A proof of AGT conjecture for general β

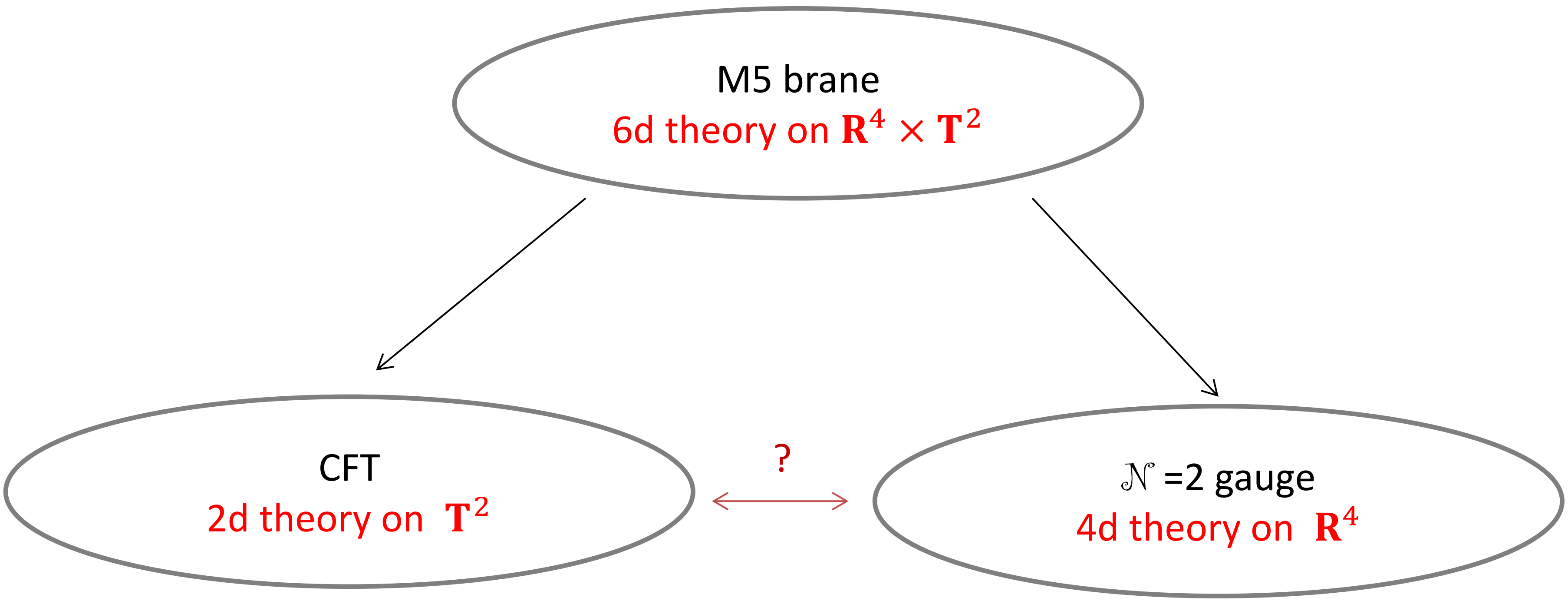
張　弘

東京大学素粒子研究室

S. Kanno, Y. Matsuo and H. Zhang, ArXiv:1304.????

AGT conjecture

Inspired by the idea that the 6d theory can be compactified in two different way



[*Gaiotto '09*] [*Alday-Gaiotto-Tachikawa '09*]

General idea

2D CFT

4D Gauge

Conformal blocks

AGT conjecture

Partition function

$$\langle\langle \vec{Y} | V | \vec{W} \rangle\rangle = Z$$

Satisfy Ward identity

Constrained by a recursion Relation

$$\sum \hat{O} \langle\langle \vec{Y} | V | \vec{W} \rangle\rangle = 0$$

$$\sum \hat{O} Z = 0$$

MAIN RESULT

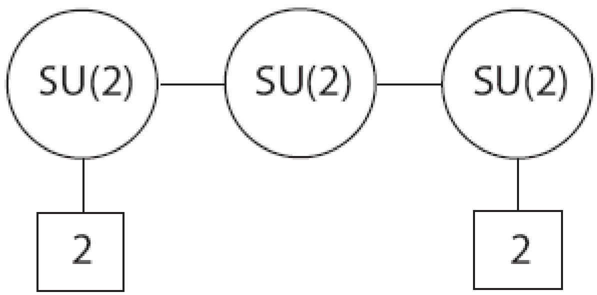
Selberg Integral and SU(N) AGT Conjecture

