Constructive Wall-Crossing & Seiberg-Witten

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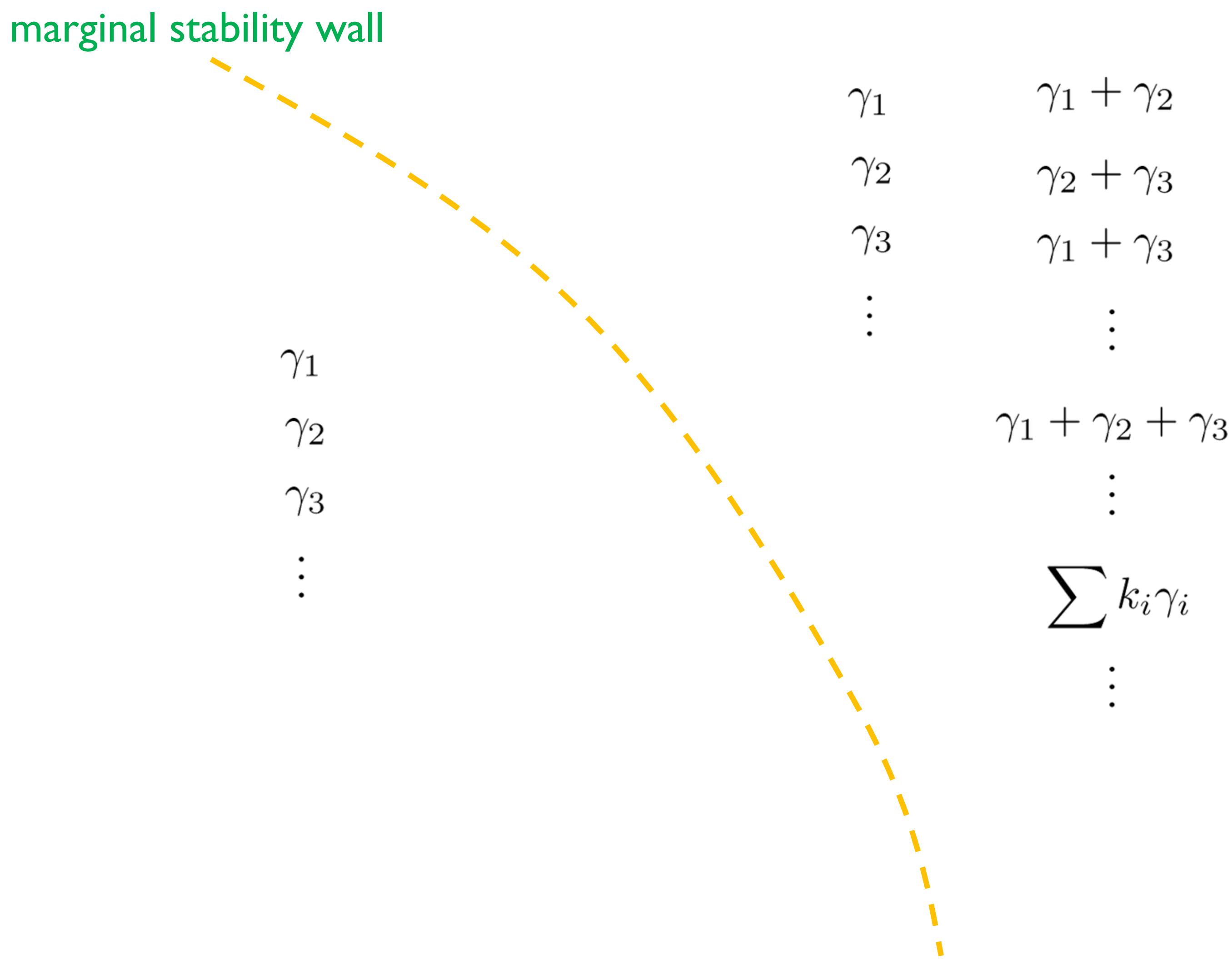
1102.1729 with Sungjay Lee
1107. 0723 with Heeyeon Kim, Jaemo Park, and Zhaolong Wang

wall-crossing is discontinuity of charged & particle-like BPS states in vacuum moduli space / parameter space

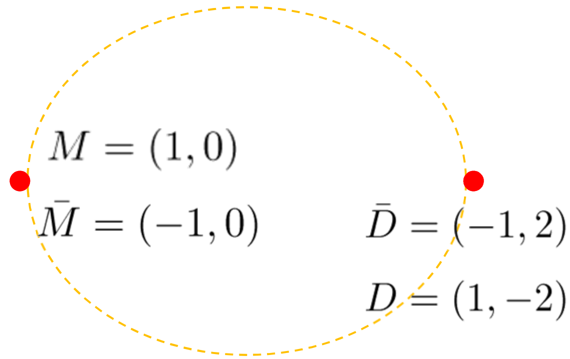


prototype : N=2 Seiberg-Witten SU(2) → U(1)

$$W^{\pm} = (0, \pm 2)$$

$$D_n = (1, 2n) = M + nW^{+}$$

$$\bar{D}_n = (-1, 2n) = \bar{M} + nW^{+}$$



1996 Bilal + Ferarri
self-consistent minimal set of BPS states
in SU(2) Seiberg-Witten, from monodromy

Kontsevich-Soibelman, 2008

(also Gaiotto-Moore-Neitzke, 2008-2009)

$$[V_\alpha, V_\beta] = (-1)^{\langle\alpha,\beta\rangle}\langle\alpha,\beta\rangle V_{\alpha+\beta} \qquad K_\gamma \equiv \exp\left(\sum_n \frac{V_{n\gamma}}{n^2}\right)$$

marginal stability wall

+ side

$$\prod_\gamma K_\gamma^{\Omega^+(\gamma)} = \prod_{\gamma'} K_{\gamma'}^{\Omega^-(\gamma')}$$

– side

where $\Omega^{\pm}$ are 2nd helicity trace of one-particle Hilbert spaces

$$SO(4) = SU(2)_{\text{rotation}} \times SU(2)_{\text{R-symmetry}}$$

$$J \qquad\qquad I$$

$$\Omega = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_3)^2$$

2nd helicity trace

$$\Leftarrow_{y=1}$$

$$\Omega(y) = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_3)^2 y^{2I_3+2J_3}$$

protected spin character

Gaiotto, Moore, Neitzke 2010

$$\to (-1)^{2l} \times (2l+1)$$

on [a spin ½ + two spin 0]
x [angular momentum l multiplet]

$$[V_\alpha, V_\beta] = (-1)^{\langle \alpha, \beta \rangle} \langle \alpha, \beta \rangle V_{\alpha+\beta}$$

$$\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$$

rational invariant

marginal stability wall

+ side

$$\prod_{\gamma} e^{\bar{\Omega}^+(\gamma) V_\gamma} = \prod_{\gamma'} e^{\bar{\Omega}^-(\gamma') V_{\gamma'}}$$

– side

why wall-crossings happen ?

does Kontsevich-Soibelman work for all N=2 theories ?

why rational invariants ? $\bar{\Omega}(\Gamma) = \sum_{p|\Gamma} \Omega(\Gamma/p)/p^2$

or more generally, is it possible to derive wall-crossing rules entirely from an elementary field theory computation ?

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1998 Lee + P.Y.

N=4 SU(n) ¼ BPS states via multi-center classical dyon solitons

1999 Bak + Lee + Lee + P.Y.

N=4 SU(n) ¼ BPS states via multi-center monopole dynamics

1999-2000 Gauntlett + Kim + Park + P.Y. / Gauntlett + Kim + Lee + P.Y.

N=2 SU(n) BPS states via multi-center monopole dynamics

a generic BPS state with 4 supercharges preserved
is a loose bound state of many charge centers

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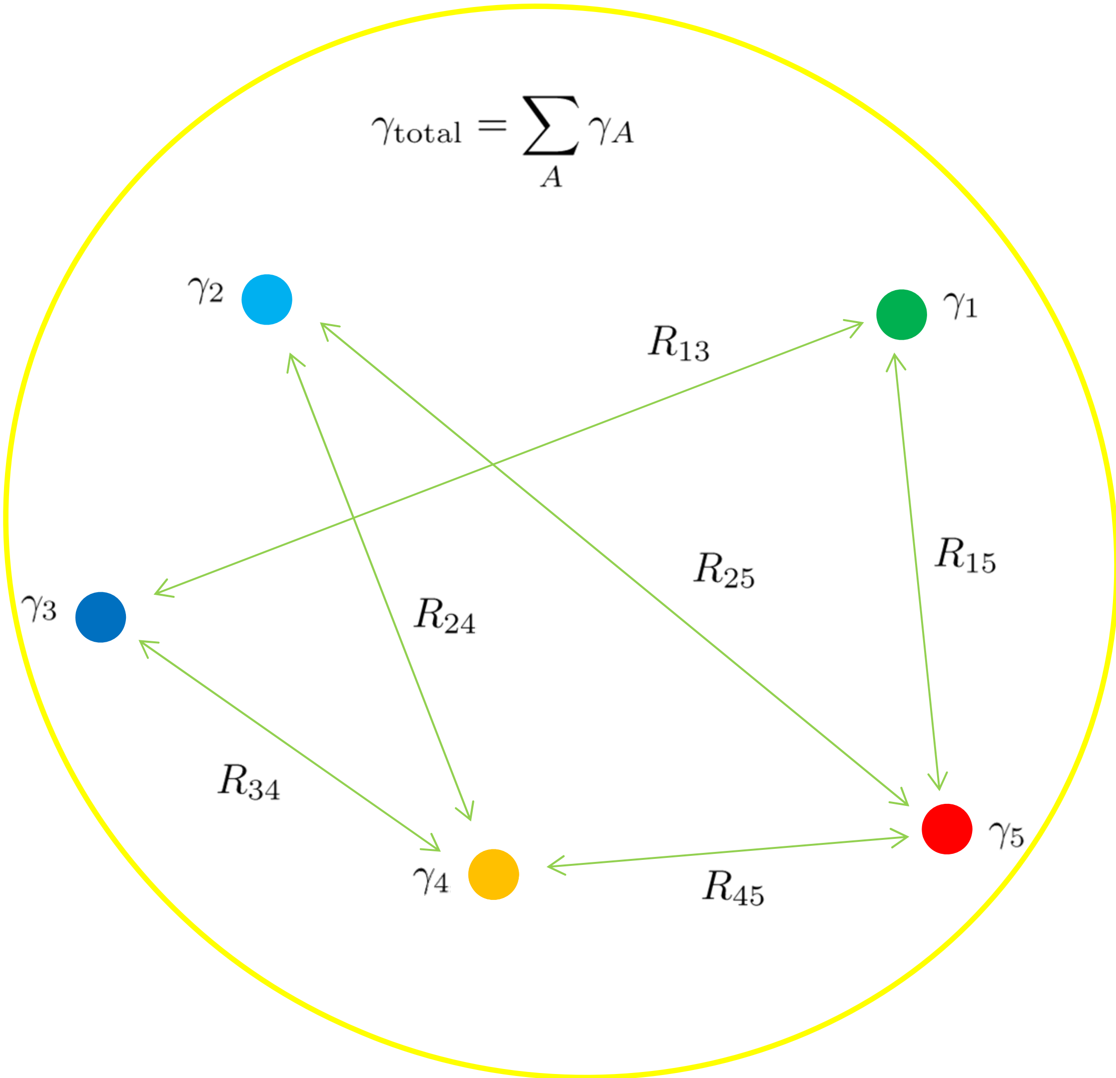
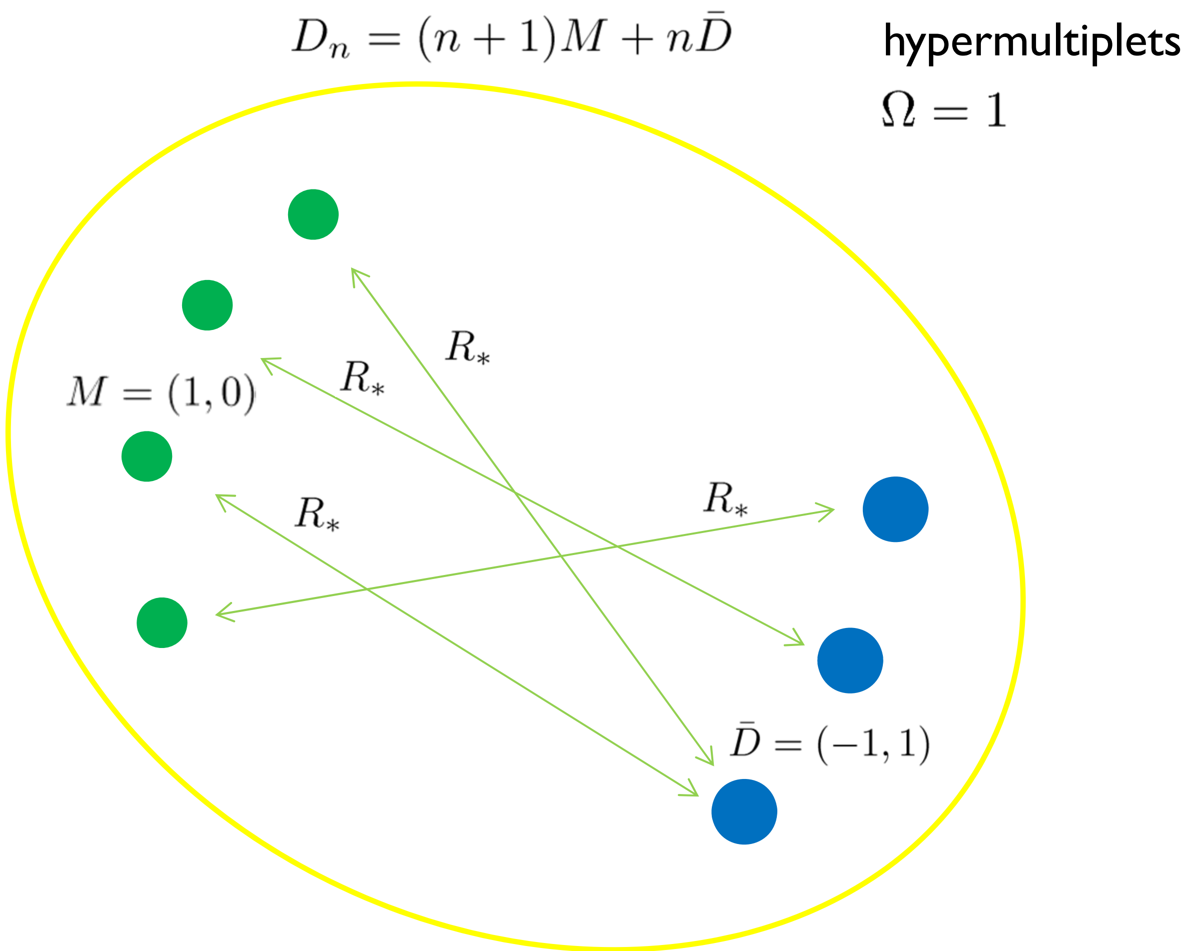


