Bound States in the Mirror TBA

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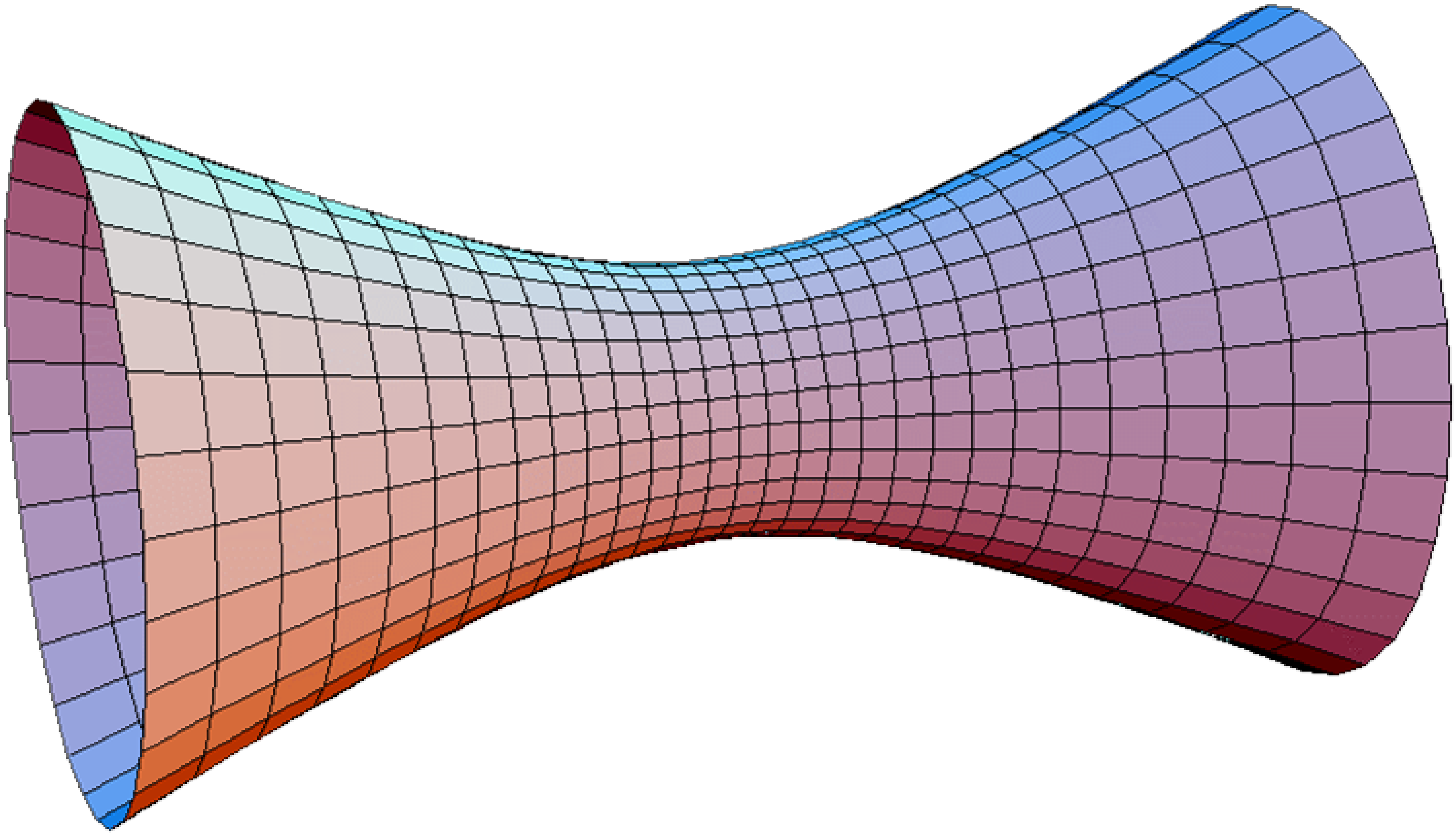
Outline

- *Ground state TBA*
- *Excited states TBA*
- *Implementation of symmetries* **Frolov & G.A `11**
- *TBA for bound states* **Frolov, van Tongeren & G.A `11**

Ground state TBA

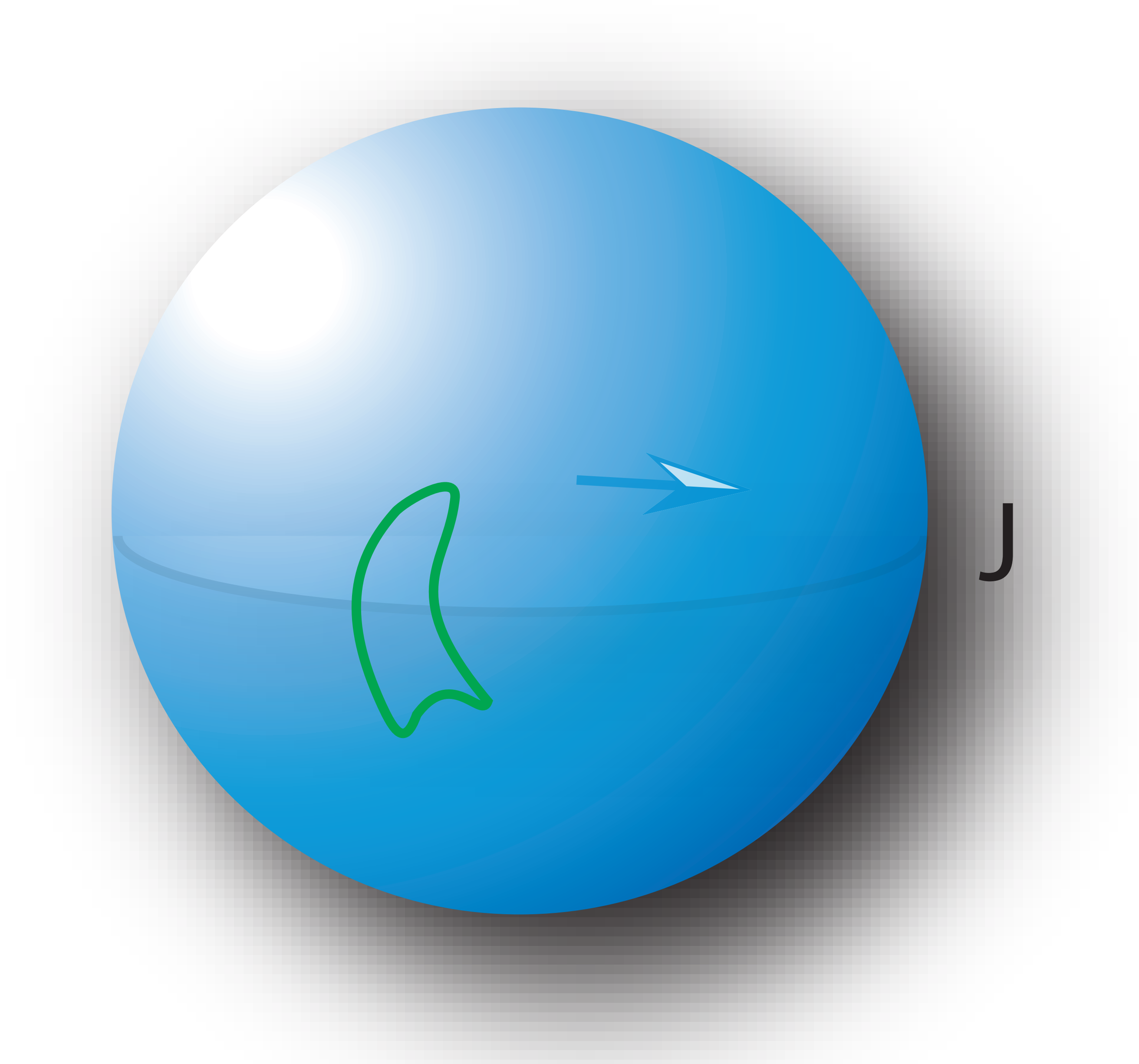
String Energy E



$$X_0^2 + X_5^2 - X_1^2 - X_2^2 - X_3^2 - X_4^2 = R^2, \quad R = 1$$

String energy E is a conserved Noether charge corresponding to the $\mathrm{SO}(2)$ subgroup of the isometry (conformal) group $\mathrm{SO}(4,2)$