Virasoro constraint for Nekrasov instanton partition function

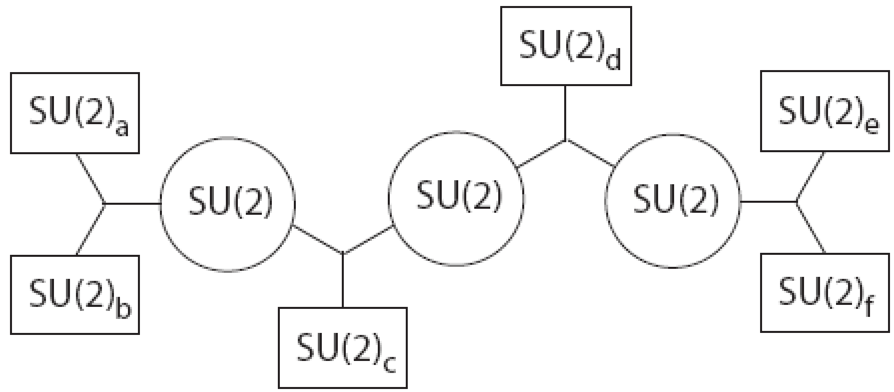
$\mathcal{N}$ =2 theories

[*Gaiotto '09*]

4d $\mathcal{N}$ =2 theories are constructed by wrapping M5 branes over a Riemann surface.

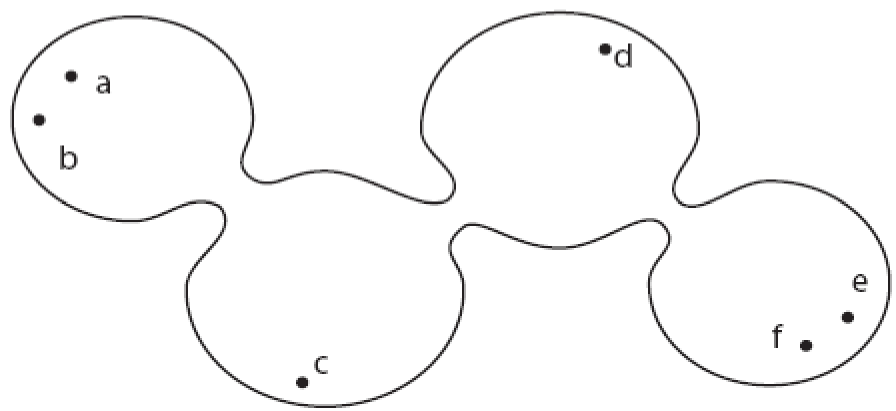


Circles indicate gauge groups, and squares fundamental flavors.



the same theory depicted as a generalized quiver, with the SU(2) factors of the flavor group made evident and labeled.

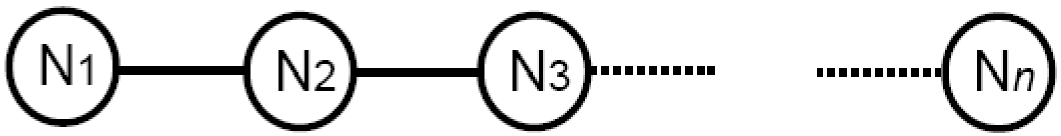
The corresponding space of gauge couplings is parameterized by a sphere with six punctures.



Nekrasov partition function

The low energy effective Limit of 4d N=2 SUSY gauge theory is described by Seiberg-Witten prepotential. Its generating function is given by Nekrasov.

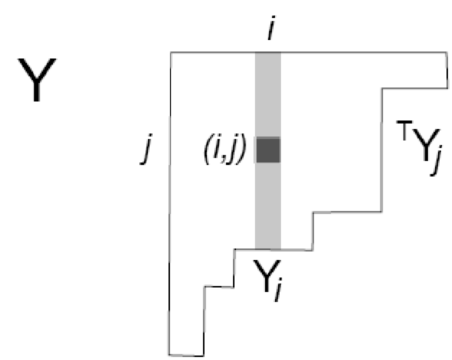
we focus on the linear quiver gauge theories with gauge group $SU(N_1)\times\cdots\times SU(N_n)$.