in particular, for SU(2) Seiberg-Witten, we expect

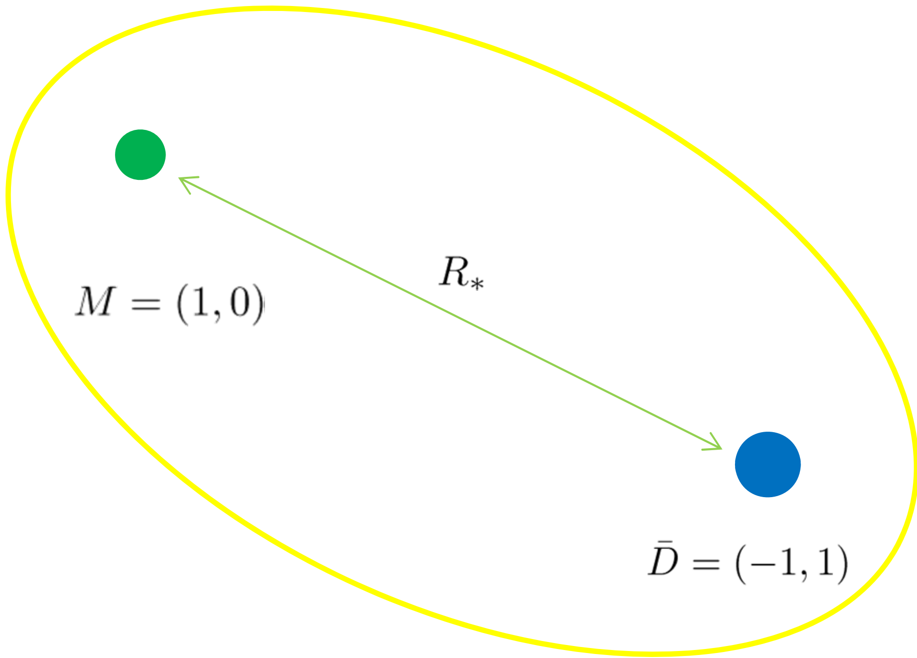


in particular, for SU(2) Seiberg-Witten, we expect

$$W^+ = M + \bar{D}$$

vector multiplet

$\Omega = -2$



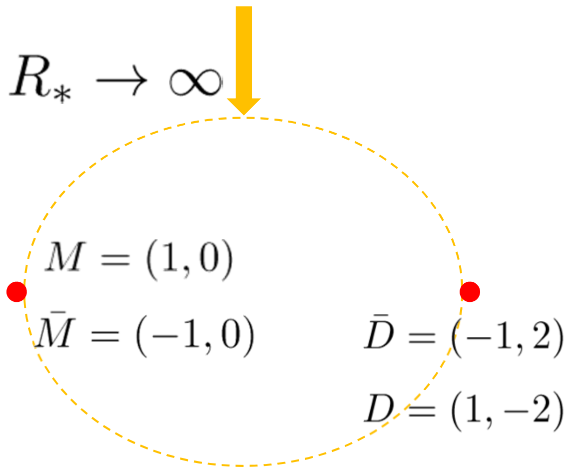
wall-crossing ← dissociation of supersymmetric bound states

$$W^+ = M + \bar{D}$$

$$W^- = \bar{M} + D$$

$$D_n = (n+1)M + n\bar{D}$$

$$\bar{D}_n = (n-1)M + n\bar{D}$$



$R_* \to 0$

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N=2 SU(n) BPS states via multi-center monopole dynamics

2001 Denef:

N=2 supergravity via classical multi-center black holes attractor solutions

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{R_{AB}}$$

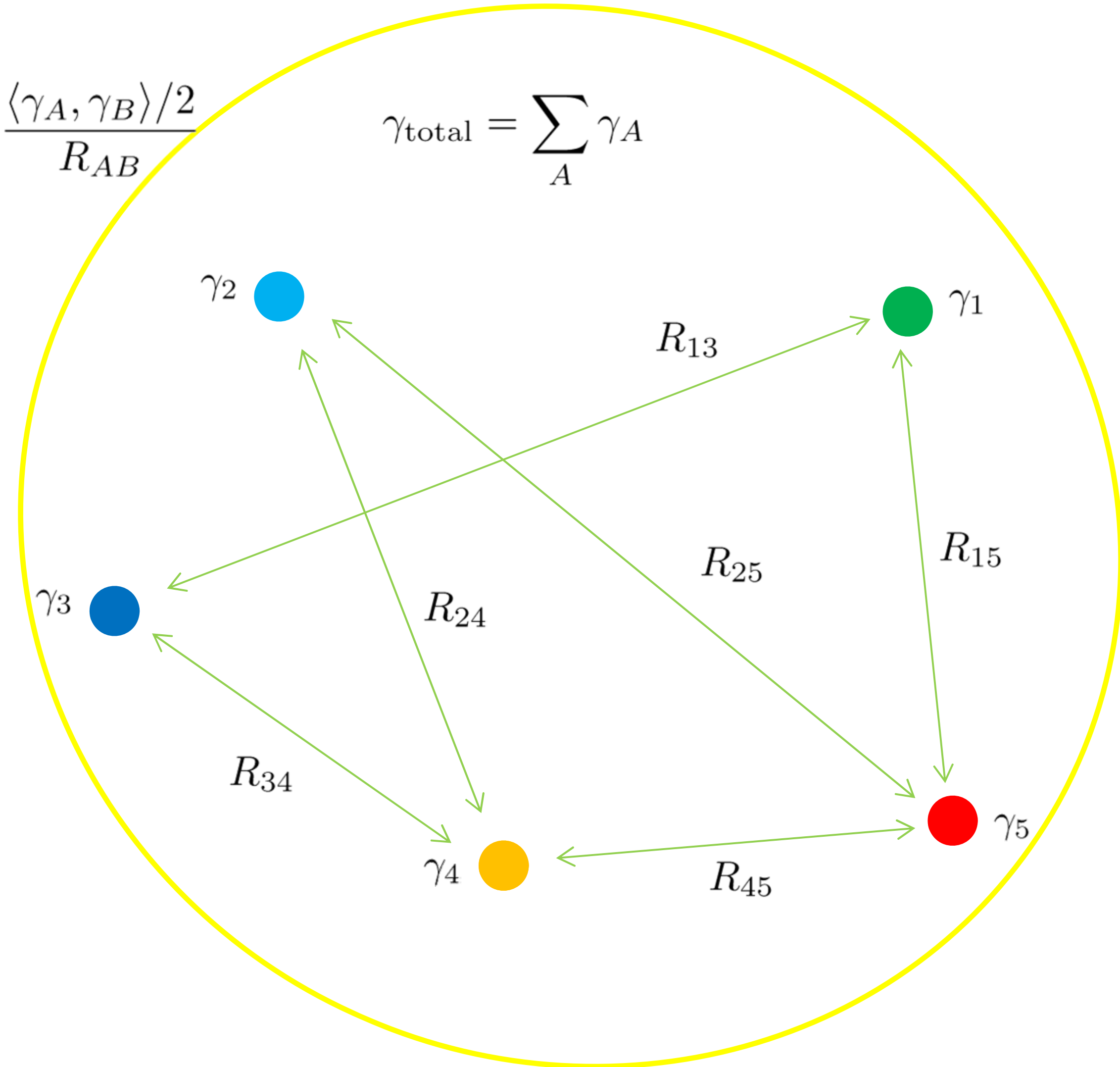
$$\zeta \equiv \frac{\sum_A Z_{\gamma_A}}{|\sum_A Z_{\gamma_A}|}$$

a generic BPS black hole with 4 supercharges preserved
is a loose bound state of many single-centered black holes

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle /2}{R_{AB}}$$

$$\zeta \equiv \frac{\sum_A Z_{\gamma_A}}{|\sum_A Z_{\gamma_A}|}$$

$$\gamma_{\text{total}} = \sum_A \gamma_A$$



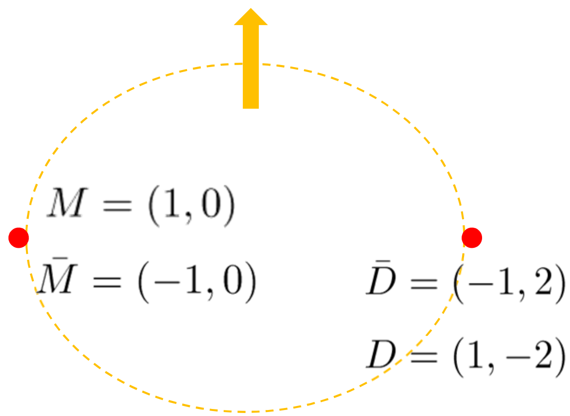
wall-crossing → emergence of supersymmetric bound states

