{12}(t))$$

Note that $\vec{X}$ **is homogeneous and it means that**

$$\vec{X}(\vec{R}) = \vec{X}(\lambda\vec{R})$$

For arbitrary λ

We put $\vec{S} \equiv \frac{\vec{R}}{R}$ and $R \equiv |\vec{R}|$.

where $\vec{S} \cdot \vec{S} = 1$

Note that $\vec{X}(\vec{R}) = \vec{X}(\vec{S})$ **because of homogeneity.**

We write $\vec{X} \equiv \vec{X}(S)$.

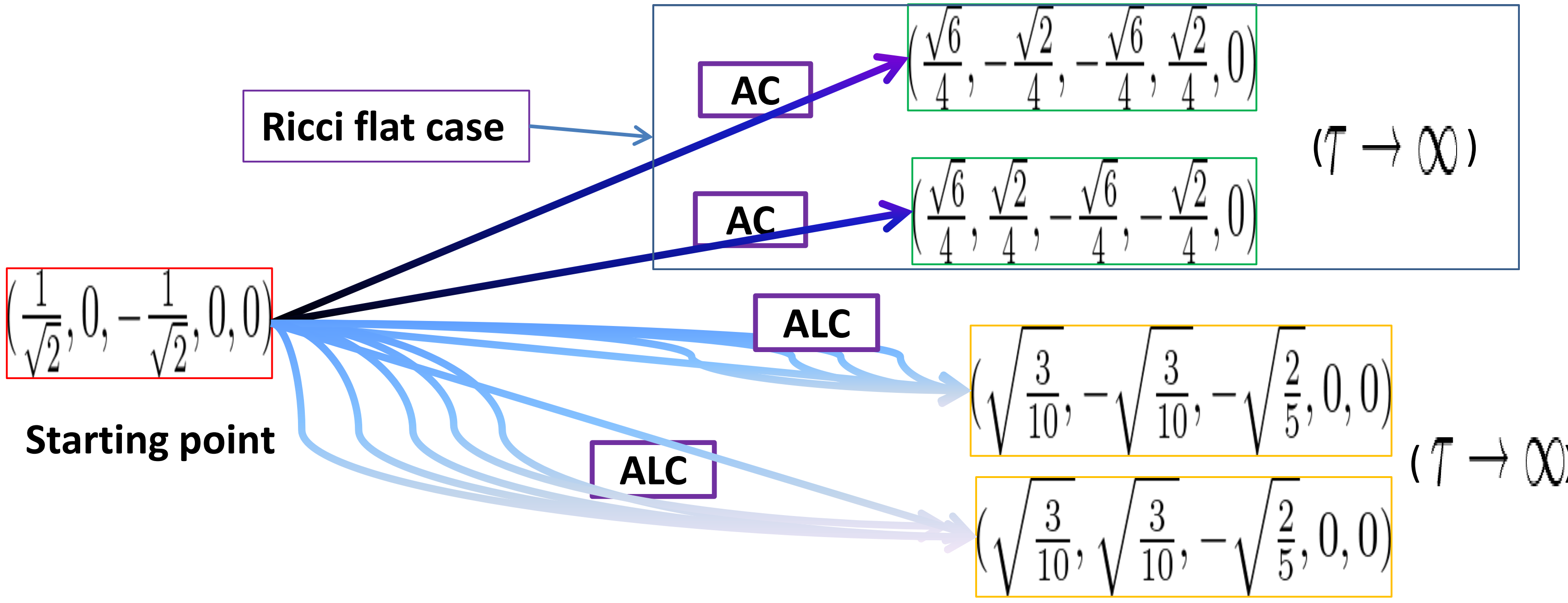
The first order equations are regarded as the flow equations on S^4 by using the variable $\vec{S}$.

Then flow equations on S^4 are

$$\frac{d\vec{S}}{d\tau} = \vec{X} - (\vec{X} \cdot \vec{S})\vec{S}$$

($\frac{1}{R}\frac{dR}{d\tau} = \vec{X} \cdot \vec{S}$)

where, $d\tau \equiv \frac{1}{R} dt$.