For this case, Nekrasov partition function is written in the form of matrix multiplication

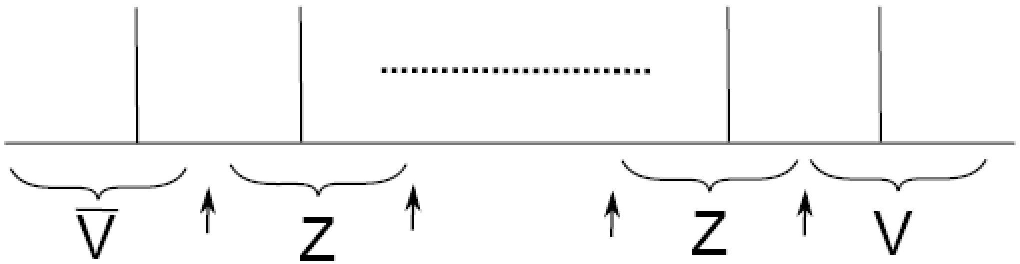
$$Z^{\mathrm{Nek}} = \sum_{\vec{Y}^{(1)},\cdots,\vec{Y}^{(n)}} q_i^{|\vec{Y}^{(i)}|} \bar{V}_{\vec{Y}^{(1)}} \cdot Z_{\vec{Y}^{(1)}\vec{Y}^{(2)}} \cdots Z_{\vec{Y}^{(n-1)}\vec{Y}^{(n)}} \cdot V_{\vec{Y}^{(n)}}$$

$$\begin{aligned} Z_{\vec{Y}^{(i)}\vec{Y}^{(i+1)}} &= Z(\vec{a}^{(i)}, \vec{Y}^{(i)}; \vec{a}^{(i+1)}, \vec{Y}^{(i+1)}; \mu^{(i)}), \\ \bar{V}_{\vec{Y}^{(1)}} &= Z(\vec{\lambda}, \vec{\emptyset}; \vec{a}^{(1)}, \vec{Y}^{(1)}; \mu^{(0)}), \\ V_{\vec{Y}^{(n)}} &= Z(\vec{a}^{(n)}, \vec{Y}^{(n)}; \vec{\lambda}', \vec{\emptyset}; \mu^{(n)}), \end{aligned}$$



2D CFT

We illustrate the conformal block as below. It can be reduced to the multiplication of three point functions by inserting a complete basis of the Hilbert space at the intermediate channel. Insertion points of such operators are depicted by arrows.

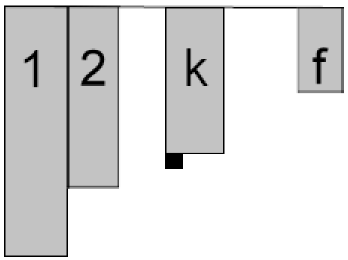


The basis of the Hilbert space is labeled by N Young tables $\vec{Y}$.Then it may be possible to choose such basis so that the factor Z may be rewritten as $\langle\langle \vec{Y}|V|\vec{W}\rangle\rangle = Z$ with some vertex operator V .

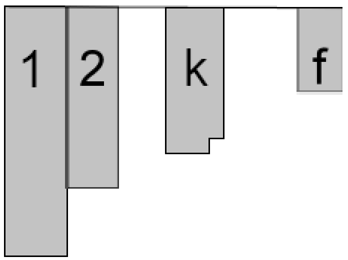
Method: Variation of Nekrasov formula

Through former works, we get familiar with the action of adding one box

$Y^{(k,+)} =$



$Y^{(k,-)} =$



$$\frac{\langle \vec{Y}^{(+)}|V(1)|\vec{W}\rangle}{\langle \vec{Y}|V(1)|\vec{W}\rangle}$$

Construction of the constraints

We would like to establish that the partition function is written as an inner product,

$$Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu) = \langle \vec{a},\vec{Y}|V(1)|\vec{b},\vec{W}\rangle$$

