Instanton Effects in ABJM Theory

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[arXiv:1207.4283] Y. Hatsuda, S. Moriyama, KO
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Introduction

- ABJM theory describes the dynamics of multiple M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$

  ABJM = 3d $U(N)_k \times U(N)_{-k}$ Chern-Simons-matter theory

- Holographic dual

  ABJM on $S^3 \leftrightarrow$ M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$

- ABJM theory on S^3 can be thought of as a nonperturbative definition of M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$

- The partition function $Z(N,k)$ of ABJM theory on S^3 can be obtained by the localization technique [Kapustin-Willett-Yaakov]
  - 't Hooft limit ($k, N \to \infty, N/k = \text{fixed}$) $\Rightarrow$ Type IIA on $AdS_4 \times \mathbb{CP}^3$
  - We are interested in the M-theory regime $k =$ finite

ABJM Matrix Model

- Partition function of ABJM theory on S^3

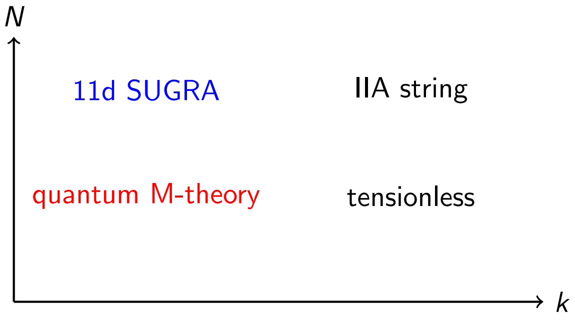
$$Z(N,k)=\int\frac{d^N\mu d^N\nu}{(2\pi)^{2N}(N!)^2}\left[\frac{\prod_{i<j}2\sinh\frac{\mu_i-\mu_j}{2}\cdot 2\sinh\frac{\nu_i-\nu_j}{2}}{\prod_{i,j}2\cosh\frac{\mu_i-\nu_j}{2}}\right]^2 e^{\frac{ik}{4\pi}(\mu^2-\nu^2)}$$

$$=\frac{1}{2^N N!}\int\frac{d^N x}{(2\pi k)^N}\prod_{i<j}\tanh^2\frac{x_i-x_j}{2k}\prod_i\frac{1}{2\cosh\frac{x_i}{2}}$$

- ABJM matrix model is simple enough to be tractable
- At the same time, it captures the essential part of quantum M-theory

ABJM matrix model = 'Ising model' of M-theory

Parameter Region



- There is no phase boundary
  - CFT dual of M-theory $\neq$ matrix model
  - (k, N) parametrizes a family of 3d CFTs

Large N Behavior of ABJM Matrix Model

- Large N limit of ABJM matrix model reproduces the $N^{3/2}$ d.o.f. of N M2-branes [Drukker-Marino-Putrov]

$$Z(N,k) \approx \exp\left(-\frac{\pi\sqrt{2k}}{3}N^{3/2}\right)$$

- Perturbative part of partition function is given by the Airy function [Fuji-Hirano-Moriyama]

$$Z_{\rm pert}(N,k) = C^{-\frac{1}{3}} e^{A} \cdot {\rm Ai}\left[C^{-\frac{1}{3}}(N-B)\right]$$
$$C = \frac{2}{\pi^2 k}, \quad B = \frac{k}{24} + \frac{1}{3k}$$

- A is the contribution of constant maps [Hanada et al]

Large N Expansion in M-theory

- M-theory expansion is quite different from the genus expansion of string theory

$$\text{string}: \quad F = \sum_g F_g N^{2(1-g)} + \mathcal{O}(e^{-N})$$

$$\text{ABJM}: \quad F = \sum_\ell c_\ell N^{\frac{3}{2}(1-\ell)} + \mathcal{O}(e^{-\sqrt{N}}) = F_{\text{pert}} + F_{\text{np}}$$

- Expansion parameter in quantum M-theory ($R_{AdS} = 1$)

$$\frac{1}{G_{11}} \sim \frac{1}{\ell_{pl}^9} \sim N^{3/2}, \qquad T_{M2} \sim \frac{1}{\ell_{pl}^3} \sim \sqrt{N}$$