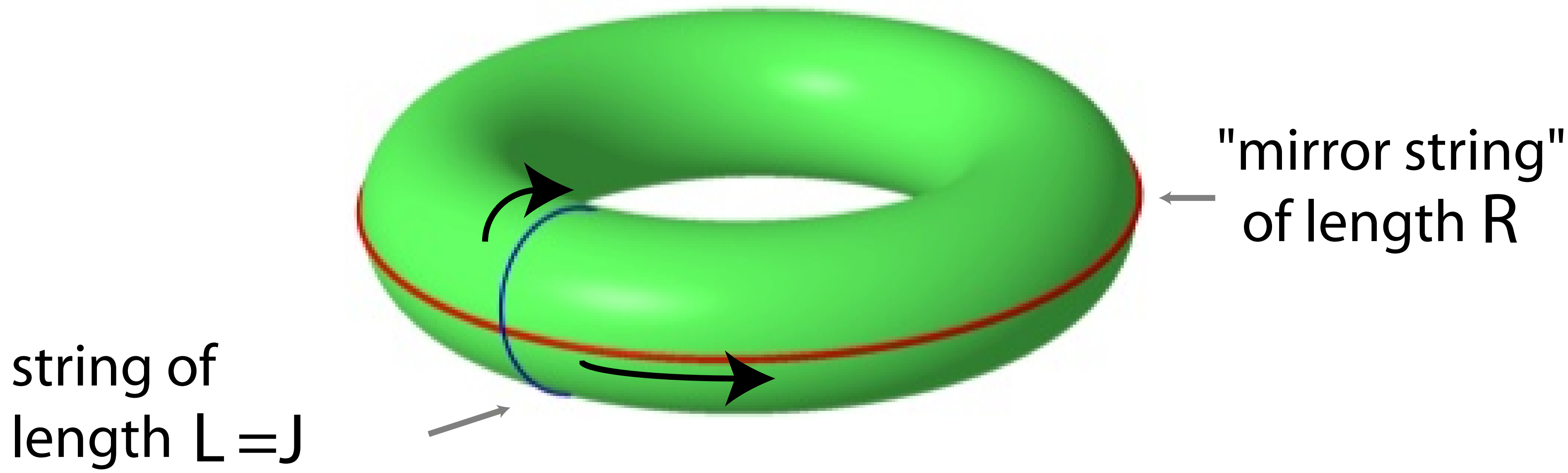
Charge J



J is a conserved Noether charge corresponding to one of the $\mathrm{SO}(2)$ subgroups of the isometry group $\mathrm{SO}(6)$

Mirror Theory

Frolov and G.A. '07



- One Euclidean theory – two Minkowski theories. One is related to the other by the double Wick rotation:

$$\tilde{\sigma} = -i\tau\,, \qquad \tilde{\tau} = i\sigma$$

The Hamiltonian $\tilde{\mathrm{H}}$ w.r.t. $\tilde{\tau}$ defines the *mirror theory*.

Ground state energy is related to the free energy of its mirror

$$E(L) = \lim_{R\to\infty} \frac{L}{R} F(L) = L\mathcal{F}$$

- J – momentum carried by string along the equator of S^5,
  L – "length" (will be related to J)
- p – momentum of a string particle
- $\mathcal{E}$ – energy of a string particle: $\mathcal{E} = \sqrt{1 + 4g^2 \sin^2 \frac{p}{2}}$
- $\tilde{p}$ – momentum of a mirror particle
- $\tilde{\mathcal{E}}$ – energy of a mirror particle: $\tilde{\mathcal{E}} = 2\,\mathrm{arcsinh}\left(\frac{1}{2g}\sqrt{1+\tilde{p}^2}\right)$
- String S-matrix $S(p_1, p_2)$
- Mirror S-matrix $\tilde{S}(\tilde{p}_1, \tilde{p}_2)$

Bethe-Yang equations for the mirror model

Mirror Bethe-Yang equations for fundamental particles ($\alpha = 1, 2$)

Length of the mirror circle

$$1 = e^{i\widetilde{p}_k R} \prod_{\substack{l=1\\ l\neq k}}^{K^{\rm I}} S(\tilde{p}_k, \tilde{p}_l) \prod_{\alpha=1}^{2} \prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{x_k^- - y_l^{(\alpha)}}{x_k^+ - y_l^{(\alpha)}} \sqrt{\frac{x_k^+}{x_k^-}}$$

$$-1 = \prod_{l=1}^{K^{\rm I}} \frac{y_k^{(\alpha)} - x_l^-}{y_k^{(\alpha)} - x_l^+} \sqrt{\frac{x_l^+}{x_l^-}} \prod_{l=1}^{K^{\rm III}_{(\alpha)}} \frac{v_k^{(\alpha)} - w_l^{(\alpha)} - \frac{i}{g}}{v_k^{(\alpha)} - w_l^{(\alpha)} + \frac{i}{g}}$$

$$1 = \prod_{l=1}^{K^{\rm II}_{(\alpha)}} \frac{w_k^{(\alpha)} - v_l^{(\alpha)} + \frac{i}{g}}{w_k^{(\alpha)} - v_l^{(\alpha)} - \frac{i}{g}} \prod_{\substack{l=1\\ l\neq k}}^{K^{\rm III}_{(\alpha)}} \frac{w_k^{(\alpha)} - w_l^{(\alpha)} - \frac{2i}{g}}{w_k^{(\alpha)} - w_l^{(\alpha)} + \frac{2i}{g}}$$

follow from $\tilde{S}(\tilde{p}_1, \tilde{p}_2)$

Auxiliary roots – w^α, y^α; $\quad v = y + 1/y$

$(K^{\rm III}_-, K^{\rm II}_-, K^{\rm I}, K^{\rm II}_+, K^{\rm III}_+)$ $\qquad psu(2|2) \oplus psu(2|2)$

The spectrum of TBA particles

String hypothesis suggests the existence of
nine types of TBA vacuum particles ($\alpha = 1, 2$):

- Q-particles (Q-particle bound states) carrying momentum $\tilde{p}_Q$ $\Longrightarrow$ $Y_Q^{(\alpha)}(u)$
- $y^{\pm(\alpha)}$-particles corresponding to fermionic Bethe roots $\Longrightarrow$ $Y_\pm^{(\alpha)}(u)$, $|u| < 2$
- $M|vw^{(\alpha)}$-strings $\Longrightarrow$ $Y_{M|vw}^{(\alpha)}(u)$
- $M|w^{(\alpha)}$-strings $\Longrightarrow$ $Y_{M|w}^{(\alpha)}(u)$

$$\widetilde{p}^Q(u) = g\, x(u - \frac{iQ}{g}) - g\, x(u + \frac{iQ}{g}) + iQ\,,$$

$$\widetilde{\mathcal{E}}^Q(u) = \log \frac{x(u - \frac{iQ}{g})}{x(u + \frac{iQ}{g})} = 2\,\mathrm{arcsinh}\Big(\frac{1}{2g}\sqrt{Q^2 + \widetilde{p}^2}\Big)$$

Simplified TBA equations for the ground state

- $M|w$-strings: $$\log Y^{(\alpha)}_{M|w} = \log(1+Y^{(\alpha)}_{M-1|w})(1+Y^{(\alpha)}_{M+1|w})\star s + \delta_{M1}\,\log\frac{1-\frac{1}{Y^{(\alpha)}_{-}}}{1-\frac{1}{Y^{(\alpha)}_{+}}}\,\hat{\star}\, s$$

- $M|vw$-strings: $$\log Y^{(\alpha)}_{M|vw} = \log(1+Y^{(\alpha)}_{M-1|vw})(1+Y^{(\alpha)}_{M+1|vw})\star s - \log(1+Y_{M+1})\star s + \delta_{M1}\,\log\frac{1-Y^{(\alpha)}_{-}}{1-Y^{(\alpha)}_{+}}\,\hat{\star}\, s$$