We use the following trivial identity of operator O,

$$\begin{aligned} 0 &= \langle \vec{a},\vec{Y}|\mathcal{O}V_\kappa(1)|\vec{b},\vec{W}\rangle - \langle \vec{a},\vec{Y}|V_\kappa(1)\mathcal{O}|\vec{b},\vec{W}\rangle - \langle \vec{a},\vec{Y}|[\mathcal{O},V_\kappa(1)]|\vec{b},\vec{W}\rangle \\ &= \sum_{\vec{Y}',\vec{W}'} \hat{\mathcal{O}}_{\vec{Y},\vec{W}}^{\vec{Y}',\vec{W}'} \langle \vec{a},\vec{Y}'|V_\kappa(1)|\vec{b},\vec{W}'\rangle\,. \end{aligned}$$

Then the Nekrasov function should also satisfy the relation, namely,

$$\sum_{\vec{Y}',\vec{W}'} \hat{\mathcal{O}}_{\vec{Y},\vec{W}}^{\vec{Y}',\vec{W}'} Z(\vec{a},\vec{Y}';\vec{b},\vec{W}';\mu) = 0$$

Previous work

there exist simple recursion formulae for Nekrasov's instanton partition function for SU(N) gauge theories

$$\sum_{\vec{Y}',\vec{W}'} (\hat{J}_n)_{\vec{Y},\vec{W}}^{\vec{Y}',\vec{W}'} Z_{\vec{Y}',\vec{W}'} = 0\,, \qquad \sum_{\vec{Y}',\vec{W}'} (\hat{L}_n)_{\vec{Y},\vec{W}}^{\vec{Y}',\vec{W}'} Z_{\vec{Y}',\vec{W}'} = 0$$

- Extend the constraint to W1+∞ algebra.

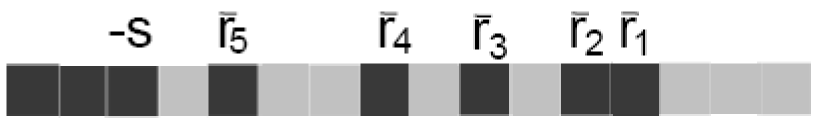
- Consider general β.

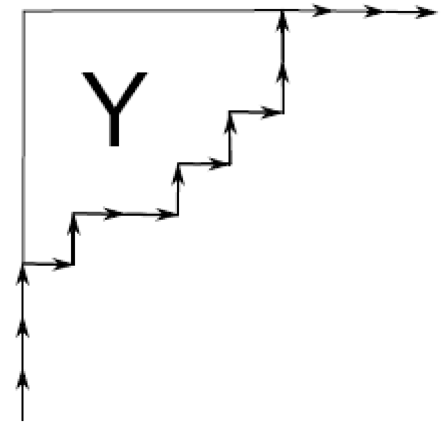
  modied vertex operator

  Evaluate the action on bra and ket basis

β = 1 case: Action on bra and ket basis

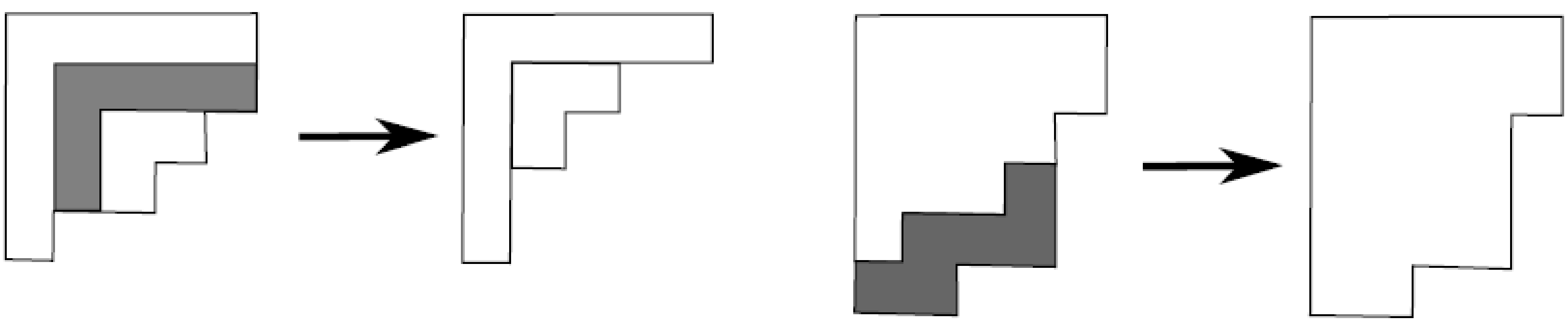
$$|Y,\lambda\rangle = \bar\psi_{-\bar r_1}\bar\psi_{-\bar r_2}\cdots\bar\psi_{-\bar r_s}\bar\psi_s\bar\psi_{s+1}\cdots\bar\psi_L|-L,\lambda\rangle$$