This figure represents the solutions as flows on S^4.

- **Arrows represent flows on S^4. Stating point is given by zeroth order Taylor expansion.**
- **ALC solutions have two free parameters, and these parameters give directions of flows.**
- **AC solutions have one parameter, and their flows are one dimensional.**
- **Final points are stationary points of the flow equation.**

The equation has discrete symmetry;

There are eight starting points, so AC solutions are totally 16.

Eight starting points on S^4

$$\left\{\begin{array}{llll}
(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}},0,0) & (-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}},0,0) & (\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}},0,0) & (-\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}},0,0) \\
(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0) & (0,-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,0) & (0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,0) & (0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,0)
\end{array}\right.$$

AC

$$\left\{\begin{array}{llll}
(\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4},0) & (-\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4},-\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4},0) & (\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4},-\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4},0) & (\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4},-\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4},0) \\
(\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4},0) & (\frac{-\sqrt{6}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4},0) & (\frac{-\sqrt{6}}{4},-\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4},0) & (\frac{-\sqrt{6}}{4},-\frac{\sqrt{2}}{4},-\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4},0) \\
(\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4},0) & (\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4},-\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4},0) & (\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4},-\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4},0) & (\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4},0) \\
(\frac{-\sqrt{6}}{4},-\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4},0) & (\frac{-\sqrt{6}}{4},\frac{\sqrt{2}}{4},-\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4},0) & (\frac{-\sqrt{6}}{4},\frac{\sqrt{2}}{4},-\frac{\sqrt{6}}{4},\frac{\sqrt{2}}{4},0) & (\frac{-\sqrt{6}}{4},\frac{\sqrt{2}}{4},\frac{\sqrt{6}}{4},-\frac{\sqrt{2}}{4},0)
\end{array}\right.$$

Contents

We will construct SUSY solutions of 8-dim abelian heterotic SUGRA by using the similar way to case of 7 dimension.

First order equations

$$\begin{cases} T^{(3)} = *d\Omega^{(4)} - \frac{7}{6} * \left(\Theta \wedge \Omega^{(4)}\right) & \Theta = -\frac{1}{7} * \left(*d\Omega^{(4)} \wedge \Omega^{(4)}\right) \\ H_{(3)} = T^{(3)} & d\Phi = -\frac{7}{6}\Theta \\ dH_{(3)} = F_{(2)} \wedge F_{(2)} & *(\Omega^{(4)} \wedge F_{(2)}) = F_{(2)} \\ dF_{(2)} = 0 & \end{cases}$$

metric on $M_8 = \boldsymbol{R} \times \mathcal{P}$ **;** $\quad g^{(8)} = dt^2 + \sum_{i=1}^{3} a_i(t)^2 \boldsymbol{\varphi}_i(t)^2 + f(t)^2 g_{M_4}$

$\mathcal{P}$ **; principal SO(3) fiber bundle on M_4** $\quad$ **global 1-form on** $\mathcal{P}$ **;** $\quad \boldsymbol{\varphi} = \sum_{i=1}^{3} \boldsymbol{\varphi}^i E^i$

global curvature 2-form on $\mathcal{P}$ **;** $\quad \boldsymbol{\omega} = \sum_{i=1}^{3} \boldsymbol{\omega}^i E^i \qquad \boldsymbol{\omega}^i = d\boldsymbol{\varphi}^i + \frac{1}{2}\epsilon_{ijk}\boldsymbol{\varphi}^j \wedge \boldsymbol{\varphi}^k$

