Schrödinger Sigma Models and the Classical Jordanian Twist

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work in progress with

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based on `arXiv:1304.XXXX` (in preparation).

Motivation

What are the Schrödinger Sigma Models?

The two-dim. classical field theory whose target space is the null-like ("pp-wave") deformation of AdS_3, so called the Schrödinger spacetime.

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Why AdS_3?

- Simpler than $AdS_5, S^5, SO(4,2), SU(4), SU(2|2), SU(2,2|4)$...
- Schrödinger spacetime could not be obtained from squashed S^3.

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In this talk: We will show that the Jordanian twist provides a comprehensive understanding for the integrable structures of the Schrödinger sigma models, such as the Lax connections, the monodromy and the algebraic symmetries.

Plan

Schrödinger Spacetime - 1/2

The metric of 3-dim Schrödinger spacetime is a null-like deformation of AdS_3, which is given by [Detournay, Orlando, Petropoulos, Spindel] [Anninos, Li, Padi, Song, Strominger]

$$ds^2 = \underbrace{d\rho^2 - 2e^{-2\rho}dudv}_{AdS_3} - \underbrace{Ce^{-4\rho}dv^2}_{\text{deformation}}$$

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The deformation by C causes the breaking of the global symmetry of AdS_3;

$$SL(2,\mathbb{R})_L \times SL(2,\mathbb{R})_R \quad \rightarrow \quad SL(2,\mathbb{R})_L \times U(1)_R$$

Schrödinger Spacetime - 2/2

The isometry $SL(2,\mathbb{R})_L \times U(1)_R$ of the Schrödinger spacetime is more obvious if it is expressed in terms of the Maurer-Cartan 1-form

$$J = g^{-1}dg \qquad \text{with} \qquad g = \mathrm{e}^{2vT^+}\mathrm{e}^{2\rho T^2}\mathrm{e}^{2uT^-}$$

where $T^a (a = 2, \pm)$ are fund. repr. of $\mathfrak{sl}(2)$ generators given by

$$T^2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T^+ = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T^- = \frac{-1}{\sqrt{2}}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

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Then, the metric is written as

$$ds^2 = \frac{1}{2}\left[\mathrm{Tr}(J^2) - 2C\,\mathrm{Tr}(T^-J)^2\right]$$

This is invariant under $SL(2,\mathbb{R})_L \times U(1)_R \ni (\mathrm{e}^{xT^a}, \mathrm{e}^{yT^-})$ action,

$$g \mapsto \mathrm{e}^{xT^a} g\, \mathrm{e}^{yT^-} \qquad J \mapsto \mathrm{e}^{-yT^-} J\, \mathrm{e}^{yT^-} \ .$$

Schrödinger Sigma Models - 1/3

Field $g(t,x)$ is an embedding of the worldsheet $\mathcal{W}$ to the target space AdS_3

$$\mathcal{W} = \left\{ x^\mu = (t,x) \middle| dx^2 = \eta^{\mu\nu} dx_\mu dx_\nu,\ \eta^{\mu\nu} = \mathrm{diag}(-1,+1) \right\}$$

$$g : \mathcal{W} \ni (t,x) \mapsto g(t,x) \in SL(2,\mathbb{R}) \simeq AdS_3.$$

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Current $J(t,x) = g^{-1}dg$ is the $\mathfrak{sl}(2)$-valued 1-form

$$J = J_\mu dx^\mu, \qquad J_\mu = -J_\mu^+ T^- - J_\mu^- T^+ + J_\mu^2 T^2$$

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Action $S[J]$ of the Schrödinger sigma models are given by

$$S[J] = -\eta^{\mu\nu} \iint_{-\infty}^{\infty} dt dx [\mathrm{Tr}(J_\mu J_\nu) - 2C\,\mathrm{Tr}(T^- J_\mu)\mathrm{Tr}(T^- J_\nu)].$$

Schrödinger Sigma Models - 2/3

From the action of Schrödinger sigma models

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we obtain the Equation of Motion $\delta S/\delta g = 0$;

$$\partial^\mu J_\mu = C\left(\partial^\mu J_\mu^- T^- + J_\mu^- [J^\mu, T^-]\right) .$$

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$$\partial^\mu J_\mu = C\big(\partial^\mu J_\mu^- T^- + J_\mu^- [J^\mu, T^-]\big) \;.$$

In addition, it satisfies Flatness condition due to the def. $J = g^{-1}dg$,

$$dJ + J \wedge J = 0 \;.$$

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  $\Rightarrow$ Two descriptions based on $SL(2,\mathbb{R})_L$ and $U(1)_R$ respectively.

