Irregular states and isolated superconformal field theory

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Gaiotto's construction and isolated SCFT

Twisted compactification of 6 dim. $\mathcal{N} = (2,0)$ theory on a punctured Riemann surface $\Sigma_{g,n}$

$\longrightarrow$ Large class (called class $\mathcal{S}$) of $\mathcal{N} = 2$ SCFT's in 4 dim. [Gaiotto '09]

Gauge coupling = marginal deformations $\simeq \mathcal{M}_{g,n}$ (moduli of $\Sigma_{g,n}$), on which the S-duality group acts.

There exists another class of $\mathcal{N} = 2$ SCFT's.
They are "isolated" : they allow no marginal deformations.

Typically they appear as an IR fixed point
on the Coulomb branch of asymptotically free theories.
(Originally found by Argyres-Douglus).

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Isolated SCFT from irregular singularities

Question : Can we obtain isolated $\mathcal{N} = 2$ SCFT's via "the Gaiotto construction" ?

Recent proposals : Use a Riemann sphere with irregular singularities.

[Ceccoti-Neitzke-Vafa 1006.3435; Bonelli-Maruyoshi-Tanzini 1112.1691 Gaiotto-Teschner 1203.1052; D.Xie 1204.2270]

We are going to follow the approach of Gaiotto-Teschner, where they construct the irregular state corresponding to the irregular puncture and introduce the irregular conformal block.

$\longrightarrow$ AGT-like relation for isolated $\mathcal{N} = 2$ SCFT's ?

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Irregular singularity and the Hitchin system

What is the irregular singularity (or irregular puncture) ?

On $\Sigma_{g,n}$ there lives the Hitchin system;

the Higgs field $\varphi(z)$: $\mathfrak{g}(= A_r)$-valued one form $\leftarrow$ by the twist.

$$\det\,(x - \varphi(z)) = x^{r+1} + \phi_2(z)x^{r-1} + \cdots + \phi_r(z)x + \phi_{r+1}(z)$$

the spectral curve $\subset T^*\Sigma_{g,n} \ni (z, x)$ is identified as the SW curve of the corresponding $\mathcal{N} = 2$ theory with the SW diffrential $\lambda_{SW} = x\ dz$.

- Regular singularity : $\varphi(z)$ has a simple pole. $\mathrm{Res}\ \varphi$ gives mass parameters for the flavor symmetry associated with the puncture.
- Irregular singularity : $\varphi(z)$ has a higher order pole. The pole order of the k-differential $\phi_k(z)$ is higher than k.

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AGT-W dictionary and irregular states

AGT-W dictionary
[Alday-Gaiotto-Tachikawa 0906.3219; Wyllard 0907.2189]

$$\phi_2(z) = \langle T(z) \rangle, \qquad \phi_3(z) = \langle W(z) \rangle, \qquad \cdots$$

Mode expansion arround a puncture (at $z = 0$)

$$T(z) = \sum_{k \in \mathbb{Z}} T_k \, z^{-k-2}, \qquad W(z) = \sum_{k \in \mathbb{Z}} W_k \, z^{-k-3}, \qquad \cdots$$

The CFT state $|0\rangle$ associated with a puncture on a Riemann surface,

- Regular singularity : $T_n|0\rangle = W_n|0\rangle = \cdots = 0, \ \ \forall n > 0$
  $\longrightarrow |0\rangle$ is a primary state.
- Irregular singularity : $T_n|0\rangle \neq 0, \ W_n|0\rangle \neq 0, \ \cdots \ \exists n > 0$
  $\longrightarrow |0\rangle$ is NOT primary any more !

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Confluence of regular singularities (I)

We have already encountered "mild" irregular states (the Gaiotto state) whose norm (or inner product) gives the Nekrasov partition function for some asymptotically free theories. [Gaiotto 0908.0307]

For isolated SCFT's we need more "wild" irregular state.

The irreguler singularity arises from a result of the confluence of several regular singularities. (This is familier in the ODE of Fuchsian type; Confluenced hypergeometric functions)

- Confluence of two (full + simple) punctures
  $\longrightarrow$ "Mild" irregular state for asymptotically free theories.
- More than three (one full + others simple) punctures
  $\longrightarrow$ "Wild" irregular state for isolated SCFT's.

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Confluence of regular singularities (II)

The irreguler singularity from the confluence of regular singularities.

Basic idea is illustrated by the following simple identity;

$$\varphi(y)dy \sim \left(\sum_{i=1}^{n} \frac{\alpha_i}{y - z_i} + \frac{\alpha_0}{y} \right) dy = \frac{P_n(y)dy}{y \prod_{i=1}^{n}(y - z_i)}$$

$P_n(y) = c_0 y^n + c_1 y^{n-1} + \cdots + c_n$. ($c_k$ is linear in α_i and order k in z_i.)