4-form which has one to one correspondence with Spin(7)-structure :

$$\begin{aligned} \Omega^{(4)} \quad := \quad & a_1(t)a_2(t)a_3(t)dt \wedge \boldsymbol{\varphi}^{123} - f(t)^4 vol(M_4) \\ & + f(t)^2(a_1(t)dt \wedge \boldsymbol{\varphi}^1 - a_2(t)a_3(t)\boldsymbol{\varphi}^2 \wedge \boldsymbol{\varphi}^3) \wedge \boldsymbol{\omega}^1 \\ & + f(t)^2(a_2(t)dt \wedge \boldsymbol{\varphi}^2 - a_3(t)a_1(t)\boldsymbol{\varphi}^3 \wedge \boldsymbol{\varphi}^1) \wedge \boldsymbol{\omega}^2 \\ & + f(t)^2(a_3(t)dt \wedge \boldsymbol{\varphi}^3 - a_1(t)a_2(t)\boldsymbol{\varphi}^1 \wedge \boldsymbol{\varphi}^2) \wedge \boldsymbol{\omega}^3 \end{aligned}$$

$F_{(2)}$ and $H_{(3)}$ are restricted by $dH_{(3)} = F_{(2)} \wedge F_{(2)}$ **;**

$$\Rightarrow \quad H_{(3)} = H_{123}\varphi^{123} - H_{123}\varphi^{1} \wedge \omega^{1} \qquad F_{(2)} = \frac{dF_{\omega_1}}{dt} dt \wedge \varphi^{1} + F_{\omega_1}\left(\omega^{1} - \varphi^{2} \wedge \varphi^{3}\right)$$

The first order equations reduce to

$$\begin{cases} \frac{da}{dt} = \frac{a^2}{f^2} - \frac{a^2}{2b^2} + \left(\frac{1}{f^2} - \frac{1}{2b^2}\right) H_{123} \\ \frac{db}{dt} = \frac{b^2}{f^2} - 1 + \frac{a}{2b} - \frac{H_{123}}{2ab} \\ \frac{df}{dt} = -\frac{a}{2f} - \frac{b}{f} + \frac{H_{123}}{2af} \\ \frac{dH_{123}}{dt} = 4aH_{123}\left(\frac{1}{f^2} - \frac{1}{2b^2}\right) \\ \frac{d\Phi}{dt} = \frac{2}{a}\left(\frac{1}{2b^2} - \frac{1}{f^2}\right) \end{cases}$$

$$a(t) \equiv a_1(t)$$

Where, we imposed $b(t) \equiv a_2(t) = a_3(t)$ **in order to get regular solution.**

solution

$$a(t) = \frac{C_1 A^{-1}}{1+C_1 A^{-2}} \qquad C_1 > 0$$

A , b and f satisfy first order equations when there are no matter fields.

$$A^2 = \frac{4(v-2)zf}{(1+z)v^3} \qquad b^2 = \frac{(v-2)zf}{(1+z)v}$$

$$f^2 = \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}} exp\left[\int^z \frac{1}{(1-z'^2)v} dz'\right]$$

$$v = \frac{2k\sqrt{z}}{(1-z^2)^{\frac{1}{4}}} - 2zF\left[1,\ \frac{1}{2};\ \frac{5}{4};\ 1-z^2\right]$$

hypergeometric function

$$dr = bdt \qquad dr = \frac{f^2 dz}{(1-z^2)v}$$

Also, matters are

$$H_{123} = -\frac{C_1}{(1+C_1 A^{-2})^2} \qquad {F_{\omega_1}}^2 = -H_{123} \qquad \Phi = -\frac{1}{2}\int \frac{1}{a^2}\frac{d}{dt}\log[H_{123}]dt$$

Contents

	G_2	Spin(7)
Free parameters	2	2
stationary point on S^4	**2 (up to discreet sym)**	**2 (up to discreet sym)**
AC metrics exist only Ricci-flat case	**○**	**○**
Exact solution	**○ (numerical)**	**○ (analytic)**

summary

We constructed SUSY solutions on 7 and 8 dimensional manifold;
They are G_2 structure and Spin(7) structure, respectively.

- We would like to challenge to solve general solution in G_2 case as in Spin(7) case whose solution is written by Hyper-geometric function.

- We would like to understand the G_2-structure associated with supergravity from a half-flat SU(3)-structure.

- We would like to construct supersymmetric solutions in non-abelian heterotic theory.