$$W^+ = M + \bar{D}$$

$$W^- = \bar{M} + D$$

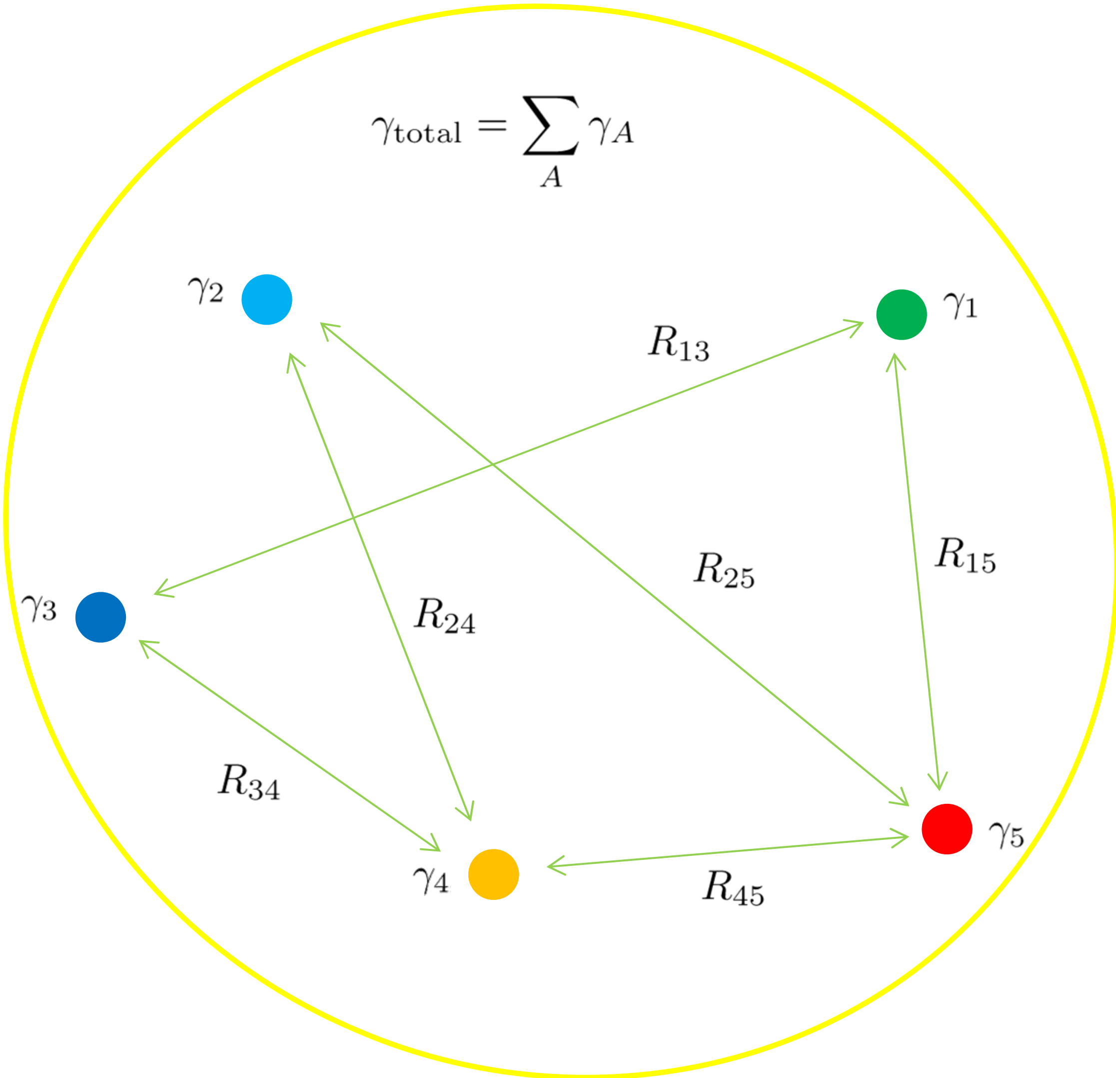
$$D_n = (n+1)M + n\bar{D}$$

$$\bar{D}_n = (n-1)M + n\bar{D}$$



wall-crossing problem

= how to count & classify such supersymmetric bound states



2000 Stern + P.Y.
wall-crossing formula for simple magnetic charges;
weak coupling regime

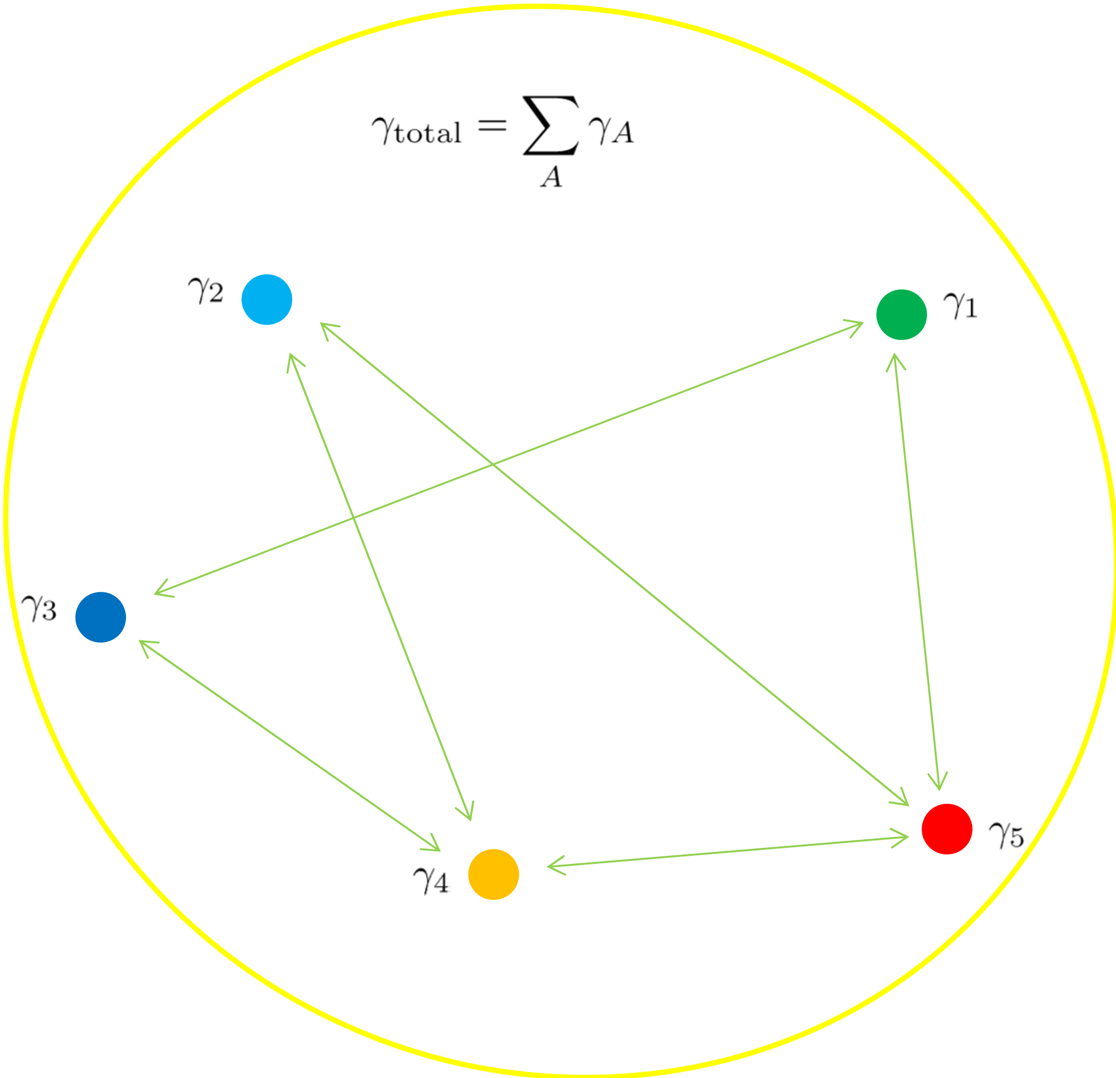
2002 Denef:
quiver dynamics representation of N=2 supergravity BH's

⋮

2008 de Boer + El Showk + Messamah + van den Bleeken
2- & 3- particle bound state conjecture

2010/2011 Manschot + Pioline + Sen:
general n-particle conjecture & evaluation

→ how to count & classify such supersymmetric bound states, entirely from low energy dynamics of constituent particles

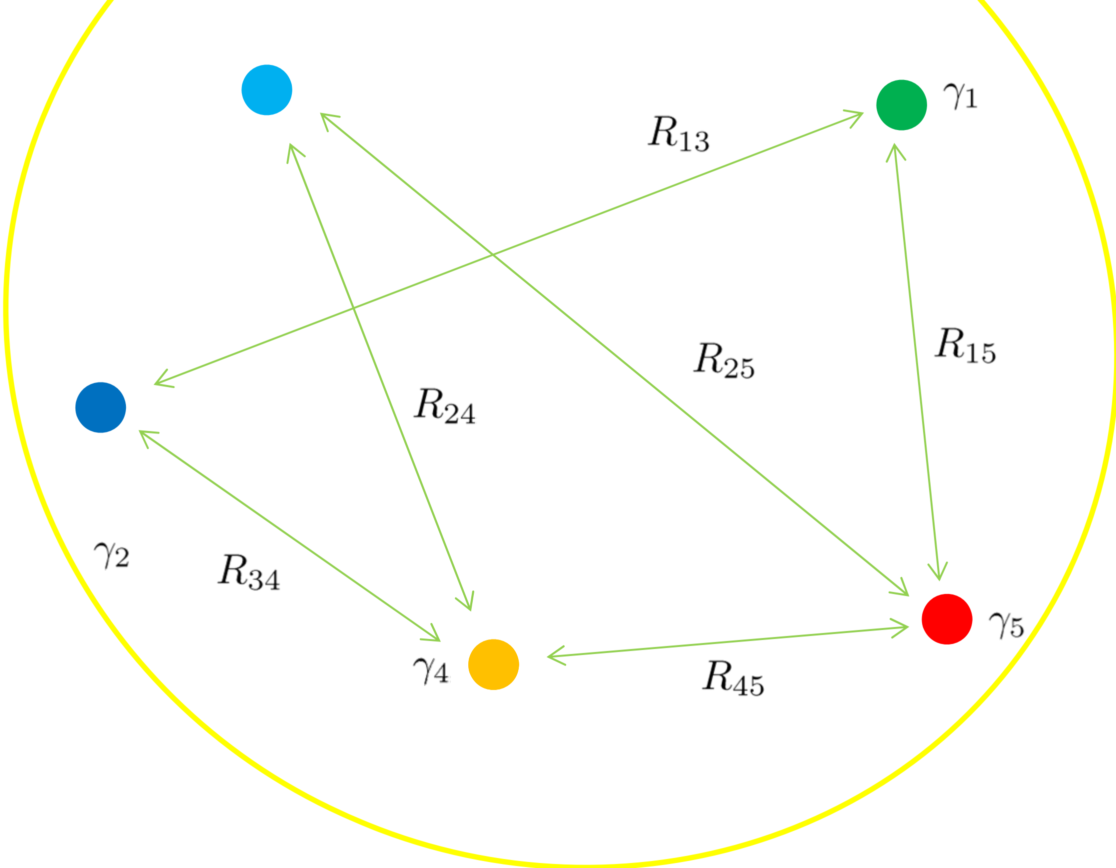


yet, for supersymmetric solutions only with constrained distances,
physically sensible quantum mechanics with four supercharges do not exist

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{R_{AB}}$$

$$\gamma_{\mathrm{total}} = \sum_A \gamma_A$$

$$\zeta \equiv \frac{\sum_A Z_{\gamma_A}}{|\sum_A Z_{\gamma_A}|}$$



2011 S. Lee + P.Y. / H. Kim + J. Park + Z. Wang + P.Y.
reformulation and full solution via generic dyon/BH dynamics

real space dynamics among dyons,
index theorems,
& universal wall-crossing formulae.

removed almost all hypotheses/conjectures that have been invoked in the past
& in some sense "proves" Konsevitch-Soibelman proposal

again, with 2nd helicity trace (or protected spin character)

$$SO(4) = SU(2)_{\rm rotation} \times SU(2)_{\rm R-symmetry}$$

$$J \qquad\qquad\qquad I$$

$$\boxed{\Omega = -\frac{1}{2}{\rm tr}\,(-1)^{2J_3}(2J_3)^2} \quad \underset{y=1}{\Leftarrow} \quad \Omega(y) = -\frac{1}{2}{\rm tr}\,(-1)^{2J_3}(2J_3)^2 y^{2I_3+2J_3}$$

$$\rightarrow (-1)^{2l} \times (2l+1)$$

on [a spin ½ + two spin 0]
x [angular momentum l multiplet]

(1) index for n-body problems

Kim+Park+P.Y.+Wang 2011

$$\Omega_{\text{before statistics}} = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\boxed{I_n(\{\gamma_A\})}\prod_A \Omega(\gamma_A)$$

$$\begin{aligned} I_n(\{\gamma_A\}) &= \int_{\mathcal{K}_A=0} ch(\mathcal{F}) \\ &= \frac{1}{(2\pi)^{n-1}(n-1)!}\int_{\mathcal{K}_A=0}\mathcal{F}^{n-1} \end{aligned}$$

$$\mathcal{K}_A = \text{Im}[\zeta^{-1}Z_{\gamma_A}] - \sum_{B\neq A}\frac{\langle\gamma_A,\gamma_B\rangle/2}{|\vec{x}_A-\vec{x}_B|} \qquad \mathcal{F} \equiv \sum_{A>B}\frac{\langle\gamma_A,\gamma_B\rangle}{2}\,\mathcal{F}^{4\pi}_{Dirac}(\vec{x}_A-\vec{x}_B)\Big|_{\mathcal{K}_A=0}$$

(2) the wall-crossing formula from real space dynamics of dyons with Bose/Fermi statistics

Kim+Park+P.Y.+Wang 2011

Manschot, Pioline, Sen 2010/2011

$$\Omega^-\left(\sum\gamma_A\right)-\Omega^+\left(\sum\gamma_A\right)=(-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\frac{\prod_A\bar{\Omega}^+(\gamma_A)}{|\Gamma|}I_n(\{\gamma_A\})$$

$$\vdots$$

$$+(-1)^{\sum_{A'>B'}\langle\gamma'_{A'},\gamma'_{B'}\rangle+n'-1}\frac{\prod_{A'}\bar{\Omega}^+(\gamma'_{A'})}{|\Gamma'|}I_{n'}(\{\gamma'_{A'}\})$$

$$\vdots$$

$$\sum_{A=1}^{n}\gamma_A=\cdots=\sum_{A'=1}^{n'}\gamma'_{A'}=\cdots$$

$$\bar{\Omega}(\gamma)=\sum_{p|\gamma}\Omega(\gamma/p)/p^2$$

with all charges γ_A on a single plane of charge lattice,
this has been shown to be equivalent to
the Kontsevich-Soibelman proposal