The generator flips one black box at ℓ to white and one white box at −ℓ − n to black.

$$\mathcal{W}(z^n e^{xD}) = \sum_\ell e^{x(\ell+\lambda)} :\bar\psi_{\ell+n}\psi_{-\ell}:$$



Commutator with the vertex

$$[J_n, V_\kappa(1)] = \kappa V(1), \quad [L_n, V_\kappa(1)] = \frac{\kappa^2(n+1)}{2}V_\kappa(1) + \partial V(1)$$

SH^c algebra

to evaluate the action on bra and ket basis, different from the former *β = 1 case, we need the introduction of* SH^c: algebra

$$
\begin{aligned}
[D_{0,l}, D_{1,k}] &= D_{1,l+k-1} \quad l \geq 1 \\
[D_{0,l}, D_{-1,k}] &= -D_{-1,l+k-1}, \quad l \geq 1 \\
[D_{-1,k}, D_{1,l}] &= E_{k+l} \quad l,k \geq 1
\end{aligned}
$$

where c_l *are central charges and* E_k *are some nonlinear combinations of D.*

$$\mathrm{SH}^c \supset \mathrm{U(1)} \oplus \mathrm{Virasoro} \oplus \mathrm{W}$$

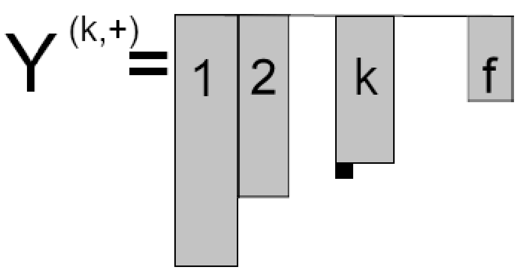
Definition of generators on the basis

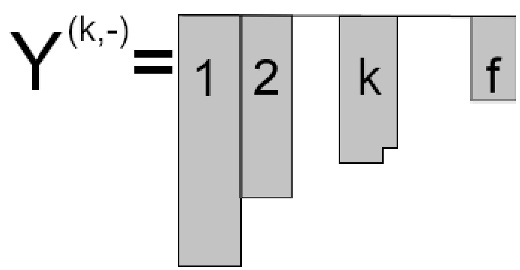
The action *on the ket state basis is defined as*

$$D_{-1,l}|\vec{b},\vec{W}> \;=\; \sum_{q=1}^{N}\sum_{t=1}^{\tilde{f}_q} \mathrm{Coefficients}\times|\vec{b},\vec{W}^{(t,-),q}>$$

$$D_{1,l}|\vec{b},\vec{W}> \;=\; \sum_{q=1}^{N}\sum_{t=1}^{\tilde{f}_q+1} \mathrm{Coefficients}\times|\vec{b},\vec{W}^{(t,+),q}>$$

$$D_{0,l+1}|\vec{b},\vec{W}> \;=\; \sum_{q=1}^{N} \mathrm{Coefficients}\times|\vec{b},\vec{W}> .$$





Virasoro and U(1) algebra as subalgebras of SHc

U(1) current and Virasoro generators are embedded in SHc as,

$$J_l = (-x)^{-l} D_{-l,0}, \quad J_{-l} = y^{-l} D_{l,0}, \quad J_0 = E_1/\beta,$$
$$L_l = (-x)^{-l} D_{-l,1}/l + (1-l)c_0 \xi J_l/2,$$
$$L_{-l} = y^{-l} D_{l,1}/l + (1-l)c_0 \xi J_{-l}/2 \quad L_0 = [L_1, L_{-1}]/2$$