Take $z_i \to 0$, while keeping $c_0, \cdots, c_n$ finite.

$$\varphi(y)dy \sim \left(\frac{c_n}{y^{n+1}} + \frac{c_{n-1}}{y^n} + \cdots + \frac{c_0}{y} \right) dy$$

c_k parametrize relevant deformations of isolated SCFT.

"Mild" case ($n = 1$) : $c_0 = \alpha_0 + \alpha_1, \quad c_1 = \alpha_0 z_1$

$\alpha_0 \to \infty \longleftrightarrow$ Decoupling of a hypermultiplet.

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$SU(3)$ case (W_3 Ward identity for simple puncture)

In A_2 theory, we have two types of regular singularities;

- simple type with $U(1)$ flavor symmetry
- full type with $SU(3)$ flavor symmetry

What is "nice" for punctures of simple type is that they allow a null state at level one on $|\alpha_s\rangle := V_{\vec{\alpha_s}}(0)|0\rangle,\ \vec{\alpha_s} = (\alpha, -\frac{Q}{2})$;

$$W_{-1}|\alpha_s\rangle = \frac{3w}{2\Delta} L_{-1}|\alpha_s\rangle = \frac{3\sqrt{\kappa}}{2}\partial_z|\alpha_s\rangle.$$

It helps to simplify the W_3 Ward identity for $|R_1\rangle := V_{\vec{\alpha_s}}(z_1)|\alpha_f\rangle$ such as

$$W_+(y)|R_1\rangle = \left[\frac{w_s}{(y-z_1)^3} + \frac{w_f}{y^3} + \frac{z_1^2}{y^2(y-z_1)^3} W^{(1)}_{-1}\right.$$
$$\left. - \frac{w_s + w_f}{y^2(y-z_1)} + \frac{W_0}{y^2(y-z_1)} + \frac{W_{-1}}{y^2} + \frac{W_{-2}}{y}\right] |R_1\rangle$$

Characterizing properties of the A_2 irregular state

By looking at the W_3 Ward identities in the collision limit, we find the irregular state $|I_n\rangle_{A_2}$ from the collision of n simple punctures with a single puncture of full type satisfies;

- $|I_n\rangle_{A_2}$ is annihilated by higher modes $L_{\ell>2n}$ and $W_{\ell>3n}$.
- $|I_n\rangle_{A_2}$ is a simultaneous eigenstate of $L_n, L_{n+1}, \cdots, L_{2n}$ and $W_{2n}, W_{2n+1}, \cdots, W_{3n}$. Their eigenvalues are polynomials in c_i. Note that this is consistent thanks to $[L_n, W_{2n}] = 0$.
- $L_0, L_1, \cdots, L_{n-1}$ and $W_n, \cdots, W_{2n-1}$ act on $|I_n\rangle_{A_2}$ as first order differential operators in c_i. From the viewpoint of SCFT side, they are conjugate to the relevant deformations.
- We have no simple expressions for $W_0, W_1, \cdots, W_{n-1}$. This means we lose some of information in the limit.

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Degeneration of the Seiberg-Witten curve (I)

Let us try to identify the corresponding isolated SCFT as an IR fixed point on the Coulomb moduli of some $\mathcal{N} = 2$ theory.
For $|I_n\rangle_{A_2}$, it is natural to start with $SU(3)^{n-1}$ quiver theory.

M theory curve of $SU(3)$ theory with $N_f = 6$ for $|I_2\rangle_{A_2}$;

$$\prod_{a=1}^{3}(v + m_a)t^2 - (1 + z)(v^3 + Pv^2 + Qv + R)t + z\prod_{b=4}^{6}(v - m_b) = 0.$$

P is fixed such that the two punctures at $t = 1, z$ are of simple type.
Q and R correspond to the Coulomb moduli of $SU(3)$ theory.

The curve in the Gaiotto form $x^3 + \phi_2(t)x + \phi_3(t) = 0$,
which is a cubic cover of $\mathbb{CP}^1$ with a coordinate t.
$x := v/t$ is a coordinate along the fible of $T^*\mathbb{CP}^1$.