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Rational description	based on $SL(2,\mathbb{R})_L$ symmetry
Anisotropic description	based on $U(1)_R$ symmetry

Plan

Left Flat Conserved Current

The flat and conserved left current was introduced by [Kawaguchi Yoshida]

$$j^{L,\pm} = dg\, g^{-1} - 2C\,\mathrm{Tr}(T^- J) g T^- g^{-1} \pm \sqrt{C} * d(g T^- g^{-1}).$$

The last term could be added for the flatness $dj^{L,\pm} + j^{L,\pm} \wedge j^{L,\pm} = 0$ **without** violating the conservation law $d * j^{L,\pm} = 0$.

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The flat conserved current allows us to introduce the Lax connection [a],

$$L^L(\lambda_L) = \frac{j^L - \lambda_L * j^L}{1 - \lambda_L^2} \quad \Rightarrow \quad dL^L(\lambda_L) + L^L(\lambda_L) \wedge L^L(\lambda_L) = 0$$

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We could also define the monodromy matrix as

$$U^L(\lambda^L) = \mathrm{Pexp} \int_{\mathcal{C}} L^L(\lambda_L) \quad \Rightarrow \quad \frac{d}{dt} U^L(\lambda^L) = 0 \quad \text{(conserved)}$$

Thanks to the spectral param. λ_L, there are infinitely many conserved charges!

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Algebraic Structure

The algebra of the higher conserved charges $Q_{(n\geq 0)}$ introduced by $U^L(\lambda_L) =: \exp\left(\sum_{n=0}^{\infty} Q_{(n)} \lambda_L^{-n-1}\right)$ could be computed by requiring

$$\{X(x), \Pi_X(y)\}_{\mathrm{P}} \equiv \delta(x-y) \qquad \text{for} \qquad X \in (v, \rho, u)$$

where $\quad \Pi_X(y) = \dfrac{\partial \mathcal{L}}{\partial \dot{X}}$ $\qquad$ Conjugate momentum.

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The algebra does not depend on deform. param. C $\left(Q^a = \frac{1}{2}\mathrm{Tr}(T^a Q^a)\right)$

$$\{Q^a_{(0)}, Q^b_{(0)}\}_{\mathrm{P}} = \epsilon^{ab}{}_c\, Q^c_{(0)} \qquad \{Q^a_{(0)}, Q^b_{(1)}\}_{\mathrm{P}} = \epsilon^{ab}{}_c\, Q^c_{(1)}$$

$$\{Q^a_{(1)}, Q^b_{(1)}\}_{\mathrm{P}} = -\frac{1}{12}\epsilon^{ab}{}_c\, Q^c_{(0)} (Q_{(0)})^2 + \epsilon^{ab}{}_c\, Q^c_{(2)}.$$

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$$\{Q^a_{(1)}, Q^b_{(1)}\}_{\mathrm{P}} = -\frac{1}{12}\epsilon^{ab}{}_c Q^c_{(0)} (Q_{(0)})^2 + \epsilon^{ab}{}_c Q^c_{(2)}.$$

The last term does **Not** affect the Yangian Serre relation,

$$\left\{\{Q^+_{(1)}, Q^-_{(1)}\}_{\mathrm{P}}, Q^2_{(1)}\right\}_{\mathrm{P}} = \frac{1}{4}(Q^+_{(1)} Q^-_{(0)} - Q^+_{(0)} Q^-_{(1)}) Q^2_{(0)}.$$

Thus, the Yangian $\mathcal{Y}(\mathfrak{su}(2))$ is still realized. [Kawaguchi Yoshida] ([MacKay 1992] for $C = 0$)