(Ashoke Sen, December 2011)

D=4 N=2 Seiberg-Witten

$$\mathcal{L}_{U(1)^r}\Big|_{\mathcal{N}=2} = \mathcal{K}\left[\Psi^m, \bar{\Psi}^{\bar{k}}\right]\Big|_{\theta^2\bar{\theta}^2} + \frac{1}{8\pi}\, Im\left[\tau_{kl}(\Psi^m)\,\epsilon^{\alpha\beta} W_\alpha^{(k)} W_\beta^{(l)}\Big|_{\theta^2}\right]$$

$$(\Psi_D)_m = \frac{\partial \mathcal{F}}{\partial \Psi^m}$$

$$\mathcal{F} = \mathcal{F}(\Psi) \qquad \mathcal{K}(\Psi, \bar{\Psi}) = -\frac{i}{2\pi}\left(\bar{\Psi}^{\bar{k}}(\Psi_D)_k - \Psi^k(\bar{\Psi}_D)_k\right)$$

$$\tau_{kl}(\Psi) = \frac{\partial(\Psi_D)_k}{\partial \Psi^l}$$

D=4 N=2 Seiberg-Witten

$$SU(r+1) \ \rightarrow U(1)^r$$

Ψ

$$\Phi = \begin{pmatrix} \phi^{(1)} & 0 & \cdots & 0 \\ 0 & \phi^{(2)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \phi^{(r+1)} \end{pmatrix}$$

V

$$A_\mu = \begin{pmatrix} A_\mu^{(1)} & 0 & \cdots & 0 \\ 0 & A_\mu^{(2)} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_\mu^{(r+1)} \end{pmatrix}$$

$$\phi^{(1)} + \phi^{(2)} + \cdots + \phi^{(r+1)} = 0$$

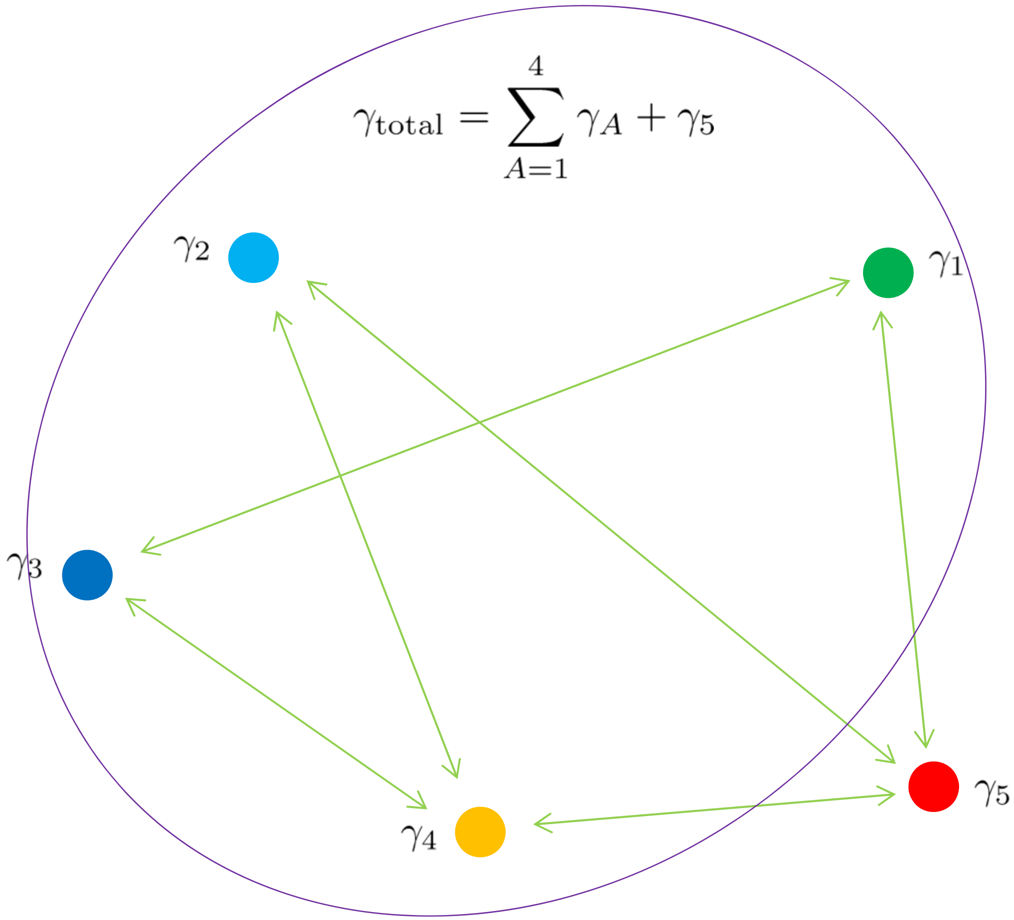
$$A_\mu^{(1)} + A_\mu^{(2)} + \cdots + A_\mu^{(r+1)} = 0$$

all massive off-diagonal fields are integrated out we will represent BPS particles as

semi-classical solutions with non-Abelian cores cut-off,

which is sufficient

when the field theory vacuum sits near a marginal stability wall as a preliminary step, treat one dyon dynamical at a time



BPS states as semiclassical & asymptotic solutions

with electric charges n^i and magnetic charges m^i

$$\mathcal{E} = |Z| \qquad Z \equiv \left\langle m^i \phi_D^i + n^i \phi^i \right\rangle = |Z|\zeta$$

$$F_a^i = i\zeta^{-1}\partial_a \phi^i$$

$$(F_D)_a^i = i\zeta^{-1}\partial_a \phi_D^i$$

$$\mathrm{Re}\int_{S^2} F^i = 4\pi m^i \qquad F_a^i \equiv B_a^i + iE_a^i$$

$$\mathrm{Re}\int_{S^2} F_D^i = -4\pi n^i \qquad (F_D)_a^i \equiv \tau^{ij} F_j^a$$

central charge & central charge function

$$Z \equiv \left\langle m^i \phi_D^i + n^i \phi^i \right\rangle = \langle \mathcal{Z} \rangle$$

$$\mathcal{Z}(x) \equiv m^i \phi_D^i(x) + n^i \phi^i(x)$$

a probe charge to a system of background "core" dyons

$$\gamma_h = (p, 2q) \qquad \sum_{A\neq h} \gamma_A = \sum_{A\neq h} (m_A, 2n_A)$$

background dyons:

$$F^i_a = i\zeta^{-1}\partial_a \phi^i \qquad \mathrm{Re}\int_{S^2} F^i = 4\pi m^i$$

$$(F_D)^i_a = i\zeta^{-1}\partial_a \phi^i_D \qquad \mathrm{Re}\int_{S^2} F^i_D = -4\pi n^i$$

central charge function of probe dyon:

$$q^i F^i_a + p^i (F_D)^i_a = i\zeta^{-1}\partial_a \left(q_i \phi^i + p^i \phi^i_D\right)$$

$$\equiv i\zeta^{-1}\partial_a \mathcal{Z}_{\gamma_h}(x)$$

a probe charge to a system of background "core" dyons

$$q^i F^i_a + p^i \tau_{ij} F^j_a = i\zeta^{-1}\partial_a \left(q_i \phi^i + p^i \phi^i_D\right) \equiv i\zeta^{-1}\partial_a \mathcal{Z}_{\gamma_h}(x)$$

$$F^i_a = i\zeta^{-1}\partial_a \phi^i$$

$$F^i_a \equiv B^i_a + iE^i_a$$

$$\mathrm{Im}[\zeta^{-1}\mathcal{Z}_{\gamma_h}] = \mathrm{Im}[\zeta^{-1}Z_{\gamma_h}] - \sum_{A\neq h} \frac{\langle \gamma_h, \gamma_A\rangle/2}{|\vec{x}-\vec{x}_A|}$$

$$\mathrm{Re}[\zeta^{-1}\mathcal{Z}_{\gamma_h}] = \mathrm{Re}[\zeta^{-1}Z_{\gamma_h}] - \sum_{A\neq h} \frac{I_{Ah}}{|\vec{x}-\vec{x}_A|}$$

$$I_{Ah} \equiv \langle \mathrm{Re}\tau^{ij}\rangle p^i m^j_A + \langle \mathrm{Re}\tau^{-1}_{ij}\rangle (q_i + \langle \mathrm{Im}\tau_{ik}\rangle p_k)(n^j_A + \langle \mathrm{Im}\tau_{jl}\rangle m^l_A)$$

a probe charge to a system of background "core" dyons

Sungjay Lee+P.Y. 2011

$$\mathcal{L}_{probe} = -|\mathcal{Z}_h|\sqrt{1-\dot{\vec{x}}^2} + \mathrm{Re}[\zeta^{-1}\mathcal{Z}_h] - \dot{\vec{x}}\cdot\vec{W}$$

$$\vec{\partial}\times\vec{W} \equiv \vec{\partial}\,\mathrm{Im}\,[\zeta^{-1}\mathcal{Z}_h]$$

a probe charge to a system of background "core" dyons

Sungjay Lee+P.Y. 2011

$$\mathcal{L}_{probe} = -|\mathcal{Z}_h|\sqrt{1-\dot{\vec{x}}^2} + \mathrm{Re}[\zeta^{-1}\mathcal{Z}_h] - \dot{\vec{x}}\cdot\vec{W}$$

$$\simeq \frac{1}{2}\,|\mathcal{Z}_h|\dot{\vec{x}}^2 - (|\mathcal{Z}_h| - \mathrm{Re}[\zeta^{-1}\mathcal{Z}_h]) - \dot{\vec{x}}\cdot\vec{W}$$

$$\vec{\partial}\times\vec{W} \equiv \vec{\partial}\,\mathrm{Im}\,[\zeta^{-1}\mathcal{Z}_h]$$

a probe charge to a system of background "core" dyons

Sungjay Lee+P.Y. 2011

$$\mathcal{L}_{probe} = -|\mathcal{Z}_h|\sqrt{1-\dot{\vec{x}}^2} + \mathrm{Re}[\zeta^{-1}\mathcal{Z}_h] - \dot{\vec{x}}\cdot\vec{W}$$

$$\simeq \frac{1}{2}\,|\mathcal{Z}_h|\dot{\vec{x}}^2 - (|\mathcal{Z}_h| - \mathrm{Re}[\zeta^{-1}\mathcal{Z}_h]) - \dot{\vec{x}}\cdot\vec{W}$$

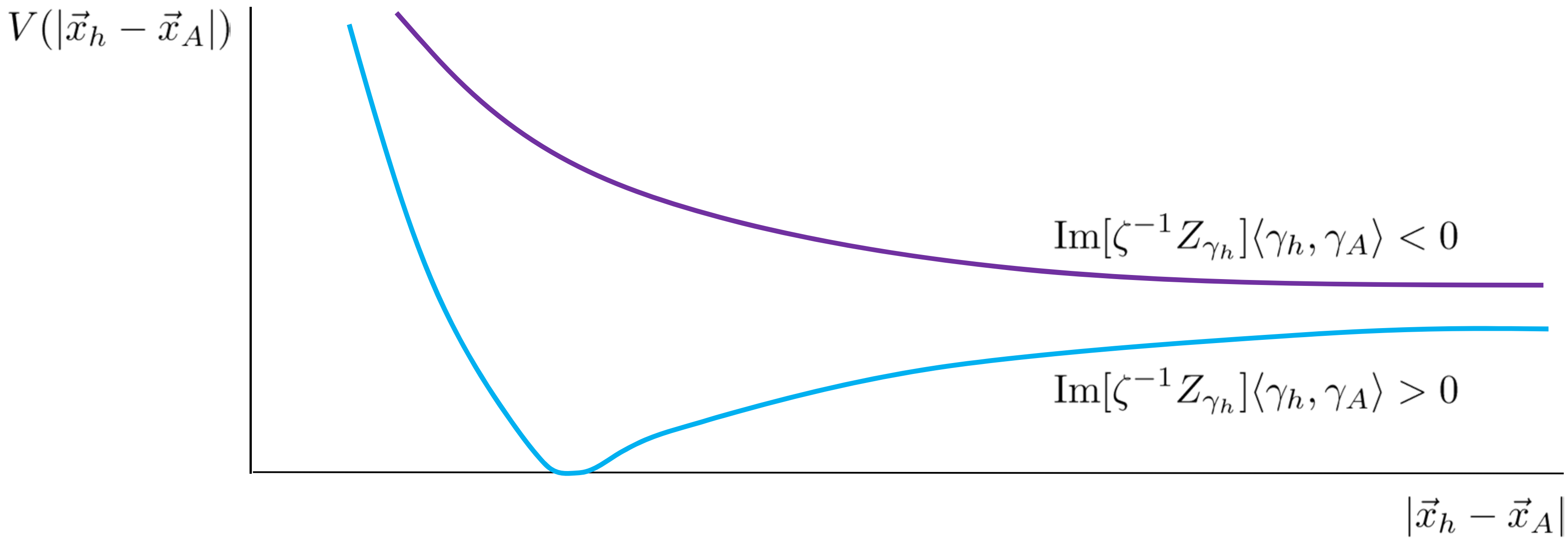
$$\simeq \frac{1}{2}\,|\mathcal{Z}_h|\dot{\vec{x}}^2 - \frac{(\mathrm{Im}[\zeta^{-1}\mathcal{Z}_h])^2}{2|\mathcal{Z}_h|} - \dot{\vec{x}}\cdot\vec{W}$$

$$\vec{\partial}\times\vec{W} \equiv \vec{\partial}\,\mathrm{Im}\,[\zeta^{-1}\mathcal{Z}_h]$$

$$\zeta^{-1}\mathcal{Z}_h = |\mathcal{Z}_h|e^{i\alpha}, \quad |\alpha| \ll 1$$

wall-crossing is mostly a classical phenomena !