Non-Perturbative Corrections

- We are interested in the non-perturbative part of free energy $F_{\rm np}$
- $F_{\rm np}$ comes from instanton effects

  Worldsheet instanton $\sim e^{-\sqrt{N/k}}$
  Membrane instanton $\sim e^{-\sqrt{kN}}$

- To study the behavior of instanton corrections, we compute $Z(N,k)$ exactly at finite $N \sim \ell_{pl}^{-6}$

  finite $N \leftrightarrow$ quantum regime of M-theory

Exact Values of $Z(N,k)$

- $N=2$ at arbitrary k [KO]
- up to $N=9$ at $k=1$ [Hatsuda-Moriyama-KO]
- up to $N=19$ at $k=1$ [Putrov-Yamazaki]
- Larger N at various k [Hatsuda-Moriyama-KO]
  up to $N=44,20,18,16,14$ at $k=1,2,3,4,6$, respectively

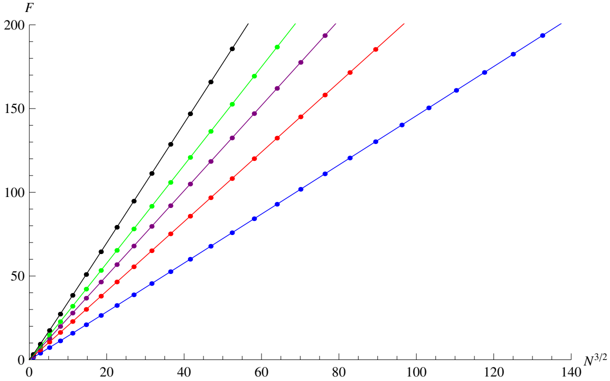
- For example, $Z(N)$ at $k=1$ are

$$Z(1)=\frac{1}{2^2},\quad Z(2)=\frac{1}{2^4\pi},\quad Z(3)=\frac{\pi-3}{2^6\pi},$$
$$Z(4)=\frac{-\pi^2+10}{2^{10}\pi^2},\qquad Z(5)=\frac{-9\pi^2+20\pi+26}{2^{12}\pi^2},$$
$$Z(6)=\frac{36\pi^3-121\pi^2+78}{2^{14}3^2\pi^3},\quad Z(7)=\frac{-75\pi^3+193\pi^2+174\pi-126}{2^{16}3\pi^3},$$
$$Z(8)=\frac{1053\pi^4-2016\pi^3-4148\pi^2+876}{2^{21}3^2\pi^4},\quad \cdots$$

Free Energy of ABJM Theory

- Free energy exhibits the $N^{3/2}$ behavior even at small N

$$F \approx \frac{\pi\sqrt{2k}}{3} N^{3/2}$$



HMO Cancellation Mechanism

- Free energy should be a smooth function of string coupling $g_s = \frac{2}{k}$

  $$\text{small } g_s \ \ \text{(IIA)} \quad \longrightarrow \quad \text{large } g_s \ \ \text{(M)}$$

- However, the worldsheet instanton coefficient ($\sim \dfrac{1}{\sin^2 \pi m g_s}$) diverges at some rational number $g_s = \frac{n}{m}$

- This divergence is precisely cancelled by the membrane instanton [Hatsuda-Moriyama-KO]

$$\lim_{g_s \to \frac{n}{m}} \Big[(\textit{worldsheet inst}) + (\textit{membrane inst}) \Big] = \text{finite}$$

- Mariño coined this mechanism as "HMO cancellation"

- HMO cancellation + small k expansion $\Rightarrow$ find membrane instanton

Membrane is still mysterious

Grand Canonical Partition Function

- The measure factor is not the Vandermonde determinant, but the Cauchy determinant
- A useful way to analyze ABJM matrix model is to consider the grand canonical ensemble [Marino-Putrov, KO]

$$\Xi(\mu) = \sum_{N=0}^{\infty} e^{N\mu} Z(N)$$

- Grand partition function is written as a Fredholm determinant

$$\Xi(\mu) = \mathrm{Det}(1 + e^{\mu - H}) = \prod_{n=0}^{\infty} (1 + e^{\mu - E_n})$$