Plan

q-Poincaré Symmetry

The $U(1)_R$ symmetry enhances to the classical q-Poincaré algebra! [Kawaguchi Yoshida]

The non-trivial conserved currents are given by

$$j^{R,+} = -\mathrm{e}^{\sqrt{C}\chi}(J^{+} + CJ^{-} + \sqrt{C} * J^{2})$$

$$j^{R,2} = -\mathrm{e}^{\sqrt{C}\chi}(J^{2} + \sqrt{C} * J^{-}), \qquad j^{R,-} = -J^{-}$$

where $\chi(x)$ is a non-local field defined by ($\epsilon(x)$ is the signature fun.)

$$\chi(x) := \frac{1}{2}\int_{-\infty}^{\infty} dy\, \epsilon(x-y) j_t^{R,-}(y) \,.$$

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The Poisson alg. for $Q^R = \int *j^R$ reads the q-Poincaré algebra ($\xi := \frac{\sqrt{C}}{2}$)

$$\{Q^{R,2}, Q^{R,+}\}_{\mathrm{P}} = Q^{R,+}\cosh(\xi Q^{R,-}), \quad \{Q^{R,-}, Q^{R,+}\}_{\mathrm{P}} = Q^{R,2}$$

$$\{Q^{R,2}, Q^{R,-}\}_{\mathrm{P}} = -\frac{1}{\xi}\sinh(\xi Q^{R,-}),$$

which is knows as a non-standard quantization of $U(\mathfrak{sl}(2))$. [Ch.Ohn 1992]

Affine Extension

We have another set of q-Poincaré alg. by flipping the sign $\sqrt{C} \to -\sqrt{C}$.
Under this map, the currents, charges are mapped as $j^R \mapsto \tilde{j}^R$, $Q^R \mapsto \widetilde{Q}^R$.
For example,

$$j^{R,2} = -\mathrm{e}^{\sqrt{C}\chi}(J^2 + \sqrt{C} * J^-) \mapsto \tilde{j}^{R,2} = -\mathrm{e}^{-\sqrt{C}\chi}(J^2 - \sqrt{C} * J^-).$$

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Q. What's the whole algebra including the mixed comm. $\{Q^R, \widetilde{Q}^R\}_{\mathrm{P}}$?

A. It leads an infinite dimensional Exotic Symmetry! which only has the positive levels such as Yangian. $\Rightarrow$ Kawaguchi-san's Talk! [Kawaguchi Yoshida]

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c.f. In the squashed sigma models, this procedure enhances quantum group $U_q(\mathfrak{su}(2))$ to quantum affine algebra $U_q(\widehat{\mathfrak{su}}(2))$. [Kawaguchi Yoshida TM]

Anisotropic Lax Connection

The right anisotropic Lax connection was obtain by [Kawaguchi Yoshida] [Faddeev Reshetikhin]

$$L^R(x;\lambda_R) = \frac{1}{1-\lambda_R^2}\Big[T^+(J^- - \lambda_R * J^-)$$
$$+ T^-(J^+ - \lambda_R(*J^+ \pm\sqrt{C}J^2 + C * J^-) + \lambda_R^2(\pm\sqrt{C} * J^2 + CJ^-))$$
$$- T^2(J^2 - \lambda_R(*J^2 \pm\sqrt{C}J^-) \pm\sqrt{C}\lambda_R^2 * J^-)\Big],$$

The zero-curv. cond. $0 = dL^R + L^R \wedge L^R$ yields both the flatness and EoM of J. The associated monodromy $U^R(\lambda^R) = \mathrm{Pexp}\int_{\mathcal{C}} L^R(\lambda^R)$ leads infinitely many conserved charges.

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On the other hand, we expect that there exists isotropic (or rational) Lax conn.

because

- “Exotic sym.” has the semi-infinite levels such as Yangians
- the spectral param. relation $\lambda_L = 1/\lambda_R$ is same to PCM.
- taken the null (pp-wave) limit for the warped AdS_3.