- y-particles $$\log\frac{Y^{(\alpha)}_{+}}{Y^{(\alpha)}_{-}} = \log(1+Y_Q)\star K_{Qy}\,,$$ $$\log Y^{(\alpha)}_{+}Y^{(\alpha)}_{-} = \log\,(1+Y_Q)\star(-K_Q + 2K^{Q1}_{xv}\star s) + 2\log\frac{1+Y_{1|vw}}{1+Y_{1|w}}\star s$$

- Q-particles for $Q\geq 2$ $$\log Y_Q = \log\frac{\left(1+\frac{1}{Y^{(1)}_{Q-1|vw}}\right)\left(1+\frac{1}{Y^{(2)}_{Q-1|vw}}\right)}{\left(1+\frac{1}{Y_{Q-1}}\right)\left(1+\frac{1}{Y_{Q+1}}\right)}\star s$$

Most complicated piece -- it depends on TBA length L and on the dressing phase

- $Q=1$-particle $$\log Y_1 = \log\frac{\left(1-\frac{1}{Y^{(1)}_{-}}\right)\left(1-\frac{1}{Y^{(2)}_{-}}\right)}{1+\frac{1}{Y_2}}\star s - \Delta(L)\,\check{\star}\, s\,,\qquad s(u)=\frac{g}{4\cosh\frac{g\pi u}{2}}$$

$$E(L) = -\frac{1}{2\pi}\sum_{Q=1}^{\infty}\int_{-\infty}^{\infty}\mathrm{d}u\,\frac{d\tilde{p}^Q}{du}\log(1+Y_Q)$$

Frolov & G.A. `09

Excited states TBA

Large L solution of TBA

$L \to \infty$: Bethe-Yang (all $1/L$ powers) + Lüscher corrections (leading e^{-mL} corrections)

(standing 1-particle states) Lüscher '86

(general N-particle states) Bajnok, Janik '08

$$Y_Q^o(v) \quad = \quad \Upsilon_Q(v)\, T_{Q,-1}(v)\, T_{Q,1}(v)$$

- Transfer matrix

$$T_{Q,1}(u) = \mathrm{Tr}_Q\Big[S_{Q,1}(u,u_1)\dots S_{Q,N}(u,u_N)\Big]$$

- The prefactor

$$\Upsilon_Q^+ \Upsilon_Q^- = \Upsilon_{Q-1}\Upsilon_{Q+1}\,, \qquad \Upsilon_Q(v) \sim e^{-J\widetilde{\mathcal{E}}_Q(v)}$$

- Bethe-Yang equations are equivalent to

$$Y_{1_*}^o(u_k) = -1\,, \qquad k = 1,\dots,N$$

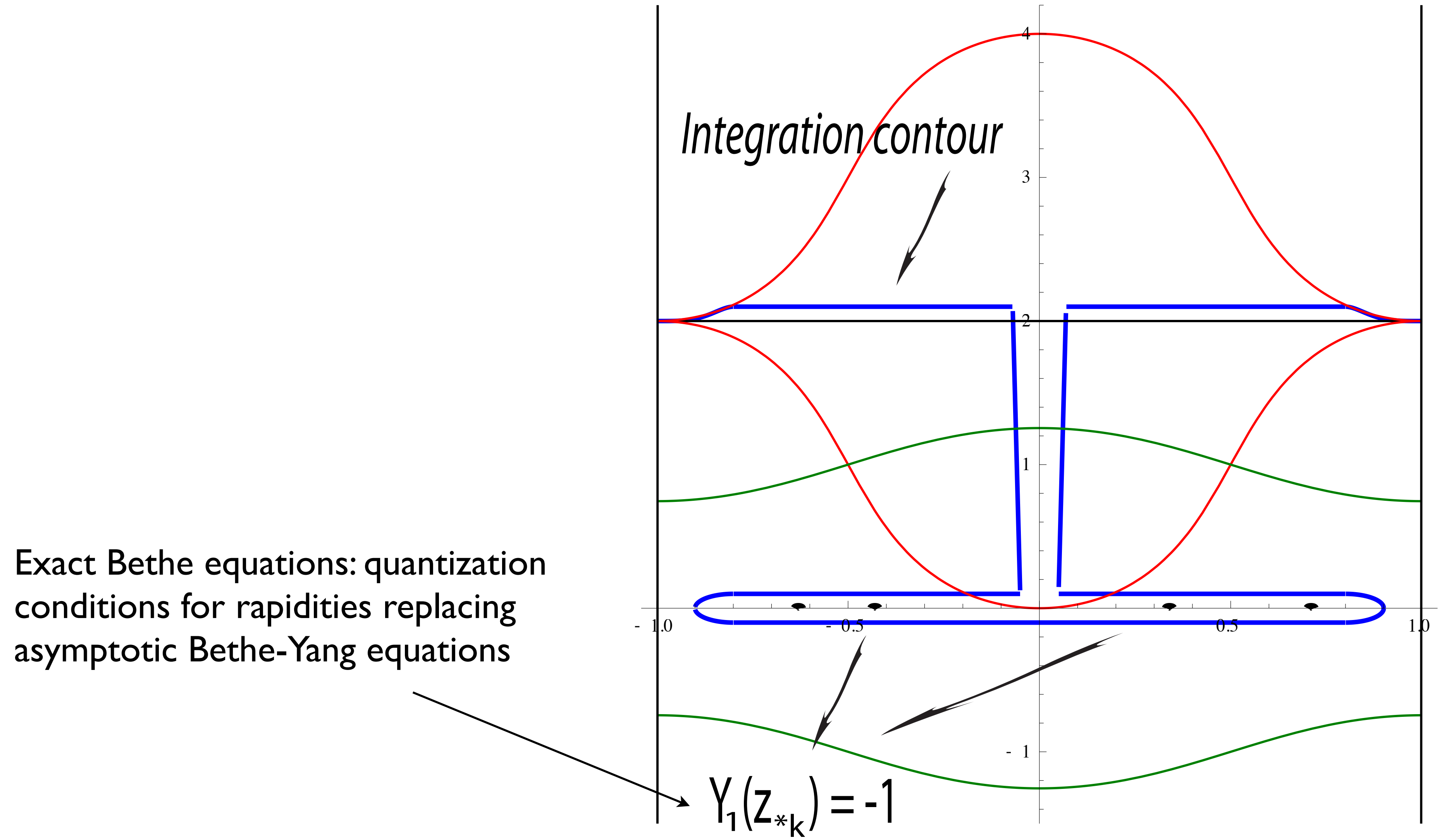
- All other Y-functions $Y_\pm^o$, $Y_{M|vw}^o$, $Y_{M|w}^o$ are found from the TBA and Y_Q^o

Relation between the TBA length L and the charge J

The asymptotic TBA equation for Y_1 is satisfied by the asymptotic solution provided

$$L=J+2$$

Contour deformation trick



- **TBA's for excited states differ only by a choice of the integration contour**

- Taking the contour back to the real mirror line produces extra contributions $-\log S(z_*, z)$ from $\log(1+Y_1) \star K$, where $K(w,z) = \frac{1}{2\pi i}\frac{d}{dw}\log S(w,z)$

General strategy

1. Solve the BY equations for a fixed set of integers

$$J\,, \qquad N = K^{\rm I}\,, \qquad (K_-^{\rm III},\, K_-^{\rm II},\, K_+^{\rm II},\, K_+^{\rm III})$$

   Pick up a solution. It is characterized by a definite set of g-dependent momenta.