- This is interpreted as a system of free Fermi gas

Hamiltonian of ABJM Fermi Gas

- One-body Hamiltonian of ABJM Fermi gas

$$e^{-H} = \frac{1}{\sqrt{2\cosh\frac{x}{2}}}\frac{1}{2\cosh\frac{p}{2}}\frac{1}{\sqrt{2\cosh\frac{x}{2}}}$$

$$[x, p] = 2\pi i k$$

- Chern-Simons level $k \quad = \quad \hbar$ of Fermi gas system
  - small k expansion $\leftrightarrow$ semi-classical expansion of Fermi gas
  - large k expansion $\leftrightarrow$ genus expansion in $g_s = 2/k$

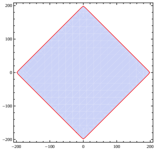
- The n-th eigenvalue E_n of the Hamiltonian H behaves as

$$E_n^2 \sim C^{-1} n \quad (n \gg 1)$$

Classical Fermi Surface

- The constant energy surface becomes a square in the large E limit

$$\left(2\cosh\frac{x}{2}\cdot 2\cosh\frac{p}{2}\right)^{-1}=e^{-E}\quad\Rightarrow\quad |x|+|p|=2E$$



- Number of states $n(E)$ below E in the semi-classical limit [Marino-Putrov]

$$n(E)\approx\frac{\mathrm{Area}}{2\pi\hbar}=\frac{(2\sqrt{2}E)^2}{2\pi\cdot 2\pi k}=\frac{2}{\pi^2 k}E^2=CE^2$$

Perturbative Grand Potential

- We consider the large μ expansion of grand potential $J(\mu)$

$$J(\mu) = \log \Xi(\mu) = \mathrm{Tr} \log(1 + e^{\mu - H})$$

- Semi-classical expression of grand potential

$$J(\mu) \approx \int_0^\infty dE \, \frac{dn(E)}{dE} \log(1 + e^{\mu - E}) \approx \int_0^\infty dE \, \frac{n(E)}{1 + e^{E-\mu}} \approx \frac{C}{3}\mu^3$$

- $N^{3/2}$ d.o.f. of M2-brane is easily reproduced from the saddle point approximation

$$Z(N) \approx \int d\mu \, e^{-N\mu + \frac{C\mu^3}{3}} \approx e^{-N\mu_* + \frac{C\mu_*^3}{3}} = e^{-\frac{2}{3}C\mu_*^3} \quad (C\mu_*^2 = N)$$

Airy Function from Fermi Gas

- [Marino-Putrov] studied the semi-classical expansion of grand potential $J(\mu)$ in the small k, large μ limit

$$J(\mu) = J_{\rm pert}(\mu) + J_{\rm np}(\mu)$$
$$J_{\rm pert}(\mu) = \frac{C}{3}\mu^3 + B\mu + A$$

- Airy function behavior of perturbative partition function is easily reproduced from the Fermi gas approach

$$Z_{\rm pert}(N) = \int d\mu\, e^{-N\mu + J_{\rm pert}(\mu)} = C^{-\frac{1}{3}} e^{A} \cdot \mathrm{Ai}\Big[C^{-\frac{1}{3}}(N-B)\Big]$$

Relation to Topological String

M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$

↕ holography

ABJM theory on S^3

↕ analytic continuation

Pure Chern-Simons theory on $S^3/\mathbb{Z}_2$

↕ geometric transition

Topological string on (diagonal) local $\mathbb{P}^1 \times \mathbb{P}^1$

$$T_{1,2} = T \pm i\pi, \quad T = \frac{4\mu}{k} = 2g_s\mu, \quad g_s = \frac{2}{k}$$

Instanton Corrections

Two types of instantons (Type IIA picture of $AdS_4 \times \mathbb{CP}^3$)