$$V = \frac{(\mathrm{Im}[\zeta^{-1}\mathcal{Z}_h])^2}{2|\mathcal{Z}_h|} \sim \left(\mathrm{Im}[\zeta^{-1}Z_{\gamma_h}] - \frac{\langle\gamma_h,\gamma_A\rangle/2}{|\vec{x}_h - \vec{x}_A|}\right)^2$$



we need to

1) elevate this to a N=4 supersymmetric quantum mechanics

2) make all dyons dynamical

N=4 SUSY QM with 3n bosons & 4n fermions ?

4n N=1 supermultiplets

$$\Phi^{Aa} = x^{Aa} - i\theta\psi^{Aa} \qquad \Lambda^A = i\lambda^A + i\theta b^A \qquad A = 1, 2, \dots, n$$

↓

position of A-th dyon

or n N=4 supermultiplet

$$\hat{\Phi}^{Aa} = -\frac{i}{4}(\epsilon\sigma^a)^{\alpha\beta}\Phi^A_{\alpha\beta} \; ; \qquad \Phi^A_{\alpha\beta} = (D_\alpha \bar{D}_\beta + \bar{D}_\beta D_\alpha)V^A$$

~ D=4 N=1 vector multiplet dimensionally reduced

N=4 SUSY QM with 3n bosons & 4n fermions ?

Kim+Park+P.Y.+Wang 2011

$$\int dt\, \mathcal{L}_{kinetic} = \int dt \int d\theta^2 d\bar{\theta}^2 F(\hat{\Phi}^{Aa})$$ ⟵ Smilga; Ivanov; Papadopoulos; …… circa1988-1991

$$\hat{\Phi}^{Aa} = -\frac{i}{4}(\epsilon\sigma^a)^{\alpha\beta}\Phi^A_{\alpha\beta}\ ; \qquad \Phi^A_{\alpha\beta} = (D_\alpha \bar{D}_\beta + \bar{D}_\beta D_\alpha)V^A$$

$$\int dt\, \mathcal{L}_{potential} = \int dt \int d\theta\ \left(i\mathcal{K}(\Phi)_A \Lambda^A - iW(\Phi)_{Aa} D\Phi^{Aa}\right)$$

$$\Phi^{Aa} = x^{Aa} - i\theta\psi^{Aa} \qquad \Lambda^A = i\lambda^A + i\theta b^A$$

N=4 SUSY QM with 3n bosons & 4n fermions ?

Kim+Park+P.Y.+Wang 2011

$$\int dt\ \mathcal{L}_{kinetic} = \int dt \int d\theta^2 d\bar{\theta}^2 F(\hat{\Phi}^{Aa})$$

is manifestly N=4 supersymmetric

$$\int dt\ \mathcal{L}_{potential} = \int dt \int d\theta\ \left(i\mathcal{K}(\Phi)_A \Lambda^A - iW(\Phi)_{Aa} D\Phi^{Aa}\right)$$

is N=4 supersymmetric if and only if $\vec{\partial}_A \cdot \vec{\partial}_B \mathcal{K}_C = 0$

$$\vec{\partial}_A \times \vec{\partial}_B \mathcal{K}_C = 0$$

$$\vec{\partial}_A \mathcal{K}_B = \frac{1}{2}\left(\vec{\partial}_A \times \vec{W}_B + \vec{\partial}_B \times \vec{W}_A\right) = \vec{\partial}_A \mathcal{K}_B$$

real space N=4 quantum mechanics for dyons

Kim+Park+P.Y.+Wang 2011

$$\int dt\, \mathcal{L}_{kinetic} = \int dt \int d\theta^2 d\bar\theta^2 F(\hat\Phi^{Aa})$$

$$\int dt\, \mathcal{L}_{potential} = \int dt \int d\theta\, \left(i\mathcal{K}(\Phi)_A\Lambda^A - iW(\Phi)_{Aa}D\Phi^{Aa}\right)$$

$$\mathcal{K}_A = \mathrm{Im}[\zeta^{-1}Z_{\gamma_A}] - \sum_{B\neq A}\frac{\langle\gamma_A,\gamma_B\rangle/2}{|\vec{x}_A-\vec{x}_B|}$$

$$\vec{W}_A = \sum_{B\neq A}\frac{\langle\gamma_A,\gamma_B\rangle}{2}\,\vec{W}^{4\pi}_{Dirac}(\vec{x}_A-\vec{x}_B)$$

$$F(\vec{x}_A) \simeq \sum_A |Z_A||\vec{x}_A|^2 = \sum_A m_A|\vec{x}_A|^2 \qquad \text{asymptotically}$$

reproduces Coulomb phase dynamics in Denef's quiver description

Denef 2002

$$\int dt\, \mathcal{L}_{potential} = \int dt \int d\theta \, \left(i\mathcal{K}(\Phi)_A \Lambda^A - iW(\Phi)_{Aa} D\Phi^{Aa}\right)$$

$$\mathcal{K}_A = \mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] - \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{|\vec{x}_A - \vec{x}_B|}$$

$$\vec{W}_A = \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle}{2} \, \vec{W}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)$$

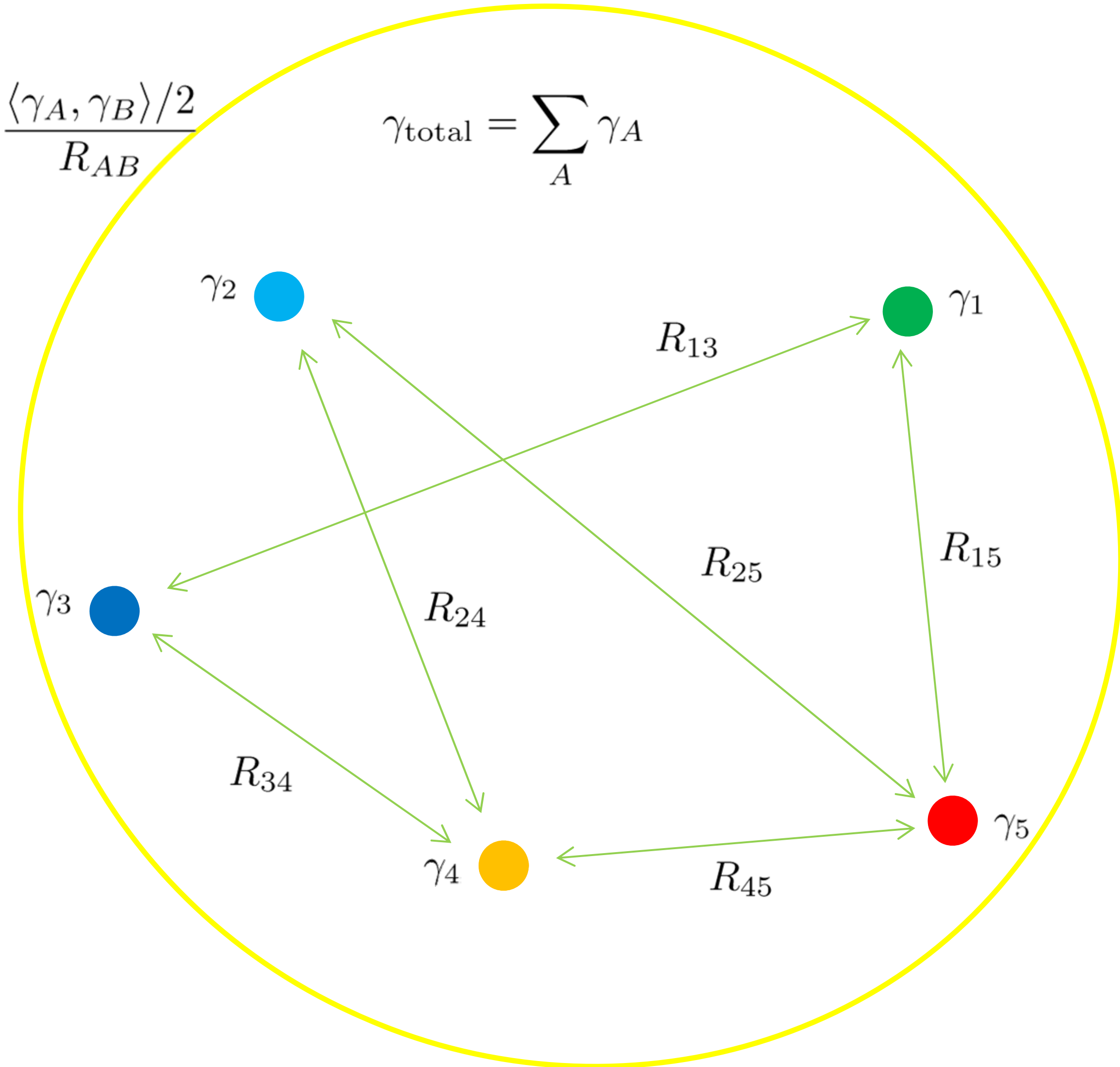
$$\vec{\partial}_A \mathcal{K}_B = \frac{1}{2}\left(\vec{\partial}_A \times \vec{W}_B + \vec{\partial}_B \times \vec{W}_A\right) = \vec{\partial}_A \mathcal{K}_B$$

and, thus, also derives Denef's distance formulae (for Black Holes)
for Seiberg-Witten BPS dyons near marginal stability walls

$$\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] = \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle /2}{R_{AB}}$$

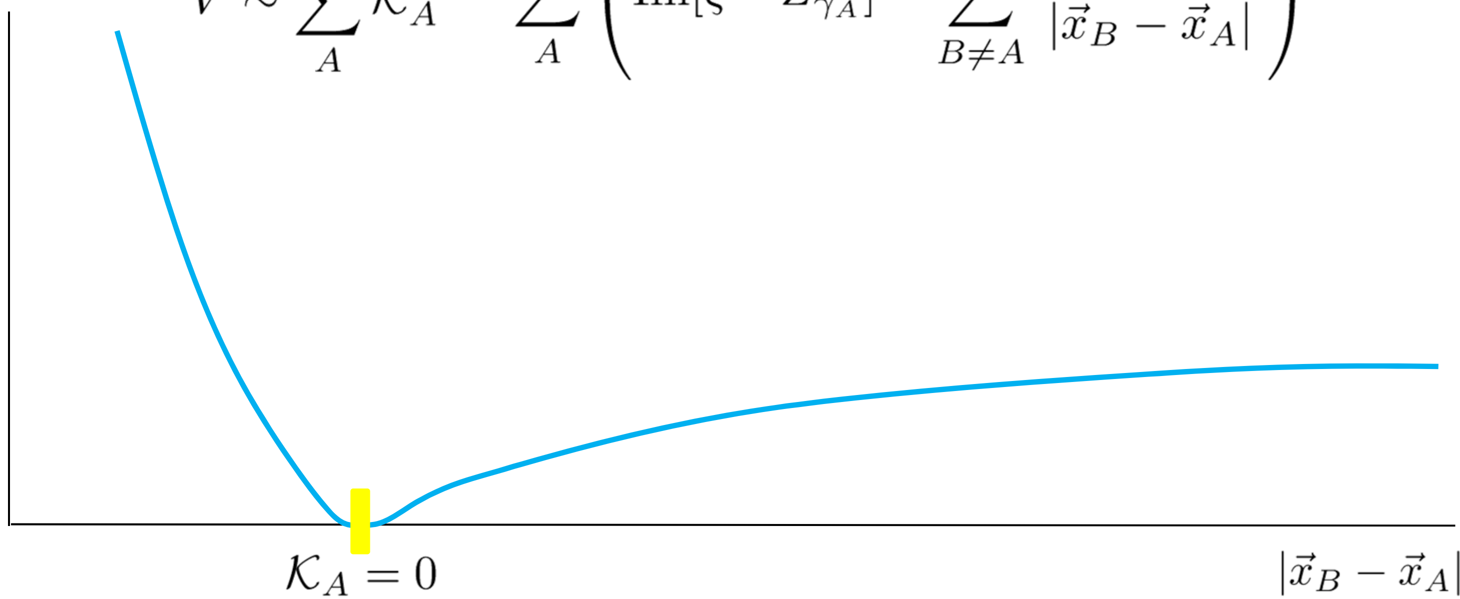
$$\zeta \equiv \frac{\sum_A Z_{\gamma_A}}{|\sum_A Z_{\gamma_A}|}$$