   Auxiliary roots are completely fixed by the momenta p_k and play no independent role in the description of the state

2. Compute asymptotic Y-functions and find zeroes and poles of $1 + Y$ and Y

3. Choose contours and engineer TBA equations for the state so that the asymptotic TBA equations obtained by dropping terms with $\log(1 + Y_Q)$ are solved by the asymptotic Y-functions

4. Exact momenta p_k are found from the *exact Bethe equations* (quantization cond.)

$$Y_{1_*}^o(p_k) = -1 \qquad \Longrightarrow \qquad Y_{1_*}(p_k) = -1$$

   derived by analytically continuing the excited state TBA equation for Y_1

General facts about symmetries

- $\mathrm{PSU}(2,2|4)$ acts on asymptotic solutions provided the level-matching is satisfied
- States can have roots $\nu^{(\alpha)} = y^{(\alpha)} + \frac{1}{y^{(\alpha)}}$ and $w^{(\alpha)}$ located at infinity
- Dynkin labels of a state are related to excitation numbers

$$\begin{aligned} q_1 &= K_-^{\mathrm{II}} - 2K_-^{\mathrm{III}} \\ p &= J - \tfrac{1}{2}(K_-^{\mathrm{II}} + K_+^{\mathrm{II}}) + K_-^{\mathrm{III}} + K_+^{\mathrm{III}} \\ q_2 &= K_+^{\mathrm{II}} - 2K_+^{\mathrm{III}} \end{aligned} \qquad \begin{aligned} s_1 &= K^{\mathrm{I}} - K_-^{\mathrm{II}} \\ s_2 &= K^{\mathrm{I}} - K_+^{\mathrm{II}} \end{aligned}$$

- States in the same multiplet must have the same anomalous dimension and canonical ones which might differ by a (half-)integer

$$E = J + \sum_{k=1}^{K^{\mathrm{I}}} \sqrt{1 + 4g^2 \sin^2 \frac{p_k}{2}}\,.$$

Putting $g = 0$ gives the canonical dimension $E = J + K^{\mathrm{I}}$. Adding particles with zero momentum $p = 0$ ($u = \infty$) changes canonical dimension.

- Superconformal primary has the lowest canonical dimension in the multiplet!

Treatment of symmetry on the asymptotic solution

- Every multiplet has a *unique regular* representative $(\vec{u}, \vec{\nu}^{(\alpha)}, \vec{w}^{(\alpha)})$ among the solutions of the Bethe ansatz equations
- All other states in a multiplet are created by adding *irregular* roots
- BY equations for all members of a multiplet must be the same

$$1 = e^{iJp_k} \prod_{l \neq k}^{K^{\mathrm{I}}} S_{\mathfrak{sl}(2)}(u_k, u_l) \prod_{l=1}^{K_-^{\mathrm{II}}} \frac{x_k^- - y_l^{(-)}}{x_k^+ - y_l^{(-)}} \sqrt{\frac{x_k^+}{x_k^-}} \prod_{l=1}^{K_+^{\mathrm{II}}} \frac{x_k^- - y_l^{(+)}}{x_k^+ - y_l^{(+)}} \sqrt{\frac{x_k^+}{x_k^-}}$$

1) Adding root with $p = 0$ does not modify BY equations, as $x^+/x^- = 1$;

2) Adding root with $y = 0$ requires a shift $J \to J + \frac{1}{2}$;

3) Adding root with $y = \infty$ requires a shift $J \to J - \frac{1}{2}$;

4) Adding irregular roots y or w does not influence auxiliary Bethe equations

Susy generators in the light-cone gauge

Susy generators of the light-cone string are divided into two groups

1. *Kinematical generators* : independent of x_-, but depend on $x^+ = \tau$
2. *Dynamical generators* : depend on x_-, but independent of $x^+ = \tau$

Since

$$\frac{dQ}{d\tau} = \frac{\partial Q}{\partial \tau} + \{\mathrm{H}, Q\}$$

dynamical generators commute with $\mathrm{H} = E - J$, while kinematical generators do not

Kinematical Poincare supercharges

Charge	Weights	ΔK^{II}_{-}	ΔK^{II}_{+}	$\Delta K^{\text{III}}_{-}$	$\Delta K^{\text{III}}_{+}$
Q^3_α	$[0,-1,1]_{(\pm\frac{1}{2},0)}$	$0_{+\frac{1}{2}}, 2_{-\frac{1}{2}}$	$1_{+\frac{1}{2}}, 1_{-\frac{1}{2}}$	$0_{+\frac{1}{2}}, 1_{-\frac{1}{2}}$	$0_{\pm\frac{1}{2}}$
Q^4_α	$[0,0,-1]_{(\pm\frac{1}{2},0)}$	$0_{+\frac{1}{2}}, 2_{-\frac{1}{2}}$	$1_{\pm\frac{1}{2}}$	$0_{+\frac{1}{2}}, 1_{-\frac{1}{2}}$	$1_{\pm\frac{1}{2}}$
$\bar{Q}_{1\dot{a}}$	$[-1,0,0]_{(0,\pm\frac{1}{2})}$	$1_{\pm\frac{1}{2}}$	$0_{+\frac{1}{2}}, 2_{-\frac{1}{2}}$	$1_{\pm\frac{1}{2}}$	$0_{+\frac{1}{2}}, 1_{-\frac{1}{2}}$
$\bar{Q}_{2\dot{a}}$	$[1,-1,0]_{(0,\pm\frac{1}{2})}$	$1_{+\frac{1}{2}}, 1_{-\frac{1}{2}}$	$0_{+\frac{1}{2}}, 2_{-\frac{1}{2}}$	$0_{\pm\frac{1}{2}}$	$0_{+\frac{1}{2}}, 1_{-\frac{1}{2}}$

- Decrease J by $-1/2$ and increase K^{I} by 1, never decrease K^{II}_α and K^{III}_α
- Action with a supercharge adds either three or one irregular y-roots
- A single y-root is at ∞, from three y-roots two at ∞ and one at 0

Dynamical Poincare supercharges

Charge	Weights	$\Delta K^{\mathrm{II}}_{-}$	$\Delta K^{\mathrm{II}}_{+}$	$\Delta K^{\mathrm{III}}_{-}$	$\Delta K^{\mathrm{III}}_{+}$
Q^1_α	$[1,0,0]_{(\pm\frac{1}{2},0)}$	$-1_{+\frac{1}{2}}, 1_{-\frac{1}{2}}$	$0_{\pm\frac{1}{2}}$	$-1_{+\frac{1}{2}}, 0_{-\frac{1}{2}}$	$0_{\pm\frac{1}{2}}$
Q^2_α	$[-1,1,0]_{(\pm\frac{1}{2},0)}$	$-1_{+\frac{1}{2}}, 1_{-\frac{1}{2}}$	$0_{\pm\frac{1}{2}}$	$0_{+\frac{1}{2}}, 1_{-\frac{1}{2}}$	$0_{\pm\frac{1}{2}}$
$\bar{Q}_{3\dot{a}}$	$[0,1,-1]_{(0,\pm\frac{1}{2})}$	$0_{\pm\frac{1}{2}}$	$-1_{+\frac{1}{2}}, 1_{-\frac{1}{2}}$	$0_{\pm\frac{1}{2}}$	$0_{+\frac{1}{2}}, 1_{-\frac{1}{2}}$
$\bar{Q}_{4\dot{a}}$	$[0,0,1]_{(0,\pm\frac{1}{2})}$	$0_{\pm\frac{1}{2}}$	$-1_{+\frac{1}{2}}, 1_{-\frac{1}{2}}$	$0_{\pm\frac{1}{2}}$	$-1_{+\frac{1}{2}}, 0_{-\frac{1}{2}}$

- J and E are increased by 1/2, K^{I} is unchanged
- Four charges (*red*) lower K^{II} by 1! These will decrease a number of (irregular) roots when acting on a superconformal primary to produce the regular state

$$E_{hws} = E_{reg} - 2\,, \qquad J_{hws} = J_{reg} - 2\,.$$

  Four y-roots at ∞!