From Anisotropic to Isotropic - 1/2

Observation 1 : Comparing to the Lax conn. of PCM $L(\lambda) = \dfrac{J - \lambda * J}{1 - \lambda^2}$, the anisotropic Lax does **not** vanish at $\lambda_R \to \infty$.

$$
\begin{aligned}
L^R(x; \lambda_R) = \frac{1}{1-\lambda_R^2}\Big[&T^+(J^- - \lambda_R * J^-) \\
&+ T^-(J^+ - \lambda_R(*J^+ \pm \sqrt{C}J^2 + C * J^-) + \lambda_R^2(\pm\sqrt{C} * J^2 + CJ^-)) \\
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\end{aligned}
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In fact, we have $L^R(x; \lambda_R = \infty) = \pm\sqrt{C}\big(T^2 * J^- - T^-(*J^2 \pm \sqrt{C}J^-)\big)$.

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In fact, we have $L^R(x;\lambda_R = \infty) = \pm\sqrt{C}\big(T^2 * J^- - T^-(*J^2 \pm \sqrt{C}J^-)\big)$.

Since $L^R(x;\infty)$ is also flat, we could introduce

$$G(x) = \mathrm{Pexp}\left[\int_{-\infty}^{x} dy\, L_y^R(y;\infty)\right]$$

as a solution of

$$(d - L^R(x;\infty))G(x) = 0\ .$$

From Anisotropic to Isotropic - 2/2

Using the formal solution $G(x)$, we can construct the vanishing Lax at $\lambda_R = \infty$ by the gauge transformation ;

$$\left[L^R(x;\lambda_R)\right]^G := G^{-1}L^R(x;\lambda_R)G - G^{-1}dG \ .$$

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$$\left[L^R(x;\lambda_R)\right]^G = \frac{\mathcal{J} - \lambda_R * \mathcal{J}}{1-\lambda_R^2} \quad \text{with} \quad \mathcal{J} := -(gG)^{-1}d(gG)\ .$$

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Q. What is the associated algebra of $Q = \int *\mathcal{J}$?

Rational, but Not Yangian....!?

It is natural to expect that the algebraic structure is the Yangian $\mathcal{Y}(\mathfrak{sl}_2)$.

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It leads, however, a slightly different algebra rather than Yangian itself,

$$\{Q^a_{(0)}, Q^b_{(n)}\}_{\mathrm{P}} = \epsilon^{ab}{}_c\, Q^c_{(n)}\big(1 + \tfrac{C}{4}(Q^-_{(0)})^2\big) \quad \text{for} \quad n = 0, 1$$

together with the Serre-type relation;

$$\begin{aligned}\big\{\{Q^+_{(1)}, Q^-_{(1)}\}_{\mathrm{P}}, Q^2_{(1)}\big\}_{\mathrm{P}} &= \tfrac{1}{4}(Q^+_{(1)}Q^-_{(0)} - Q^+_{(0)}Q^-_{(1)})Q^2_{(0)} \\ &\quad + \tfrac{C}{4}Q^2_{(1)}\{Q^-_{(0)}Q^-_{(1)}, Q^2_{(1)}\}.\end{aligned}$$

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Q. Is this a new type of the rational quantum group!? or isomorphic to $\mathcal{Y}(\mathfrak{sl}_2)$? If it is isomorphic to $\mathcal{Y}(\mathfrak{sl}_2)$, how can we see the isomorphism...??

Let us borrow the wisdom of
the quantum integrability.

Plan

Quasitriangular Hopf Algebra

Definition. A triplet $(\mathcal{A}, \Delta, \mathcal{R})$, consisting of a Hopf algebra $\mathcal{A}$, coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ and an invertible element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$, is called a quasitriangular Hopf algebra if $\mathcal{R}$ satisfies the following conditions;

$$\widetilde{\Delta}(a)\mathcal{R} = \mathcal{R}\Delta(a) \qquad \text{for any} \quad a \in \mathcal{A}$$

$$(\Delta \otimes 1)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$$

$$(1 \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12}$$

where $\widetilde{\Delta}$ is the opposite coproduct defined by $\widetilde{\Delta} := P \circ \Delta$ with $P(x \otimes y) = y \otimes x$ for any $x, y \in \mathcal{A}$. We also denote for $\mathcal{R} = \sum_i a_i \otimes b_i$,

$$\mathcal{R}_{12} = \sum_i a_i \otimes b_i \otimes 1, \ \mathcal{R}_{13} = \sum_i a_i \otimes 1 \otimes b_i, \ \mathcal{R}_{23} = \sum_i 1 \otimes a_i \otimes b_i.$$