1. Worldsheet instanton = F1 wrapping $\mathbb{CP}^1$ $\left(e^{-T} = e^{-\frac{4\mu}{k}},\ \mu \sim \sqrt{kN}\right)$

$$J_{\mathrm{WS}}(\mu) = \sum_{m=1}^{\infty} d_m(k) e^{-\frac{4m\mu}{k}}$$

2. Membrane instanton = D2 wrapping $\mathbb{RP}^3$ $\left(e^{-\frac{T}{g_s}} = e^{-2\mu}\right)$

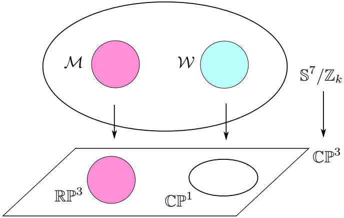
$$J_{\mathrm{D2}}(\mu) = \sum_{\ell=1}^{\infty} \Big[a_\ell(k)\mu^2 + b_\ell(k)\mu + c_\ell(k)\Big] e^{-2\ell\mu}$$

- Worldsheet instantons can be computed from topological string
- Small k expansion of membrane instanton coefficients can be computed from the semi-classical TBA [Calvo-Marino]

Bound State of Instantons

- There are additional bound states of worldsheet m-instanton and membrane ℓ-instanton
- (ℓ, m) bound state corresponds to
  M2-brane wrapping ℓ times on $\mathcal{M}$ and m times on $\mathcal{W}$

$$J_{\ell,m}(\mu) \sim \exp\left[-\left(2\ell + \frac{4m}{k}\right)\mu\right]$$



Numerical Fitting of Grand Potential

- We numerically find the instanton corrections using our exact data of $Z(N)$
- We approximate the nonperturbative part of grand potential by

$$J_{\mathrm{np}}(\mu) \approx \log\left(\sum_{N=0}^{N_{\mathrm{max}}} e^{N\mu} Z(N)\right) - \left(\frac{C}{3}\mu^3 + B\mu + A\right)$$

- $N_{\mathrm{max}} = 44, 20, 18, 16, 14$ for $k = 1, 2, 3, 4, 6,$ respectively

Instanton Corrections at $k = 1, 3, 4, 6$

$$J_{\rm np}^{(k=1)} = \left[\frac{4\mu^2+\mu+1/4}{\pi^2}\right]e^{-4\mu} + \left[-\frac{52\mu^2+\mu/2+9/16}{2\pi^2}+2\right]e^{-8\mu} + \cdots$$

$$J_{\rm np}^{(k=3)} = \frac{4}{3}e^{-\frac{4}{3}\mu} - 2e^{-\frac{8}{3}\mu} + \left[\frac{4\mu^2+\mu+1/4}{3\pi^2}+\frac{20}{9}\right]e^{-4\mu} - \frac{88}{9}e^{-\frac{16}{3}\mu} + \cdots$$

$$J_{\rm np}^{(k=4)} = e^{-\mu} + \left[-\frac{4\mu^2+2\mu+1}{2\pi^2}\right]e^{-2\mu} + \frac{16}{3}e^{-3\mu} + \cdots$$

$$J_{\rm np}^{(k=6)} = \frac{4}{3}e^{-\frac{2}{3}\mu} - 2e^{-\frac{4}{3}\mu} + \left[\frac{4\mu^2+2\mu+1}{3\pi^2}+\frac{20}{9}\right]e^{-2\mu} - \frac{88}{9}e^{-\frac{8}{3}\mu} + \cdots$$

- Blue part agrees with the prediction of topological string $d_m(k)$, but Red part does not. $(d_3(4) = 10/3, d_4(3) = d_4(6) = -8)$
- Discrepancy is due to the bound states of membrane instanton and worldsheet instanton [HMO, Calvo-Marino]

Membrane Instanton Correction

- Only little is known about the membrane instantons
- We proposed the analytic form of the membrane 1-instanton

$$J_{\mathrm{D2}}^{\text{1-inst}}(\mu) = \Big[a_1(k)\mu^2 + b_1(k)\mu + c_1(k)\Big]e^{-2\mu}$$

- We require the following three conditions for $a_1(k), b_1(k), c_1(k)$
  1. pole structure at $k = $ even

$$\lim_{k\to 2m}\Big[(WS_{m\text{-inst}})e^{-\frac{4m\mu}{k}} + (D2_{\text{1-inst}})e^{-2\mu}\Big] = \text{finite}$$