- Relation between excitation numbers

$$K^{\mathrm{I}}_{reg} = K^{\mathrm{I}}_{hws}\,, \qquad K^{\mathrm{II}}_{\alpha,reg} = K^{\mathrm{II}}_{\alpha,hws} - 2\,, \qquad K^{\mathrm{III}}_{\alpha,reg} = K^{\mathrm{III}}_{\alpha,hws} - 1$$

Typical multiplet of 2^{16} states

$$|\mathcal{O}\rangle \;=\; \prod \underbrace{(Q^d_{-\infty})^{n^d_\infty}(Q^d_{+0})^{n^d_0}}_{J\to+1/2}\,\underbrace{(Q^k_{+\infty})^{n^k_\infty}(Q^k_{+2\infty,+0})^{n^k_{\infty,0}}}_{J\to-1/2,\;\; K^{\rm I}\to+1}|\mathrm{hws}\rangle$$

- The hws has four y-roots at ∞

$$E_{hws} = E_{reg} - 2\,, \qquad J_{hws} = J_{reg} - 2$$

- J-charge

$$J = J_{hws} + \frac{1}{2}(n^d_\infty + n^d_0 - n^k_\infty - n^k_{\infty,0})$$

- Energy

$$E = E_{hws} + \frac{1}{2}(n^d_\infty + n^d_0 + n^k_\infty + n^k_{\infty,0})$$

- Number of irregular roots

$$\mathcal{K}^{\rm II}_0 \;= n^d_0 + n^k_{\infty,0}\,, \qquad \mathcal{K}^{\rm II}_\infty = 4 - n^d_\infty + n^k_\infty + 2n^k_{\infty,0}$$

From here

$$J = J_{reg} + \tfrac{1}{2}(\mathcal{K}^{\rm II}_0 - \mathcal{K}^{\rm II}_\infty)$$

psu(2,2|4) symmetry is built in TBA

$$J = J_{reg} + \frac{1}{2}(\mathcal{K}_0^{\rm II} - \mathcal{K}_\infty^{\rm II}) \qquad\qquad e^{-\tilde{\mathcal{E}}_Q} \equiv \Omega$$

Expression for the asymptotic Y_Q is universal for the whole multiplet:

$$Y_Q^o = \Omega^J T_{Q,+1} T_{Q,-1}\,, \qquad Y_Q^{o,reg} = \Omega^{J_{reg}} T_{Q,+1}^{reg} T_{Q,-1}^{reg}$$

But

$$T_{Q,+1} T_{Q,-1} = \Omega^{\frac{1}{2}(\mathcal{K}_\infty^{\rm II} - \mathcal{K}_0^{\rm II})} T_{Q,+1}^{reg} T_{Q,-1}^{reg}$$

Accordingly, the Y_Q-functions of this state can be written as

$$Y_Q^o = \Omega^{J-J_{reg}} \Omega^{\frac{1}{2}(\mathcal{K}_\infty^{\rm II} - \mathcal{K}_0^{\rm II})} Y_Q^{o,reg} = Y_Q^{o,reg}$$

For all states in a multiplet Y_Q^o-functions and, therefore, all Y^o coincide!

$Y's$ are simply invariants of $psu(2,2|4)$

TBA length L

Kinematical charges decrease J, while dynamical increase. Thus,

$$J_{hws} - 4 \leq J \leq J_{hws} + 4 = J_{reg} + 2$$

On the other hand, in the TBA studies we found that

$$L = J_{reg} + 2$$

L coincides with the maximal J-charge in a susy multiplet

Only for the vacuum $L = J_{reg}$, as the corresponding susy multiplet is atypical (short)

Excitations with complex momenta

Frolov, van Tongeren and G.A., ’11

Motivations to study such excitations

– to elucidate new features of the mirror TBA

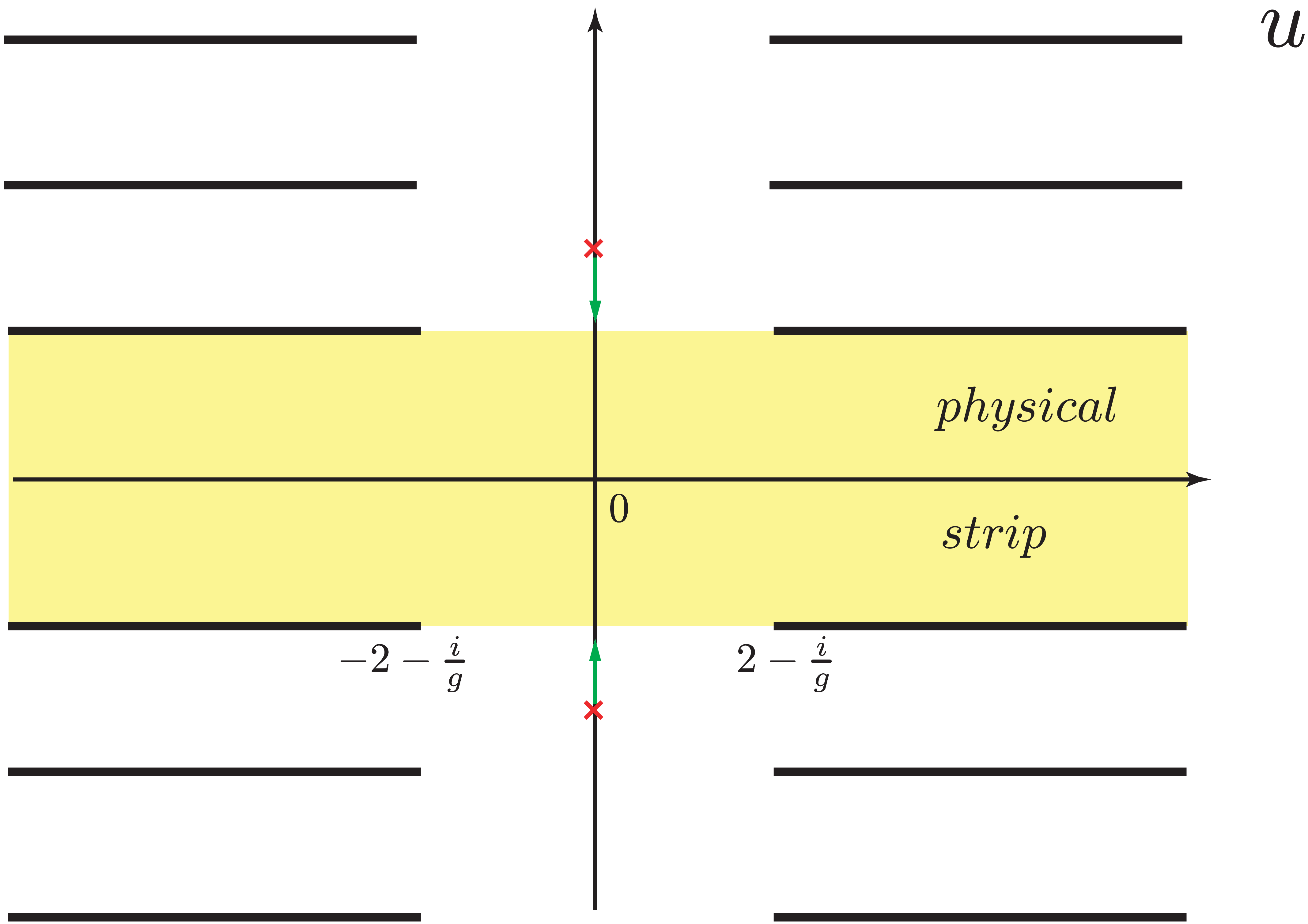
– to test the general strategy of constructing excited states

Feasible approach

- $\mathfrak{su}(2)$ sector contains particles with complex momenta, for M magnons $L = J + M$
- There are many three-particle solutions with