Quasitriangular Hopf Algebra

Proposition. Then, the universal R-matrix $\mathcal{R}$ satisfies YBE; [Drinfeld 1985]

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \ .$$

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$(\because)$ This is shown as[a]

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} \stackrel{(2)}{=} \mathcal{R}_{12}(\Delta \otimes 1)\mathcal{R} \stackrel{(1)}{=} (\tilde{\Delta} \otimes 1)\mathcal{R}\mathcal{R}_{12} \stackrel{(2)}{=} \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$$

by using the definition of the quasitriangular Hopf algebra;

$$\tilde{\Delta}(J)\mathcal{R} = \mathcal{R}\Delta(J) \tag{1}$$

$$(\Delta \otimes 1)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} \tag{2}$$

$$(1 \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{3}$$

□

[a] We could also use (3) instead of (2).

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and introduce the twisted coproduct and R-matrix by

$$\Delta^{(\mathcal{F})}(a) := \mathcal{F}\Delta(a)\mathcal{F}^{-1} \qquad \mathcal{R}^{(\mathcal{F})} := \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1} \ .$$

[a] The "cocycle condition" $\mathcal{F}_{12}(\Delta \otimes 1)\mathcal{F} = \mathcal{F}_{23}(1 \otimes \Delta)\mathcal{F}$ is sufficient for twisting.

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Theorem. $(\mathcal{A}, \Delta^{(\mathcal{F})}, \mathcal{R}^{(\mathcal{F})})$ is also a quasitriangular Hopf algebra. [Reshetikhin 1990]

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We refer to the twist as

- Reshetikhin twist if $[\mathcal{F}_{ij}, \mathcal{F}_{kl}] = 0$ and
- Jordanian twist if $[\mathcal{F}_{ij}, \mathcal{F}_{kl}] \neq 0$.

---

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Jordanian Twist for $\mathcal{Y}(\mathfrak{sl}(2))$

Let us take $\mathcal{A} = \mathcal{Y}(\mathfrak{sl}(2))$ Yangian, and consider the following Jordanian twist operator with a deformation parameter ξ, ($\xi \to 0$ is trivial limit $\mathcal{F} \to 1$.)

$$\mathcal{F} = \exp\left(\frac{1}{2} h \otimes \ln \gamma\right) = 1 + \xi h \otimes f + \frac{\xi^2}{2} h(h+2) \otimes f^2 + \cdots$$

where $\gamma := 1 - 2\xi f$ and e, h, f are the CS basis of $\mathfrak{sl}(2)$. [Gestenhaber Giantino, Schack]

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To see the relation to our model, take the Evaluation repr. ρ_u ($\simeq \frac{1}{2} \otimes [[\mathbf{u}^{-1}]]$) for the site $1, 2$, which gives [Khoroshkin Stolin, Tolstoy]

$$R_{12}^{(\mathcal{F})}(u-v) T_1^{(\mathcal{F})}(u) T_2^{(\mathcal{F})}(v) = T_2^{(\mathcal{F})}(v) T_1^{(\mathcal{F})}(u) R_{12}^{(\mathcal{F})}(u-v) \ .$$

where $T_{i=1,2}^{(\mathcal{F})} = (\rho_u \otimes \rho_v \otimes 1)\mathcal{R}_{i3}$. This is co called RTT relation.