$$\gamma_{\mathrm{total}} = \sum_A \gamma_A$$



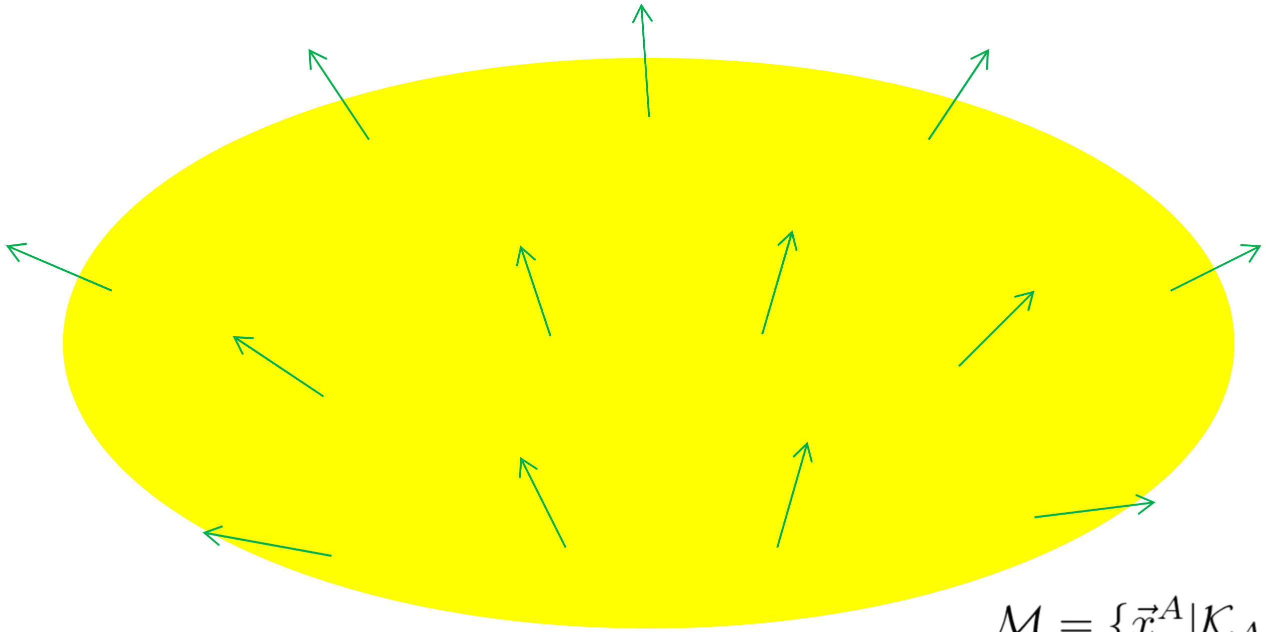
3n → 3 + 2(n-1) ?

$$V \sim \sum_A \mathcal{K}_A^2 \sim \sum_A \left(\mathrm{Im}[\zeta^{-1} Z_{\gamma_A}] - \sum_{B \neq A} \frac{\langle \gamma_A, \gamma_B \rangle / 2}{|\vec{x}_B - \vec{x}_A|} \right)^2$$



3n → 3 + 2(n-1) ?

$$\mathcal{F} \equiv \sum_A dW_A \Big|_{\mathcal{K}_A=0}$$



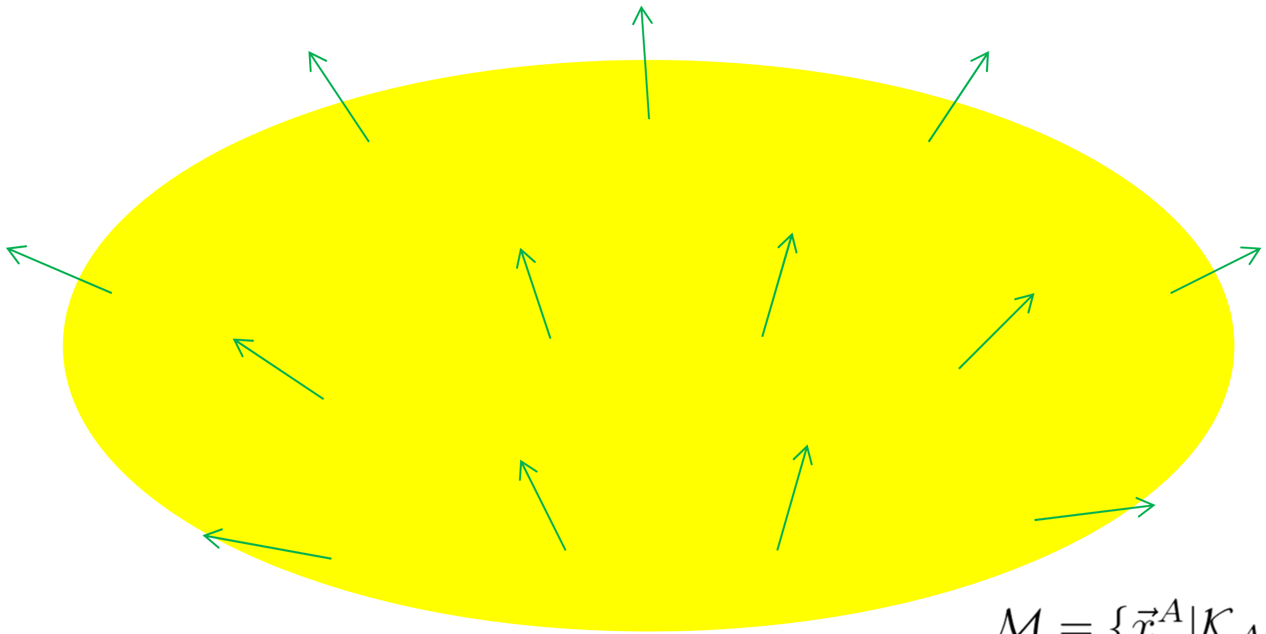
$$\mathcal{M} = \{\vec{x}^A | \mathcal{K}_A = 0\}$$

supersymmetric ground state dynamics reduces on $\mathcal{K}_A = 0$ submanifold ?

de Boer, El Showk, Messamah, van den Bleeken 2008

3n → 3 + 2(n-1) ?

$$\mathcal{F} \equiv \sum_A dW_A \Big|_{\mathcal{K}_A = 0}$$



$$\mathcal{M} = \{\vec{x}^A | \mathcal{K}_A = 0\}$$

the simple answer is NO !

3n → 3 + 2(n-1) ?

only after sacrificing all but one supersymmetries !!!

$$\vec{\partial}_A\,(\xi\cdot\mathcal{K}_B) \neq \frac{1}{2}\left(\vec{\partial}_A\times\vec{W}_B+\vec{\partial}_B\times\vec{W}_A\right)=\vec{\partial}_A\,\mathcal{K}_B$$

$$\mathcal{L}^{for\ index\ only}_{deformed}$$

$$= \int d\theta^2 d\bar{\theta}^2 F(\hat{\Phi}^{Aa}) \quad + \int d\theta\, \boxed{\left(i\xi\cdot\mathcal{K}(\Phi)_A\Lambda^A\right.} - iW(\Phi)_{Aa}D\Phi^{Aa}\Big)$$

$$\mathcal{K}_A = \mathrm{Im}[\zeta^{-1}Z_{\gamma_A}] - \sum_{B\neq A}\frac{\langle\gamma_A,\gamma_B\rangle/2}{|\vec{x}_A-\vec{x}_B|}$$

$$\vec{W}_A = \sum_{B\neq A}\frac{\langle\gamma_A,\gamma_B\rangle}{2}\,\vec{W}^{4\pi}_{Dirac}(\vec{x}_A-\vec{x}_B)$$

but this suffices for counting Witten index
or its refined generalizations

$$\Omega = -\frac{1}{2}\mathrm{tr}\,(-1)^{2J_3}(2J_2)^2 \qquad \sim \mathrm{tr}\,(-1)^F e^{-\beta Q^2}$$

$$= \#\text{Bosonic Ground States} - \#\text{Fermionic Ground States}$$

since, for index computation,
only one of the four supersymmetries is needed

counting problem reduces to a N=1 Dirac index of a nonlinear sigma model on the manifold $\mathcal{K}_A = 0$

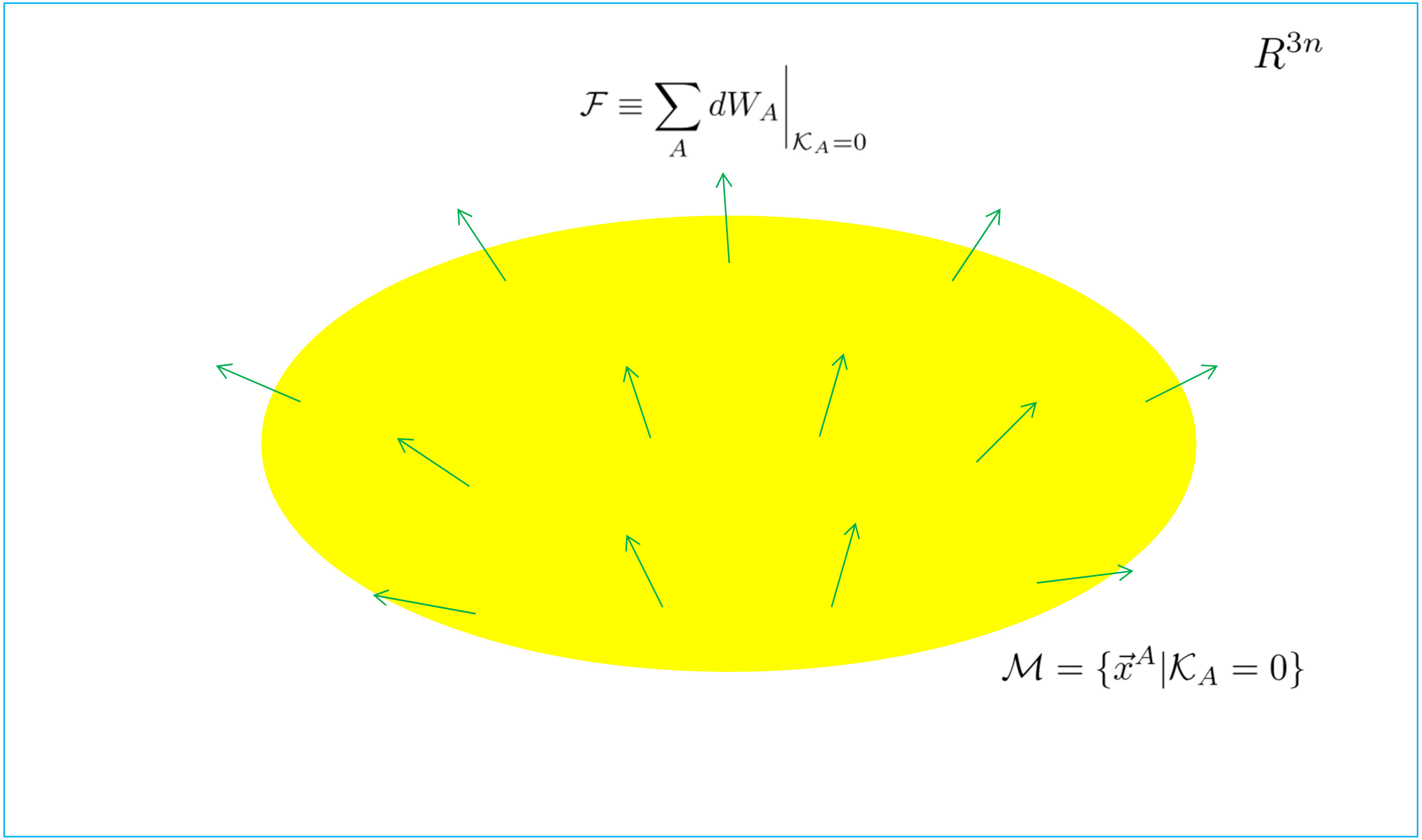
3n bosons + 4n fermions → 2(n-1) bosons + 2(n-1) fermions