$$(k-1)/g < |\mathrm{Im}(u)| < k/g\,, \qquad k = 2, 3, \ldots$$

- Explicitly consider the state $L = 7$, $M = 3$


u
physical
strip
0
$-2-\frac{i}{g}$
$2-\frac{i}{g}$

Numerical solution of the BY equation for L=7 state

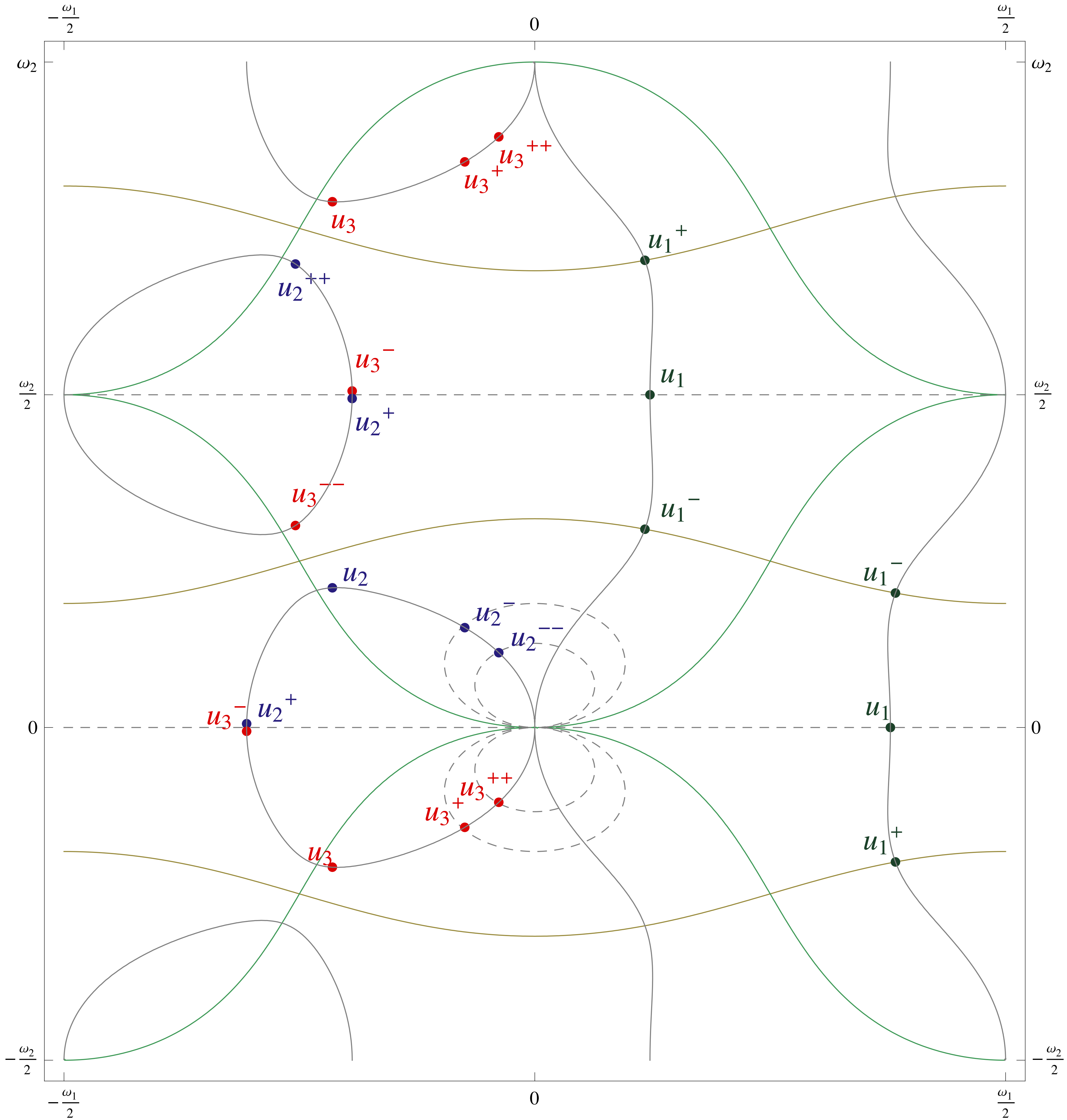
real

$\mathrm{Re}(q) > 0$

$$p_1 = p$$
$$p_2 = -\frac{p}{2} + iq$$
$$p_3 = -\frac{p}{2} - iq$$

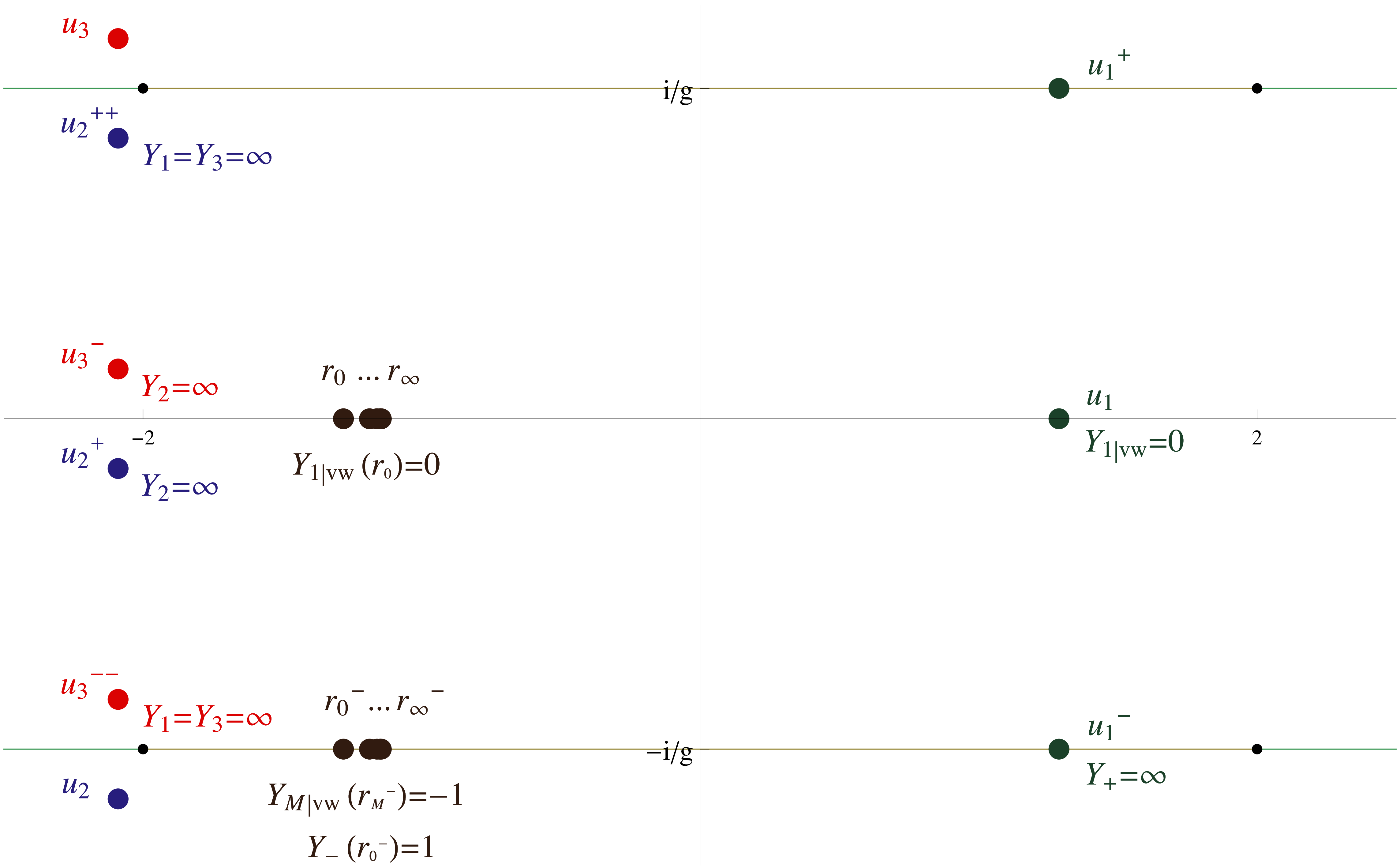
g	p	q	g	p	q
0.	2.3129	0.926075	0.5	2.24919	1.23789
0.1	2.3098	0.933177	0.51	2.24704	1.27083
0.2	2.30088	0.955744	0.52	2.2449	1.31517
0.3	2.28709	0.99838	0.53	2.24302	1.40691
0.4	2.26953	1.0737	0.5301	2.24303	1.41083
			0.5302	$2.2431 - 0.00001i$	$1.41983 - 0.001i$

BAE break down!



$-\frac{\omega_1}{2}$
0
$\frac{\omega_1}{2}$
ω_2
$\frac{\omega_2}{2}$
0
$-\frac{\omega_2}{2}$
u_3^{++}
u_3^{+}
u_3
u_1^{+}
u_2^{++}
u_3^{-}
u_1
u_2^{+}
u_3^{--}
u_1^{-}
u_2
u_2^{-}
u_2^{--}
u_3^{-}
u_2^{+}
u_3^{++}
u_3^{+}
u_3
u_1^{+}

Relevant roots/poles of the asymptotic Y's on the mirror u-plane



Y_Q with $Q \geq 3$ have poles at $u_2 + \frac{i}{g}(Q-1)\,,\ u_3 - \frac{i}{g}(Q-1)$

Analytic structure of the exact solution

The functions Y_1, Y_2 and Y_3 have poles inside the analyticity strip!