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The SW diffrential is $\lambda_{SW} = x\, dt$.
The quadratic and the cubic differentials ($|z| < 1$) are

$$\phi_2(t) = -3\left(\frac{m_+}{t-1} + \frac{\widetilde{m}_+}{t-z} + \frac{\widetilde{m}_-}{t}\right)^2 + \frac{V_2 t - (1+z)u^{(2)}}{t(t-1)(t-z)}$$

$$\phi_3(t) = 2\left(\frac{m_+}{t-1} + \frac{\widetilde{m}_+}{t-z} + \frac{\widetilde{m}_-}{t}\right)^3 + \frac{-m_+ V_2 t^3 + \cdots}{t^2(t-1)^2(t-z)^2} + \frac{V_3 t^2 - (1+z)u^{(3)} t + z\hat{M}_3}{t^3(t-1)(t-z)}$$

Take the scaling limit $z\widetilde{m}_- \to \infty$ with $t = (z\widetilde{m}_-)^{1/2} w$,
while $c_1 = (z\widetilde{m}_-)^{-1/2}\hat{c}_1$ is fixed;

$$\left(\frac{m_+}{t-1} + \frac{\widetilde{m}_+}{t-z} + \frac{\widetilde{m}_-}{t}\right) dt = \frac{c_0 t^2 + \hat{c}_1 t + z\widetilde{m}_-}{t(t-1)(t-z)} dt$$

$$\to \left(\frac{1}{w^3} + \frac{c_1}{w^2} + \frac{c_0}{w}\right) dw$$

Quadratic and cubic differential in the scaling limit

With the fine tuning of the remainder terms which involve the Coulomb moduli, we obtain

$$\phi_2 (dt)^2 \rightarrow \left[-3 \left(\frac{1}{w^3} + \frac{c_1}{w^2} + \frac{c_0}{w} \right)^2 + \frac{v_1^{(2)}}{w^3} + \frac{V_2}{w^2} \right] (dw)^2$$

$$\phi_3 (dt)^3 \rightarrow \left[2 \left(\frac{1}{w^3} + \frac{c_1}{w^2} + \frac{c_0}{w} \right)^3 - \frac{v_1^{(2)}}{w^6} + \frac{\beta^2 - V_2 - c_1 v_1^{(2)}}{w^5} \right.$$
$$\left. + \frac{v_1^{(3)} - c_1 V_2 - c_0 v_1^{(2)}}{w^4} + \frac{V_3 - m_+ V_2}{w^3} \right] (dw)^3$$

The scaling dimension of the parameters is fixed by demanding $\Delta(\lambda_{SW}) = 1$.

$$\Delta(c_1) = \frac{1}{2}, \quad \Delta(v_1^{(2)}) = \frac{3}{2}, \quad \Delta(v_1^{(3)}) = \frac{5}{2} > 2,$$
$$\Delta(c_0) = \Delta(\beta) = \Delta(m_a) = 1.$$

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Conditions for irregular states from differentials

We see that $v_1^{(3)}$ is regarded as an irrelevant parameter and $v_1^{(2)}$ can be paired with the relevant deformation parameter c_1. Hence $v_1^{(2)}$ is realized as the differential operator in c_1.

Looking at the (non-vanishing) coefficients of $w^{-6} \cdots w^{-3}$ for $\phi_2(w)$ and $w^{-9}, \cdots, w^{-4}$ for $\phi_3(w)$, we find the eigenvalues of $L_2, \cdots, L_4$ and $W_4, \cdots, W_6$. (Note that we have normalized $c_2 = 1$.) Furthermore, we see L_1, W_2 and W_3 involve $\partial/\partial c_1$.

$$W_2|I_2\rangle \sim \left(3c_0^2c_2 + 3c_0c_1^2 - 3(\beta_3^2 + 5\beta^2)c_2 + 3c_1c_2\partial_{c_1} + 3c_2^2\partial_{c_2}\right)|I_2\rangle,$$

$$W_3|I_2\rangle \sim \left(6c_0c_1c_2 + c_1^3 + 3c_2^2\partial_{c_1}/2\right)|I_2\rangle,$$

$$W_4|I_2\rangle \sim \left(3c_0c_2^2 + 3c_1^2c_2\right)|I_2\rangle,$$

$$W_5|I_2\rangle \sim 3c_1c_2^2|I_2\rangle,$$

$$W_6|I_2\rangle \sim c_2^3|I_2\rangle.$$

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Summary and Discussion

- We have derived the conditions for the W_3 irregular state by considering the collision limit of CFT Ward identities with a single full puncture and $n(> 0)$ simple punctures.
- The condition is consistent with an appropriate degeneration of the SW curve of $SU(3)$ quiver gauge theory. (We have identified the corresponding SCFT on the Coulomb moduli space of $SU(3)$ quiver gauge theory.)
- The isolated SCFT corresponding our A_2 irregular state seems to be in a rather restricted class. (The quadratic differential ϕ_2 and the cubic differential ϕ_3 are not independent.)
- It is curious to see what the inner product with irregular states (or the irregular conformal block) tells us on SCFT side. One of the issues is non-uniqueness of the irregular state.

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