Relation to Monodromy

In the RTT-relation; $R_{12}^{(\mathcal{F})}(u-v)T_1^{(\mathcal{F})}(u)T_2^{(\mathcal{F})}(v) = T_2^{(\mathcal{F})}(v)T_1^{(\mathcal{F})}(u)R_{12}^{(\mathcal{F})}(u-v)$,
the fundamental R-matrix and T-operators are explicitly computed as

$$R_{12}^{(\mathcal{F})}(u) = F_{21}F^{-1} - P_{12}u^{-1} \quad \text{with} \quad F_{21}F^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\xi & 1 & 0 & 0 \\ \xi & 0 & 1 & 0 \\ \xi^2 & -\xi & \xi & 1 \end{pmatrix}$$

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Twisted T-operator $T^{(\mathcal{F})}(u)$ should correspond to the monodromy $U^R(\lambda_R)$!

$$\begin{array}{cccc} \text{Quantum} & T(u) & \xrightarrow[\text{Jordanian twist}]{\mathcal{F}} & T^{(\mathcal{F})}(u) \\ \text{v.s.} & \| & & \| \\ \text{Classical} & [U^R(\lambda_R)]^G & \xleftarrow[\text{Non-local gauge trf.}]{G} & U^R(\lambda_R) \end{array}$$

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Boundary Terms - 1/2

Comparing the expressions, we notice it's not so symmetric... ($\gamma = 1 - 2\xi f$)

$$T(u) = \mathcal{F}_{21}^{-1} T^{(\mathcal{F})}(u) \mathcal{F} = \begin{pmatrix} 1 & 0 \\ -\xi h & 1 \end{pmatrix} T^{(\mathcal{F})}(u) \begin{pmatrix} \gamma^{-1/2} & 0 \\ 0 & \gamma^{1/2} \end{pmatrix}$$

$$[U^R(\lambda_R)]^G = G(+\infty)^{-1} U^R(\lambda_R) G(-\infty) = \begin{pmatrix} k & 0 \\ -\sqrt{2}\xi Q^{R,2} & k^{-1} \end{pmatrix} U^R(\lambda_R) 1$$

where $k := \mathrm{e}^{\xi Q^{R,-}}$ and the def. param. could be identified as $\xi = \frac{\sqrt{C}}{2}$.

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Observation2: All we need is the property: $(d - L^R(x;\infty))G(x) = 0$. It allows a right constant shift: $G(x) \mapsto G(x)M$ with any $dM = 0$. Let's improve it so that it is symmetric at the worldsheet boundaries $x = \pm\infty$!

$$\mathcal{F}(x) := G(x)M \qquad \text{with} \qquad M := \begin{pmatrix} k & 0 \\ 0 & k^{-1} \end{pmatrix}$$

Boundary Terms - 2/2

After the improvement, it seems to have a symmetric expression;

$$T(u) \simeq \begin{pmatrix} 1 & 0 \\ -\xi h & 1 \end{pmatrix} T^{(\mathcal{F})}(u) \begin{pmatrix} \gamma^{-1/2} & 0 \\ 0 & \gamma^{1/2} \end{pmatrix} \quad \text{with} \quad \gamma = 1 - 2\xi f$$

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$\Rightarrow$ Twist is worldsheet "boundary terms" of the non-local gauge-trf.

Now, we are ready to see the algebra again!

Quantum	$\mathcal{Y}(\mathfrak{sl}_2) \ni T(u)$	$\xrightarrow[\text{Jordanian twist}]{\mathcal{F}}$	$T^{(\mathcal{F})}(u) \in \mathcal{Y}_\xi(\mathfrak{sl}_2)$
v.s.	$\Vert$		$\Vert$
Classical	$\mathcal{Y}_{\mathrm{cl}}(\mathfrak{sl}_2) \ni [U^R(\lambda_R)]^{\mathcal{F}}$	$\xleftarrow[\text{Non-local gauge trf.}]{\mathcal{F}(x)}$	$U^R(\lambda_R) \in$ "Exotic"

Lie Algebra $\mathfrak{sl}(2)$

Using the improved twist $\mathcal{F}$, the current is defined by $\mathcal{I} = -(g\mathcal{F})^{-1}d(g\mathcal{F})$.
The associated charges $\mathcal{Y}_{(0)} := \int *\mathcal{I}$ are given by ($k = \mathrm{e}^{\xi Q^{R,-}}$)

$$\mathcal{Y}^{+}_{(0)} = kQ^{R,+} + \xi(kQ^{R,2})^2$$

$$\mathcal{Y}^{2}_{(0)} = kQ^{R,2}$$

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The algebra of $\mathcal{Y}_{(0)}$'s turns out to be the undeformed $\mathfrak{sl}(2)$ algebra!