- Around a pole

$$Y(u) = \frac{y(u)}{u - u_\infty}$$

- Within the analyticity strip $y(u)$ is small and of the order g^{2L-1}

- There is a point u_{-1} such that $Y(u_{-1}) = -1$ implying

$$u_{-1} - u_\infty + y(u_{-1}) = 0$$

Expanding around u_∞

$$u_{-1} \approx u_\infty - y(u_\infty) = u_\infty - \mathrm{Res}\, Y(u_\infty)$$

We conclude that u_{-1} is close to u_∞!

Very similar to SU(N) models!

Kazakov & Leurent `10
Balog (unpublished)

$$Y_Q = \Upsilon_Q \frac{T_{Q,-1}\, T_{Q,1}}{T_{Q-1,0}\, T_{Q+1,0}}\,, \qquad 1 + Y_Q = \frac{T^+_{Q,0}\, T^-_{Q,0}}{T_{Q-1,0}\, T_{Q+1,0}}$$

- Prefactor Υ_Q has poles at $u_2 + \frac{i}{g}(Q-1)$ and $u_2 + \frac{i}{g}(Q+1)$
- Asymptotically $T_{Q,0} = 1$
- For an exact solution $T_{Q,0} \neq 1$ and

$$T_{Q,0}(u_2 + \frac{i}{g}Q) = \infty\,, \quad T_{Q,0}(u_2^{(Q)} + \frac{i}{g}Q) = 0$$

This implies

$$Y_Q\left(u_2^{(Q\pm 1)} + \frac{i}{g}(Q \pm 1)\right) = \infty$$

$$1 + Y_Q\left(u_2^{(Q)} + \frac{i}{g}(Q \mp 1)\right) = 0$$

- In addition $1 + Y_1$ has zero at real u_1 which is in the string region

- $Q = 1$

$$\log \mathcal{S}_1(u_3^{(2)--}, v) - \log \mathcal{S}_1(u_3^{(1)--}, v) - \log \mathcal{S}_1(u_2^{(1)}, v) - \log \mathcal{S}_{1_*}(u_1, v)$$

- $Q = 2$

$$\begin{aligned} &+ \log \mathcal{S}_2(u_2^{(1)+}, v) + \log \mathcal{S}_2(u_3^{(3)---}, v) \\ &- \log \mathcal{S}_2(u_2^{(2)+}, v) - \log \mathcal{S}_2(u_3^{(2)---}, v) \end{aligned}$$

$\mathcal{S}_Q$ satisfies the discrete Laplace equation

$$\mathcal{S}_{Q-1}(u, v)\mathcal{S}_{Q+1}(u, v) = \mathcal{S}_Q(u^-, v)\mathcal{S}_Q(u^+, v)\,.$$

Then we take a sum over $Q \geq 3$

$$\sum_{Q=3}^{\infty} \log \frac{\mathcal{S}_Q(u_3^{(Q-1)} - \frac{i}{g}(Q-1), v)\mathcal{S}_Q(u_3^{(Q+1)} - \frac{i}{g}(Q+1), v)}{\mathcal{S}_Q(u_3^{(Q)} - \frac{i}{g}(Q-1), v)\mathcal{S}_Q(u_3^{(Q)} - \frac{i}{g}(Q+1), v)} = \log \frac{\mathcal{S}_3(u_3^{(2)--}, v)}{\mathcal{S}_2(u_3^{(3)---}, v)}\,.$$

Adding terms $Q = 1, 2$, one gets the driving terms from $\log(1 + Y_Q) \star_{C_Q} \mathcal{K}_Q$

$$- \log \mathcal{S}_{1_*}(u_1, v) - \log \frac{\mathcal{S}_1(u_2^{(1)}, v)}{\mathcal{S}_1(u_3^{(1)}, v)} + \log \frac{\mathcal{S}_2(u_2^{(1)+}, v)}{\mathcal{S}_2(u_2^{(2)+}, v)} \frac{\mathcal{S}_2(u_3^{(2)-}, v)}{\mathcal{S}_2(u_3^{(1)-}, v)}\,.$$

Compatibility of quantization conditions

$$Y_1(u_3^{(1)}) = -1 \quad \Leftrightarrow \quad Y_1(u_3^{(1)--}) = -1 \quad \Leftrightarrow \quad Y_{1*}(u_3^{(1)}) = -1$$

The exact Bethe equations representing these quantization conditions are compatible in a non-trivial manner which involves crossing symmetry

There are similar quantization conditions involving Y_2

$$Y_2(u_3^{(2)-}) = -1 \quad \Leftrightarrow \quad Y_2(u_3^{(2)---}) = -1$$

Simplified TBA equations for $Y_{M|w}$ *and* $Y_{M|vw}$

$$
\begin{aligned}
\log Y_{M|w} &= \log(1+Y_{M-1|w})(1+Y_{M+1|w})\star s \\
&+ \delta_{M1}\log\frac{1-\frac{1}{Y_-}}{1-\frac{1}{Y_+}}\,\hat{\star}\, s - \log S(r^-_{M-1}-v)S(r^-_{M+1}-v)
\end{aligned}
$$

$$
\begin{aligned}
\log Y_{M|vw} = &-\log(1+Y_{M+1})\star s + \log(1+Y_{M-1|vw})(1+Y_{M+1|vw})\star s \\
&+\delta_{M1}\log\frac{1-Y_-}{1-Y_+}\,\hat{\star}\, s \\
&+\delta_{M1}\left(\log\frac{S(u_2^{(2)+}-v)}{S(u_3^{(2)-}-v)} - \log S(u_1^- - v)S(r_0^- - v)\right)
\end{aligned}
$$

Simplified TBA equations for $Y_{\pm}$

$$\log\frac{Y_+}{Y_-} = \log(1+Y_Q)\star K_{Qy} - \sum_{i=1}^{3}\log S_{1_*y}(u_i^{(1)},v) + \log\frac{S_{2y}(u_2^{(1)+},v)}{S_{2y}(u_2^{(2)+},v)}\frac{S_{2y}(u_3^{(2)-},v)}{S_{2y}(u_3^{(1)-},v)}$$

$$\begin{aligned}\log Y_+Y_- = {} & 2\log\frac{1+Y_{1|vw}}{1+Y_{1|w}}\star s - \log(1+Y_Q)\star K_Q + 2\log(1+Y_Q)\star K_{xv}^{Q1}\star s \\ & +2\log S(r_1^- - v) - 2\log S_{xv}^{1_*1}(u_1,v)\star s + \log S_2(u_1-v)\star s \\ & -2\log\frac{S_{xv}^{11}(u_2^{(1)},v)}{S_{xv}^{11}(u_3^{(1)},v)}\star s + \log\frac{S_1(u_2^{(1)}-v)}{S_1(u_3^{(1)}-v)} \\ & -\log\frac{S_2(u_2^{(1)+}-v)}{S_2(u_2^{(2)+}-v)}\frac{S_2(u_3^{(2)-}-v)}{S_2(u_3^{(1)-}-v)} + 2\log\frac{S_{xv}^{21}(u_2^{(1)+},v)S_{xv}^{21}(u_3^{(2)-},v)}{S_{xv}^{21}(u_2^{(2)+},v)S_{xv}^{21}(u_3^{(1)-},v)}\star s\end{aligned}$$

A generalization of our construction to a three-particle state with u_2 and u_3 lying in the kth strip

Four functions $Y_{k-2}, \dots, Y_{k+1}$ will have poles in the analyticity strip, with the poles of Y_{k-2} and Y_k being closest to the real line

The driving terms in the corresponding TBA equations will depend on $u_{2,3}^{(k-1)}$ and $u_{2,3}^{(k)}$ whose locations are determined by the corresponding exact Bethe equations for Y_{k-1} and Y_k