  2. proportional to $\cos\frac{\pi k}{2}$ (no D2 1-instanton at $k = $ odd)
  3. matching of small k expansion

Membrane 1-Instanton Coefficients

- From these three conditions, we found the analytic form of the membrane 1-instanton coefficients

$$
\begin{aligned}
a_1(k) &= -\frac{4}{\pi^2 k}\cos\frac{\pi k}{2}, \\
b_1(k) &= \frac{2}{\pi}\cot\frac{\pi k}{2}\cos\frac{\pi k}{2}, \\
c_1(k) &= \left[-\frac{2}{3k}+\frac{5k}{12}+\frac{k}{2\sin^2\frac{\pi k}{2}}+\frac{1}{\pi}\cot\frac{\pi k}{2}\right]\cos\frac{\pi k}{2}
\end{aligned}
$$

- This proposal was checked up to $\mathcal{O}(k^{15})$ by [Calvo-Marino]
- Membrane $2, 3, 4, 5$-instantons were determined in a similar manner [Calvo-Marino, HMO]

Bound States and Effective Chemical Potential

- pole cancellation + matching finite pieces at $k = 1, .., 6$

$$\Downarrow$$

The effect of bound states can be incorporated into the worldsheet instantons by replacing μ by the effective chemical potential $\mu_{\rm eff}$

$$J_{\rm WS}(\mu) + (\text{bound states}) = J_{\rm WS}(\mu_{\rm eff})$$

$$\mu_{\rm eff} = \mu + C^{-1} \sum_{\ell=1}^{\infty} a_\ell(k) e^{-2\ell\mu}$$

- This replacement $\mu \to \mu_{\rm eff}$ works also for Wilson loops (open string sector) [Hatsuda-Honda-Moriyama-KO]

General Structure

- Total grand potential is simplified by rewriting in μ_{eff}

$$
\begin{aligned}
J(\mu) =& J_{\text{pert}}(\mu) + J_{\text{WS}}(\mu) + (\text{bound states}) + J_{\text{D2}}(\mu) \\
=& J_{\text{pert}}(\mu_{\text{eff}}) + J_{\text{WS}}(\mu_{\text{eff}}) + \sum_{\ell=1}^{\infty} \left[\mu_{\text{eff}} \widetilde{b}_\ell(k) + \widetilde{c}_\ell(k) \right] e^{-2\ell\mu_{\text{eff}}}
\end{aligned}
$$

- $\widetilde{b}_\ell(k)$ is written in terms of $a_\ell(k)$ and $b_\ell(k)$
- $\widetilde{c}_\ell(k)$ is determined by $\widetilde{b}_\ell(k)$

$$
\widetilde{c}_\ell(k) = -k^2 \frac{\partial}{\partial k} \left(\frac{\widetilde{b}_\ell(k)}{2\ell k} \right) = \frac{\partial}{\partial g_s} \left(g_s \frac{\widetilde{b}_\ell(k)}{2\ell} \right)
$$

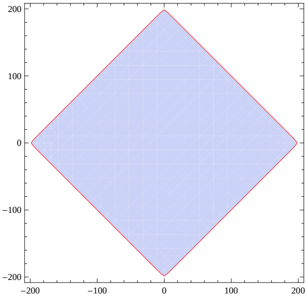
Example of $a_\ell(k)$ and $\widetilde{b}_\ell(k)$