$$\{\mathcal{Y}^2_{(0)}, \mathcal{Y}^{\pm}_{(0)}\}_{\mathrm{P}} = \pm\mathcal{Y}^{\pm}_{(0)}, \quad \{\mathcal{Y}^{-}_{(0)}, \mathcal{Y}^{+}_{(0)}\}_{\mathrm{P}} = \mathcal{Y}^2_{(0)}.$$

by using the q-Poincaré algebra:

$$\{Q^{R,2}, Q^{R,+}\}_{\mathrm{P}} = Q^{R,+}\cosh(\xi Q^{R,-})\,, \quad \{Q^{R,-}, Q^{R,+}\}_{\mathrm{P}} = Q^{R,2}$$
$$\{Q^{R,2}, Q^{R,-}\}_{\mathrm{P}} = -\frac{1}{\xi}\sinh(\xi Q^{R,-}).$$

Yangian $\mathcal{Y}(\mathfrak{sl}(2))$

Furthermore, the higher charges given by BIZZ construction are expressed in terms of both Q^R and $\widetilde{Q}^R$ $(k = \mathrm{e}^{\xi Q^{R,-}})$ [Lüscher 1978] [Brezin, Itzykson ZinnJustin, Zuber]

$$\mathcal{Y}^{+}_{(1)} = \tfrac{1}{4}k(\tfrac{1}{\xi}\{\{\widetilde{Q}^{R,2}, Q^{R,+}\}, Q^{R,+}\} + Q^{R,2}\{\widetilde{Q}^{R,2}, Q^{R,+}\} + \cdots)$$

$$\mathcal{Y}^{2}_{(1)} = \tfrac{1}{4\xi}\left(\{\widetilde{Q}^{R,2}, Q^{R,+}\} - \cosh(\xi Q^{R,-})Q^{R,+}\right) + \tfrac{1}{4}Q^{R,2}\left(\widetilde{Q}^{R,2} - Q^{R,2}\right)$$

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and they yield the undeformed Yangian algebra $\mathcal{Y}(\mathfrak{sl}(2))$! [a]

$$\{\mathcal{Y}^{a}_{(0)}, \mathcal{Y}^{b}_{(1)}\}_{\mathrm{P}} = \epsilon^{ab}{}_{c}\mathcal{Y}^{c}_{(1)}$$

[a] The cubic Serre relation is to be confirmed.

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$$\{\mathcal{Y}^{a}_{(0)}, \mathcal{Y}^{b}_{(1)}\}_{\mathrm{P}} = \epsilon^{ab}{}_{c}\mathcal{Y}^{c}_{(1)}$$

This is consistent with the following picture,

$$\mathcal{Y}_{\mathrm{cl}}(\mathfrak{sl}_2) \ni [U^R(\lambda_R)]^{\mathcal{F}} \xleftarrow[\text{Non-local gauge trf.}]{\mathcal{F}(x)} U^R(\lambda_R) \in \text{“Exotic”}$$

[a] The cubic Serre relation is to be confirmed.

Plan

Dualities

Three Lax pairs: Left(isotropic)/Right(anisotropic)/Right(isotropic), are related through the gauge-trf. by g and $\mathcal{F}$; c.f. [Kawaguchi Yoshida]

$$[L^L(x;\lambda_L)]^{g\mathcal{F}} = [L^R(x;\lambda_R)]^{\mathcal{F}} = \frac{\mathcal{I} - \lambda_R * \mathcal{I}}{1-\lambda_R^2} \ ,$$

where $\lambda_L = 1/\lambda_R$, and we have introduced the right flat current by

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$$\begin{aligned} j^{L,+} &= dg\, g^{-1} - 2C\mathrm{Tr}(T^- J) g T^- g^{-1} + \sqrt{C} * d(gT^- g^{-1}) \\ &= d(g\mathcal{F})(g\mathcal{F})^{-1} \ . \end{aligned}$$