The energy

$$E = \sum_{i=1}^{3} \mathcal{E}(u_i^{(1)}) - \frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} du \frac{d\tilde{p}_Q}{du} \log(1+Y_Q) \\ - i\tilde{p}_k\Big(u_2^{(k-1)} + (k-1)\tfrac{i}{g}\Big) + i\tilde{p}_k\Big(u_2^{(k)} + (k-1)\tfrac{i}{g}\Big) \\ - i\tilde{p}_k\Big(u_3^{(k)} - (k-1)\tfrac{i}{g}\Big) + i\tilde{p}_k\Big(u_3^{(k-1)} - (k-1)\tfrac{i}{g}\Big)$$

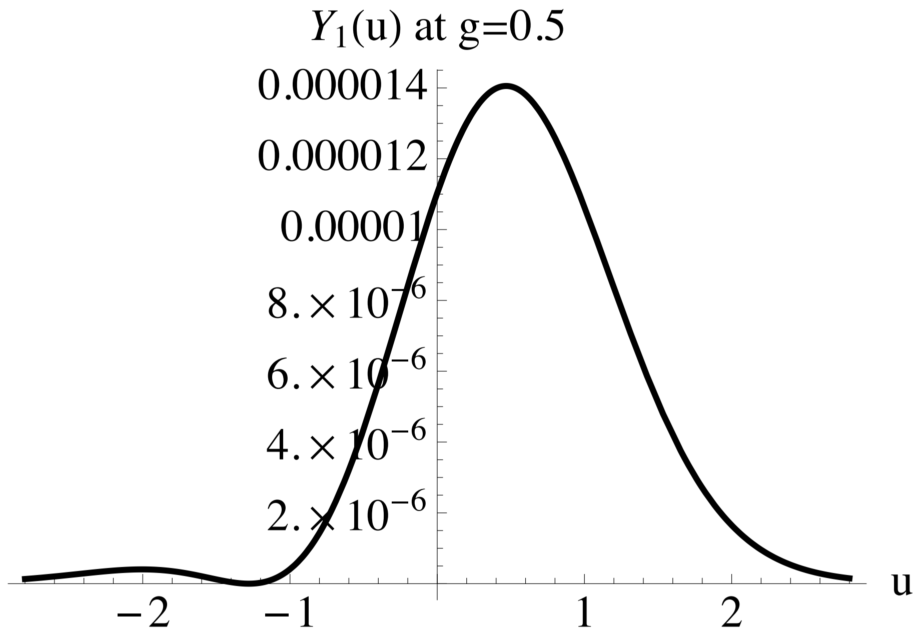
The fact that $1+Y_1$ and $1+Y_2$ functions have zeroes and poles in the analyticity strip in conjunction with the choice for the integration contours leads to the following energy formula

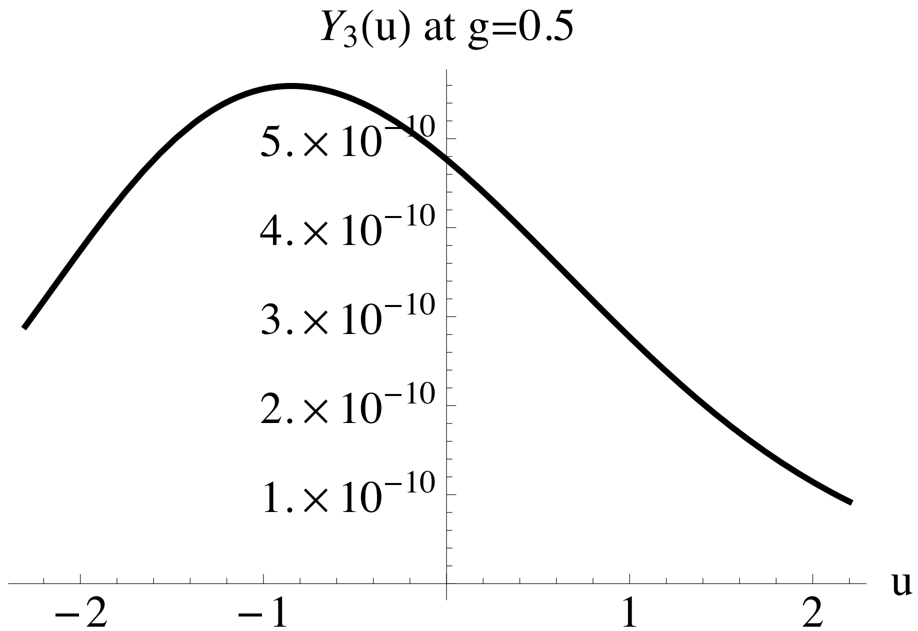
$$E = \sum_{i=1}^{3} \mathcal{E}(u_i^{(1)}) - \frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} du \frac{d\tilde{p}_Q}{du} \log(1+Y_Q) - i\tilde{p}_2(u_2^{(1)+}) + i\tilde{p}_2(u_2^{(2)+}) - i\tilde{p}_2(u_3^{(2)-}) + i\tilde{p}_2(u_3^{(1)-}),$$

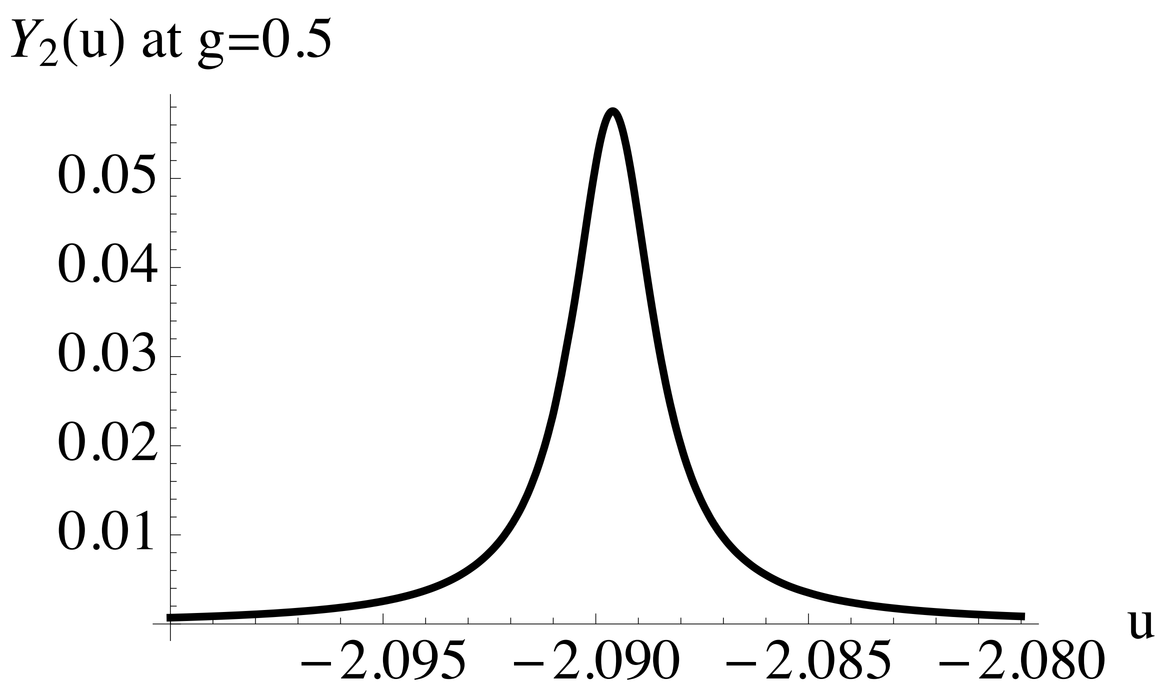
The $g \to 0$ and J finite limit provides the leading wrapping correction

$$\Delta E^{\text{wrap}} = -\frac{1}{2\pi} \sum_{Q=1}^{\infty} \int_{-\infty}^{\infty} du \frac{d\tilde{p}_Q}{du} Y_Q - i\Big[\mathrm{Res}\Big(\frac{d\tilde{p}_2}{du}(u_2^+)Y_2(u_2^+)\Big) - \mathrm{Res}\Big(\frac{d\tilde{p}_2}{du}(u_3^-)Y_2(u_3^-)\Big)\Big].$$

Fate of the bound state?








Thank you!