These satisfy the following standard relations

$$\begin{aligned} [J_m, J_n] &= Nm\delta_{m+n,0} \\ [L_m, J_n] &= -nJ_{m+n}, \\ [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \end{aligned}$$

Conformal property of modied vertex operator

We need a modified vertex operator

$$\tilde{V}_\kappa(z) = e^{\frac{i}{\sqrt{N}}(NQ-\kappa)\phi_-} e^{\frac{-i}{\sqrt{N}}\kappa\phi_+}$$

$$\phi_+ = \alpha_0 \log z - \sum_{n=1}^{\infty} \frac{\alpha_n}{n} z^{-n}, \quad \phi_- = q + \sum_{n=1}^{\infty} \frac{\alpha_{-n}}{n} z^n$$

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0}\,, \quad [\alpha_m, q] = \delta_{m,0}$$

Commutate with *U(1) current*

$$[\alpha_m, \tilde{V}_\kappa(z)] = \frac{i}{\sqrt{N}}(NQ-\kappa)z^m \tilde{V}_\kappa(z), \quad [\alpha_{-n}, \tilde{V}_\kappa(z)] = \frac{-i}{\sqrt{N}}\kappa z^{-n} \tilde{V}_\kappa(z)$$

U(1)

By using the proposal $Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu) = \langle \vec{a},\vec{Y}|V(1)|\vec{b},\vec{W}\rangle$

$$\frac{\langle \vec{Y}|J_1V(1)|\vec{W}\rangle - \langle \vec{Y}|V(1)J_1|\vec{W}\rangle}{\langle \vec{Y}|V(1)|\vec{W}\rangle} = \sum_{p=1}^{N}(a_p - b_p - \mu + \beta - 1)$$

$$\frac{\langle \vec{Y}|J_{-1}V(1)|\vec{W}\rangle - \langle \vec{Y}|V(1)J_{-1}|\vec{W}\rangle}{\langle \vec{Y}|V(1)|\vec{W}\rangle} = \sum_{p=1}^{N}(a_p - b_p - \mu)$$

These are compatible with the fact that

$$\frac{\langle \vec{Y}|[J_1, V(1)]|\vec{W}\rangle}{\langle \vec{Y}|V(1)|\vec{W}\rangle} = i(NQ - \kappa) \qquad \frac{\langle \vec{Y}|[J_{-1}, V(1)]|\vec{W}\rangle}{\langle \vec{Y}|V(1)|\vec{W}\rangle} = -i\kappa$$

So we can see k is the momentum and iQ =*β - 1*

We have also established a similar expression of J_2 case.

Virasoro

$$\langle \vec{Y}|L_1 = \sum_{p=1}^{N}\sum_{k=1}^{f_p} \langle \vec{Y}^{(k,+),p}|\tilde{A}_{(p),k}\Lambda^{(k,+),p}(\vec{Y})$$

$$L_1|\vec{W}\rangle = \sum_{q=1}^{N}\sum_{\ell=1}^{f_p} \Lambda^{(\ell,-),q}(\vec{W})D_{(q),\ell}|\vec{W}^{(\ell,-),q}\rangle$$

$$\frac{\langle \vec{Y}|L_1 V(1)|\vec{W}\rangle - \langle \vec{Y}|V(1)L_1|\vec{W}\rangle - \langle \vec{Y}|(\beta-1)V(1)J_1|\vec{W}\rangle}{\langle \vec{Y}|V(1)|\vec{W}\rangle} =$$

$$\frac{1}{2}\sum_{p=1}^{N}(a_p+\nu+\beta-1)^2 - \frac{1}{2}\sum_{p=1}^{N}(b_p+\nu+\mu)^2 + \frac{1}{2}\Big(\sum_{p=1}^{N}(a_p+\beta-1-b_p-\mu)\Big)^2 + \beta|\vec{Y}| - \beta|\vec{W}|$$

And there is a similar relation of L_{-1}.

Conclusion

The constraint equations give a direct support to (SU(N) generalization of) AGT conjecture, with arbitrary *β.*

- Extend the constraint to W1+∞ algebra.