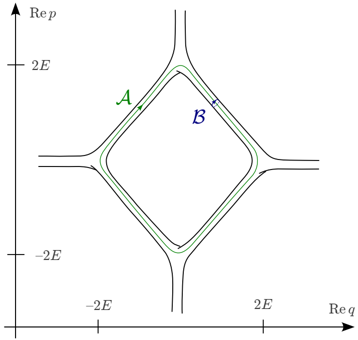
$$
\begin{aligned}
a_1(k) &= -\frac{4}{\pi^2 k}\cos\left(\frac{\pi k}{2}\right)\\
a_2(k) &= -\frac{2}{\pi^2 k}\big(4+5\cos(\pi k)\big)\\
a_3(k) &= -\frac{8}{3\pi^2 k}\cos\left(\frac{\pi k}{2}\right)\big(19+28\cos(\pi k)+3\cos(2\pi k)\big)\\
\widetilde{b}_1(k) &= \frac{2}{\pi}\cot\left(\frac{\pi k}{2}\right)\cos\left(\frac{\pi k}{2}\right)\\
\widetilde{b}_2(k) &= \frac{1}{\pi}\cot(\pi k)\big(4+5\cos(\pi k)\big)\\
\widetilde{b}_3(k) &= \frac{4}{3\pi}\cot\left(\frac{3\pi k}{2}\right)\cos\left(\frac{\pi k}{2}\right)\big(13+19\cos(\pi k)+9\cos(2\pi k)\big)
\end{aligned}
$$

- g_s-dependence: $\frac{\pi k}{2} = \frac{\pi}{g_s}$

Spectral Curve of ABJM Matrix Model

- We will see that
  $a_\ell(k) = \text{quantum A-period}, \quad b_\ell(k) = \text{quantum B-period}$
- Spectral curve of ABJM matrix model is a complexified version of the Fermi surface $H(x, p) = \mu$





Quantum Curve

- Quantum curve of local $\mathbb{P}^1 \times \mathbb{P}^1$ = Fermi surface $H(x,p) = \mu$

$$2\cosh\left(\frac{x}{2}\right) \cdot 2\cosh\left(\frac{p}{2}\right) = e^{\mu}, \qquad [x,p] = 2\pi i k$$

- By a change of variable $(x,p) \to (u,v)$, this curve is rewritten as

$$\widetilde{H}(u,v) = e^u + e^v + z_1 e^{-u} + z_2 e^{-v} - 1 = 0$$

$$[v,u] = \hbar = i\pi k = \frac{2\pi i}{g_s}$$

- The moduli z_1, z_2 of the curve are

$$z_{1,2} = e^{-\frac{T_{1,2}}{g_s}} = e^{-2\mu} q^{\pm\frac{1}{2}}, \qquad q = e^{\hbar} = e^{\frac{2\pi i}{g_s}}$$

Quantum Period

- Quantum periods are determined by the wavefunction of probe D-brane [Aganagic-Cheng-Dijkgraaf-Krefl-Vafa]

$$\widetilde{H}(u, \hbar\partial_u)\Psi(u) = 0, \quad \Psi(u) = \exp\left[\frac{1}{\hbar}S(u,\hbar)\right]$$

- Quantum periods provide us with a systematic way of computing membrane instanton coefficients $a_\ell(k), b_\ell(k)$

$$\Pi_A = \oint_A dS = C^{-1}\sum_{\ell=1}^{\infty} a_\ell(k)e^{-2\ell\mu}$$

$$\Pi_B = \oint_B dS = C^{-1}\sum_{\ell=1}^{\infty} b_\ell(k)e^{-2\ell\mu}$$

- This correctly reproduces $a_\ell(k), b_\ell(k)$ computed from the semi-classical TBA

Refined Topological String

- Refined topological string on local Calabi-Yau manifold X is a two-parameter extension of the ordinary topological string

$$Z_{\rm ref}(\epsilon_1,\epsilon_2) = Z_{\text{M-theory}}(X \times S^1 \times \mathbb{C}^2)$$
$$(y, z_1, z_2) \sim (y + 2\pi R, e^{2\pi i \epsilon_1} z_1, e^{2\pi i \epsilon_2} z_2)$$

- Unrefined and Nekrasov-Shatashvili limit of free energy are

$$F_{\rm top}(\epsilon_1) = F_{\rm ref}(\epsilon_2 = -\epsilon_1)$$
$$F_{\rm NS}(\epsilon_1) = 2\pi i \lim_{\epsilon_2 \to 0} \epsilon_2 F_{\rm ref}(\epsilon_1, \epsilon_2)$$

- We found worldsheet and membrane instantons are given by

$$WS = F_{\rm top}(\epsilon_1 = g_s), \quad D2 \sim F_{\rm NS}\left(\epsilon_1 = \frac{1}{g_s}\right)$$