$$\mathcal{L}_{deformed}^{for\ index\ only}\Big|_{\xi\to\infty} \quad \to \quad \mathcal{L}_{index}$$

$$\mathcal{L}_{index} \simeq \frac{1}{2} g_{\mu\nu}\dot{z}^{\mu}\dot{z}^{\nu} - \dot{x}^{\mu}\cdot\mathcal{A}_{\mu} + \frac{i}{2} g_{\mu\nu}\psi^{\mu}\left(\dot{\psi}^{\nu} + \dot{z}^{\alpha}\Gamma^{\nu}_{\alpha\beta}\psi^{\beta}\right) + i\mathcal{F}_{\mu\nu}\psi^{\mu}\psi^{\mu}$$

$$\mathcal{F} \equiv \sum_{A} dW_A\Big|_{\mathcal{K}_A=0} = d\mathcal{A}$$

3n vector dynamics → 3 + 2(n-1) nonlinear sigma model



index theorem

Kim+Park+P.Y.+Wang 2011

$$I_n(\{\gamma_A\}) = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right] = \mathrm{tr}\left[(-1)^F e^{-\beta Q^2}\right]$$

$$= \int_{\mathcal{M}=\{\vec{x}_A \mid \mathcal{K}_A=0\}} ch(\mathcal{F}) \wedge \boxed{\mathbf{A}(\mathcal{M})}$$

trivial for a complete intersection in flat ambient space

$$= \int_{\mathcal{M}=\{\vec{x}_A \mid \mathcal{K}_A=0\}} ch(\mathcal{F})$$

$$= \frac{1}{(2\pi)^{n-1}(n-1)!} \int_{\mathcal{M}_n} \mathcal{F}^{n-1}$$

$$\mathcal{F} \equiv \sum_{A>B} \frac{\langle \gamma_A, \gamma_B \rangle}{2} \mathcal{F}^{4\pi}_{Dirac}(\vec{x}_A - \vec{x}_B)\Big|_{\mathcal{K}_A=0}$$

symmetry before & after deformation

$$x^{a=1,2,3} \quad \leftarrow \quad SU(2)_J$$

$$\psi^{\mu=1,2,3,4} \quad \leftarrow \quad SO(4) = SU(2)_J \times SU(2)_I$$

$$\mathcal{M} = \{\vec{x}^A | \mathcal{K}_A = 0\} \qquad\qquad \mathcal{J} = J + I$$

$$\theta, \phi \quad \leftarrow \quad SO(3)_{\mathcal{J}}$$

$$\psi^{\theta,\phi} \quad \leftarrow \quad SO(3)_{\mathcal{J}}$$

connecting to the 2nd helicity trace / protected spin character

Kim+Park+P.Y.+Wang 2011

$$\Omega = -\frac{1}{2}\mathrm{tr}\left[(-1)^{2J_3}(2J_3)^2 y^{2(J_3+I_3)}\right]$$

$$H = H_{\text{center of mass}} \otimes H_{\text{reduced}}$$

$$\Omega = \mathrm{tr}_{H_{\text{reduced}}}\left[(-1)^{2L_3+2(S_3-I_3)}(-1)^{2I_3}y^{2(J_3+I_3)}\right]$$

deformation

$$\Omega = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\mathrm{tr}((-1)^F y^{2\mathcal{J}_3})$$

protected
spin character

equivariant
index

2nd helicity trace for BPS bound states from real space dynamics of dyons

Kim+Park+P.Y.+Wang 2011

$$\Omega = \Omega(y=1) = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\mathrm{tr}((-1)^F)$$

$$= (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1} I_n(\{\gamma_A\})\prod_A \Omega(\gamma_A)$$

$$I_n(\{\gamma_A\}) = \int_{\mathcal{M}=\{\vec{x}_A \mid \mathcal{K}_A=0\}} ch(\mathcal{F}) \;=\; \frac{1}{(2\pi)^{n-1}(n-1)!}\int_{\mathcal{M}_n}\mathcal{F}^{n-1}$$

$$\mathcal{F} \equiv \sum_{A>B}\frac{\langle\gamma_A,\gamma_B\rangle}{2}\,\mathcal{F}^{4\pi}_{Dirac}(\vec{x}_A-\vec{x}_B)\Big|_{\mathcal{K}_A=0}$$

2nd helicity trace for BPS bound states from real space dynamics of distinguishable dyons

Kim+Park+P.Y.+Wang 2011

$$\Omega = \Omega(y=1) = (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1}\mathrm{tr}((-1)^F)$$

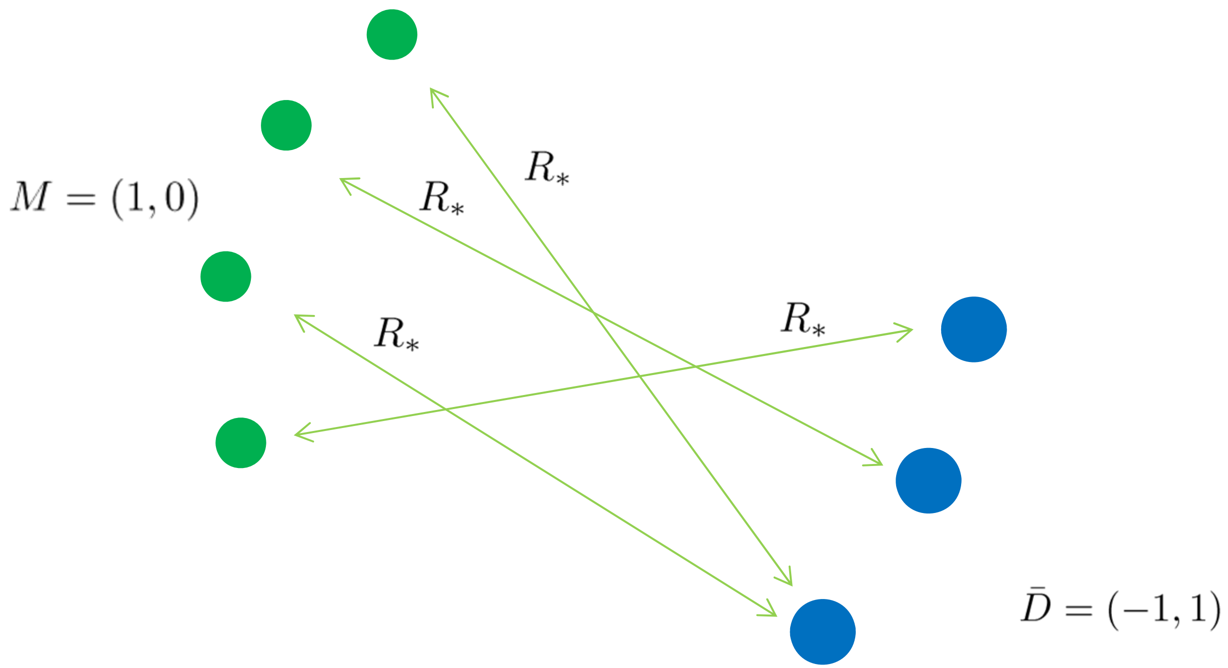
$$= (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\rangle+n-1} I_n(\{\gamma_A\})\prod_A \Omega(\gamma_A)$$

$$I_n(\{\gamma_A\}) = \int_{\mathcal{M}=\{\vec{x}_A \mid \mathcal{K}_A=0\}} ch(\mathcal{F}) = \frac{1}{(2\pi)^{n-1}(n-1)!}\int_{\mathcal{M}_n} \mathcal{F}^{n-1}$$

$$\mathcal{F} \equiv \sum_{A>B} \frac{\langle\gamma_A,\gamma_B\rangle}{2}\, \mathcal{F}^{4\pi}_{Dirac}(\vec{x}_A-\vec{x}_B)\Big|_{\mathcal{K}_A=0}$$

but Bose/Fermi statistics from identical constituent particles is essential, for example, to solve SU(2)→ U(1) problem this way

$$D_n = (n+1)M + n\bar{D}$$



incorporating Bose/Fermi statistics

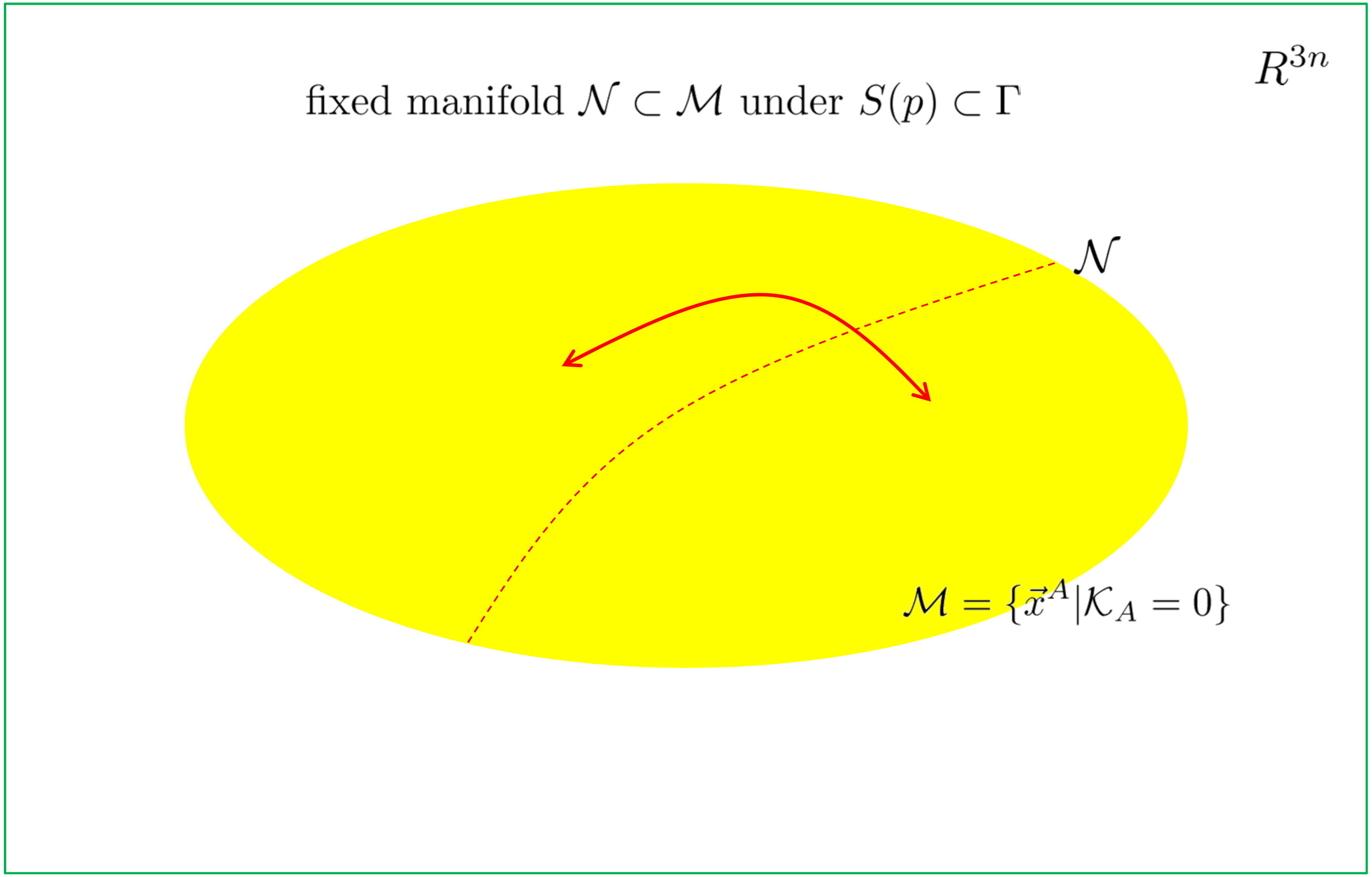
$$\mathrm{tr}\left[(-1)^F e^{-\beta Q^2}\mathcal{P}\right]$$