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Now, the picture is completely parallel with that of PCM! by the replacement

$$g \mapsto g\mathcal{F} .$$

Plan

0. Motivations
1. Set up of the Model
2. Rational Description
3. Anisotropic Description
4. Jordanian Twist
5. Relations among Lax Connections
6. Geometric Interpretation
7. Summary & Conclusion

Dipole-like Action

Using these current, the Lagrangian could be expressed in a dipole-like form,

$$\begin{aligned}\mathcal{L} &= -\eta^{\mu\nu}[\mathrm{Tr}(J_\mu J_\nu) - 2C\,\mathrm{Tr}(T^- J_\mu)\mathrm{Tr}(T^- J_\nu)] \\ &= -\eta^{\mu\nu}\mathrm{Tr}\left(j_\mu^{L,+} j_\nu^{L,-}\right) \,.\end{aligned}$$

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Since we have two left currents $j^{L,\pm}$, we actually need two kinds of twists;

$$j^{L,\pm} = dg\, g^{-1} - 2C\,\mathrm{Tr}(T^- J) g T^- g^{-1} \pm \sqrt{C} * d(gT^- g^{-1})$$
$$= d(g\mathcal{F}^\pm)(g\mathcal{F}^\pm)^{-1}\ ,$$

where the twists $\mathcal{F}^\pm$ are introduced by

$$\mathcal{F}^\pm(x) = \mathrm{Pexp}\Big[\int_{-\infty}^{x} dy\, L^{R,\pm}_y(y; \lambda_R = \infty)\Big] \mathrm{e}^{\pm\sqrt{C}T^2 Q^{R,-}}\ .$$

Dipole Coordinates

The twist could be interpreted as a change of global coordinate of AdS_3;

$$g = \mathrm{e}^{2vT^+}\mathrm{e}^{2\rho T^2}\mathrm{e}^{2uT^-} \quad \mapsto \quad g\mathcal{F}^\pm =: \mathrm{e}^{2v^\pm T^+}\mathrm{e}^{2\rho^\pm T^2}\mathrm{e}^{2u^\pm T^-}.$$

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In fact, the dipole coordinates $v^\pm, \rho^\pm, u^\pm$ are computed as ($\xi = \frac{\sqrt{C}}{2}$)

$$v^\pm = v, \quad \rho^\pm = \rho \pm \xi(\eta + Q^{R,-}), \quad u^\pm = \mathrm{e}^{\pm(\rho^+ - \rho^-)}(u \pm \xi \mathrm{e}^{\mp 2\xi\eta}\theta^\pm)$$

where we have denoted

$$\eta = \int_{-\infty}^{x} dy\, J_t^-(y), \qquad \theta^\pm = -\int_{-\infty}^{x} dy\, \mathrm{e}^{\pm 2\xi\eta}(J_t^2 \pm 2\xi J_y^-)(y).$$

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Then, the Lagrangian could be written such as PCM over "$AdS_3^+ \otimes AdS_3^-$,"

$$\begin{aligned}\mathcal{L} &= -2\eta^{\mu\nu}\left[-2e^{-2\rho}u_\mu v_\nu + \rho_\mu\rho_\nu - Ce^{-4\rho}v_\mu v_\nu\right] \\ &= -2\eta^{\mu\nu}\left[-(e^{-2\rho^+}u_\mu^+ + e^{-2\rho^-}u_\mu^-)v_\nu + \rho_\mu^+\rho_\nu^-\right].\end{aligned}$$

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Summary & Outlook

- Classical Jordanian twist gives rational right Lax from anisotropic one.

Quantum	$\mathcal{Y}(\mathfrak{sl}_2) \ni T(u)$	$\xrightarrow[\text{Jordanian twist}]{\mathcal{F}}$	$T^{(\mathcal{F})}(u) \in \mathcal{Y}_{\xi}(\mathfrak{sl}_2)$
v.s.	$\parallel$		$\parallel$
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Thank you for your attention!