Membrane Instantons and $F_{\rm NS}$

- Quantum A-period defines the generalized flat coordinate via the quantum mirror map

$$\mu_{\rm eff} = \mu + \Pi_A$$

- Quantum B-period is related to the derivative of $F_{\rm NS}$ [Aganagic et al]

$$\Pi_B - \Pi_A^2 = C^{-1} \sum_{\ell=1}^{\infty} \widetilde{b}_\ell(k) e^{-2\ell\mu_{\rm eff}} = \frac{\hbar}{2} \frac{\partial F_{\rm NS}}{\partial \mu_{\rm eff}}$$

- If we treat g_s and $T_{\rm eff} = \dfrac{4\mu_{\rm eff}}{k}$ as independent variables, then $J_{\rm D2}(\mu_{\rm eff})$ is written as

$$\sum_{\ell=1}^{\infty} \left[\mu_{\rm eff} \widetilde{b}_\ell(k) + \widetilde{c}_\ell(k) \right] e^{-2\ell\mu_{\rm eff}} = \frac{1}{2\pi i} \frac{\partial}{\partial g_s} \left[g_s F_{\rm NS}\left(\frac{T_{\rm eff}}{g_s}, \frac{1}{g_s} \right) \right]$$

Non-perturbative ABJM Grand Potential

- Non-perturbative part of grand potential is completely determined by the refined topological string on the diagonal local $\mathbb{P}^1 \times \mathbb{P}^1$

$$J_{\rm np}(\mu_{\rm eff}) = F_{\rm top}(T_{\rm eff}, g_s) + \frac{1}{2\pi i}\frac{\partial}{\partial g_s}\left[g_s F_{\rm NS}\left(\frac{T_{\rm eff}}{g_s}, \frac{1}{g_s}\right)\right]$$

$$F_{\rm top}(T_{\rm eff}, g_s) = -\sum_{m=1}^{\infty}\sum_{j_L, j_R, d} N^d_{j_L, j_R}\frac{(2j_R+1)\chi_{j_L}(q_s^m)}{(q_s^{m/2}-q_s^{-m/2})^2}\frac{e^{-md\cdot T_{\rm eff}}}{m}$$

$$F_{\rm NS}\left(\frac{T_{\rm eff}}{g_s}, \frac{1}{g_s}\right) = \sum_{n=1}^{\infty}\sum_{j_L, j_R, d} N^d_{j_L, j_R}\frac{\chi_{j_L}(q^{n/2})\chi_{j_R}(q^{n/2})}{q^{n/2}-q^{-n/2}}\frac{e^{-\frac{nd\cdot T_{\rm eff}}{g_s}}}{n^2}$$

$$q_s = e^{2\pi i g_s}, \qquad q = e^{\frac{2\pi i}{g_s}}$$

- This form guarantees the pole cancellation at $g_s = \frac{n}{m}$

Non-perturbative Completion of Topological String

- The above form of ABJM grand potential suggests a generalization to arbitrary local Calabi-Yau manifolds

- For the general local Calabi-Yau, we have to turn on a discrete B-field $B = i\pi K$ for the pole cancellation to work

$$F_{\mathrm{np}} = F_{\mathrm{top}}(T_{\mathrm{eff}} + i\pi K, g_s) + \frac{1}{2\pi i}\frac{\partial}{\partial g_s}\left[g_s F_{\mathrm{NS}}\left(\frac{T_{\mathrm{eff}}}{g_s}, \frac{1}{g_s}\right)\right]$$

- This is slightly different from the proposal by Lockhart and Vafa

Summary

- The membrane instanton coefficients $a_\ell(k)$ and $b_\ell(k)$ are determined by the quantum A- and B-period, respectively.
- Effect of bound states is incorporated into the worldsheet instantons by the quantum mirror map $\mu \to \mu_{\mathrm{eff}}$
- Membrane instanton corrections to the ABJM grand potential is given by the refined topological string in the Nekrasov-Shatashvili limit
- We proposed a non-perturbative topological string on arbitrary local Calabi-Yau manifolds