↓

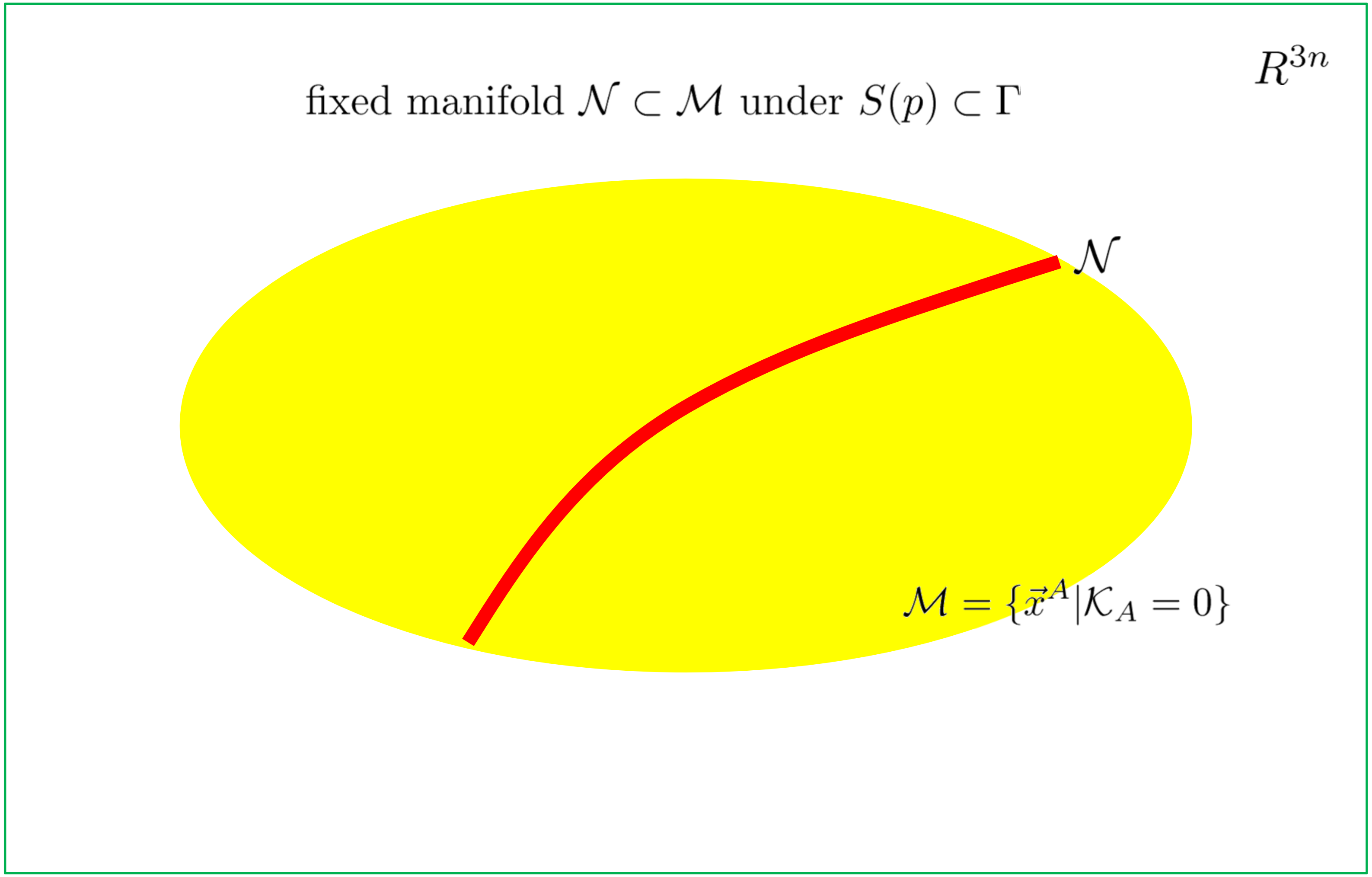
$$\mathcal{P} = \frac{1}{|\Gamma|}\sum_{\sigma\in\Gamma}(\pm 1)^{|\sigma|}\sigma$$

free bulk divided by the permutation group + sum over fixed submanifolds

incorporating Bose/Fermi statistics



incorporating Bose/Fermi statistics



incorporating Bose/Fermi statistics

$$\text{fixed manifold } \mathcal{N} \subset \mathcal{M} \text{ under } S(p) \subset \Gamma$$

$$\Gamma' = \Gamma/S(p) \qquad \mathcal{P}' = \sum_{\sigma \in \Gamma'} (\pm 1)^{|\sigma|} \sigma$$

$$\mathrm{tr}\,(-1)^F e^{-\beta H} \mathcal{P}$$

$$= \mathrm{tr}_{\mathcal{M}/\Gamma - \mathcal{N}}\,(-1)^F e^{-\beta H} \mathcal{P} + \boxed{\Delta_{\mathcal{N}}} \mathrm{tr}_{\mathcal{N}/\Gamma' - \mathcal{N}'}\,(-1)^F e^{-\beta H} \mathcal{P}' + \cdots$$

incorporating Bose/Fermi statistics

fixed manifold $\mathcal{N} \subset \mathcal{M}$ under $S(p) \subset \Gamma$

$$\Delta_{\mathcal{N}}^{(\pm)} = \mathrm{tr}_{\mathcal{N}^\perp} \left[(-1)^{F^\perp} e^{-\beta H^\perp} \mathcal{P}_{S(p)}^{(\pm)} \right]$$

= a multiplicative contribution from
the normal bundle over the fixed submanifold

= a universal numerical factor associated with
separating identical particles from each other

incorporating Bose/Fermi statistics

fixed manifold $\mathcal{N} \subset \mathcal{M}$ under $S(p) \subset \Gamma$

$$\Delta_{\mathcal{N}}^{(\pm)} = \mathrm{tr}_{\mathcal{N}^\perp} \left[(-1)^{F^\perp} e^{-\beta H^\perp} \mathcal{P}_{S(p)}^{(\pm)} \right]$$

$$= \lim_{\beta \to 0} \frac{1}{p!} \sum_{\sigma \in S(p)} \mathrm{tr}_{R^{2(p-1)};n_f=2(p-1)} \left[(-1)^{F^\perp} e^{-\beta H^\perp} \times (\pm 1)^{|\sigma|} \sigma \right]$$

$$= \lim_{\beta \to 0} \frac{1}{p!} \sum_{\sigma \in S(p)} \mathrm{tr}_{R^{2(p-1)};n_f=2(p-1)} \left[(-1)^{F^\perp} e^{\beta \partial^2/2} \times (\pm 1)^{|\sigma|} \sigma \right]$$

incorporating Bose/Fermi statistics for a pair

$$\mathcal{P}_2^{(\pm)} \; : \; x \to -x, \; \psi \to -\psi$$

$$\left. \frac{\Delta_{\mathcal{N}}^{(\pm)}}{|\Omega|} \right|_{p=2} \leftarrow \lim_{\beta \to 0} \mathrm{tr}_{R^d;n_f} \left[(-1)^{F^{\perp}} e^{\beta \partial^2/2} \mathcal{P}_2^{(\pm)} \right] / |\Omega|$$

P.Y. 1997

$$= \lim_{\beta \to 0} \int_{R^d} d^d x \, \langle -x | e^{\beta \partial^2/2} | x \rangle \times (\pm 2^{n_{fermion}/2-1})$$

$$= \lim_{\beta \to 0} \frac{\pm 2^{n_{fermion}/2-1}}{(2\pi\beta)^{d/2}} \int_{R^d} d^d x \; e^{-(x+x)^2/2\beta}$$

$$= \lim_{\beta \to 0} \frac{\pm 2^{n_{fermion}/2-1}}{2^d}$$

$$\to \quad \frac{\pm 1}{2^2}$$

n_f	=	2	4	8	16
d	=	2	3	5	9

incorporating Bose/Fermi statistics

$$\text{fixed manifold } \mathcal{N} \subset \mathcal{M} \text{ under } S(p) \subset \Gamma$$

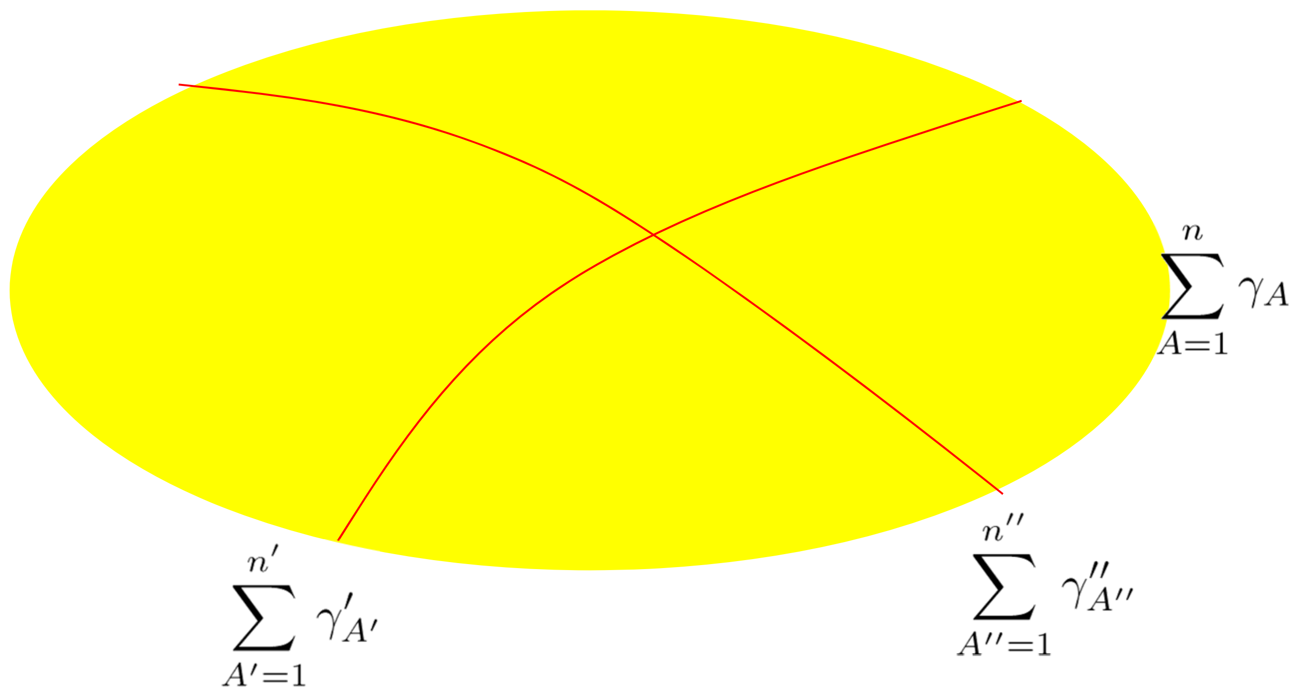
$$\Delta_{\mathcal{N}}^{(\pm)} = \mathrm{tr}_{\mathcal{N}^\perp} \left[(-1)^{F^\perp} e^{-\beta H^\perp} \mathcal{P}_{S(p)}^{(\pm)} \right] \quad = \quad \frac{\Omega}{p^2}$$

P.Y. 1997 / Green+Gutperle 1997

/ Kim+Park+P.Y.+Wang 2011

incorporating Bose/Fermi statistics

$$\sum_{A=1}^{n} \gamma_A = \cdots = \sum_{A'=1}^{n'} \gamma'_{A'} = \cdots = \sum_{A''=1}^{n''} \gamma''_{A''} = \cdots \qquad R^{3n}$$



→ universal wall-crossing formulae from real space dynamics

$$\bar{\Omega}(\gamma) = \sum_{p|\gamma} \Omega(\gamma/p)/p^2$$

$$\sum_{A=1}^{n} \gamma_A = \cdots = \sum_{A'=1}^{n'} \gamma'_{A'} = \cdots = \sum_{A''=1}^{n''} \gamma''_{A''} = \cdots$$

$$\begin{aligned}
\Omega^-\left(\sum \gamma_A\right) - \Omega^+\left(\sum \gamma_A\right) \quad &= (-1)^{\sum_{A>B}\langle\gamma_A,\gamma_B\}+n-1} \frac{\prod_A \bar{\Omega}^+(\gamma_A)}{|\Gamma|} \int_{\mathcal{M}} ch(\mathcal{F}) \\
&\vdots \\
&+(-1)^{\sum_{A'>B'}\langle\gamma'_{A'},\gamma'_{B'}\}+n'-1} \frac{\prod_{A'} \bar{\Omega}^+(\gamma'_{A'})}{|\Gamma'|} \int_{\mathcal{M}'} ch(\mathcal{F}') \\
&\vdots \\
&+(-1)^{\sum_{A''>B''}\langle\gamma''_{A''},\gamma''_{B''}\}+n''-1} \frac{\prod_{A''} \bar{\Omega}^+(\gamma''_{A''})}{|\Gamma''|} \int_{\mathcal{M}''} ch(\mathcal{F}'') \\
&\vdots
\end{aligned}$$

summary

- 3n-D real space dynamics for n Seiberg-Witten BPS dyons:
  arbitrarily accurate near marginal stability wall

- 3n → 2(n-1) : index-preserving but unphysical

- 2^{nd} helicity trace (protected spin character) ← (equivariant) index

- rational invariants from permutation orbifolding

- "proves" Kontsevich-Soibelman (via Sen, 2011) as a physical statement

wall-crossing is by now a mature problem
with diverse technologies and viewpoints
that emerged over the last decade and half,

yet many aspects remain to be clarified and explored