Localization
&
Exact Quantum Entropy

Atish Dabholkar

CERN, Geneva

Centre National de la Recherche Scientifique
Université Pierre et Marie Curie Paris VI

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Based on

- A.D. João Gomes, Sameer Murthy, *"Localization and Exact Holography,"* arXiv:1111.1163
- A.D. João Gomes, Sameer Murthy, *"Quantum Black Holes, Localization, and the Topological String,"* arXiv:1012.0265
- A.D., Sameer Murthy, Don Zagier; *"Quantum Black Holes, Wall-crossing, and Mock Modular Forms,"* arXiv:12mm.nnnn

Two Related Motivations

Entropy of black holes remains one of the most important and precise clues about the microstructure of quantum gravity.

Can we compute exact quantum entropy of black holes including all corrections both microscopically and macroscopically?

Holography has emerged as one of the central concepts regarding the degrees of freedom of quantum gravity.

Can we find simple example of AdS/CFT holgraphy where we might be able to 'prove' it exactly?

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Black Hole Entropy

Bekenstein [72]; Hawking[75]

For a BPS black hole with charge vector (q, p), *for large charges*, the leading Bekenestein- Hawking entropy precisely matches the logarithm of the degeneracy of the corresponding quantum microstates

$$\frac{A(q,p)}{4} = \log(d(q,p)) + O(1/Q)$$

Strominger & Vafa [96]

This beautiful approximate agreement raises two important questions:

- What exact formula is this an approximation to?
- Can we systematically compute corrections to both sides of this formula, perturbatively and nonperturbatively in $1/Q$ and may be even exactly for arbitrary finite values of the charges?

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Finite size effects

- We do not know which phase of string theory might correspond to the real world. For such a theory under construction, a useful strategy is to focus on *universal* properties that must hold in all phases. One universal requirement for a quantum theory of gravity is that in *any* phase of the theory that admits a black hole, it must be possible to interpret black hole as a statistical ensemble of quantum states.
- Finite size corrections to the entropy, unlike the leading area formula, depend on the details of the phase, and provide a sensitive probe of short distance degrees of freedom.

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Nonperturbative Spectrum

- Conversely, we can use the knowledge of exact black hole results from the bulk to learn about the exact nonperturbative spectrum of the theory.
- One can compare any putative counting fromula against the black hole entropy. This has already proved to be a useful guide in understanding for example the exact spectrum of quarter-BPS dyons in N=4 theories. In particular to fix subtle issues about dependence of the counting formula on wall-crossing, arithmetic duality invariants.

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Wald entropy and beyond

Wald[94]; Iyer & Wald [94]; Jacobson, Kang, & Myers [93]

- Wald entropy can incorporate the corrections to Bekenstein-Hawking entropy from all higher-derivative *local* terms in the effective action.
- This does not include quantum effects from integrating out massless fields. These are in general essential for example for duality invariance.

For a systematic comparison one needs a manifestly duality covariant formalism that generalizes Wald entropy. Such a formalism has been proposed recently by Sen using AdS_2/CFT_1 holography.

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Quantum Entropy and AdS_2/CFT_1 Sen [08]

- Near horizon geometry of a BPS black hole is $AdS_2 \times S^2$. Quantum entropy can be defined as as partition function of the AdS_2.

$$W(q,p) = \left\langle \exp\left[-i\, q_i \int_0^{2\pi} A^i \, d\theta \right] \right\rangle_{\mathrm{AdS}_2}^{\text{finite}} .$$

Functional integral over all string fields. The Wilson line insertion is necessary so that classical variational problem is well defined.

The black hole is made up of a complicated brane configuration. The worldvolume theory typically has a gap. Focusing on low energy states gives CFT_1 with a degenerate, finite dimensional Hilbert space. Partition function $d(q,p)$ is simply the dimension of this Hilbert space.

Functional Integral Boundary Conditions

- For a theory with some vector fields A^i and scalar fields ϕ^a, we have the fall-off conditions

$$
\begin{aligned}
ds_0^2 &= v\left[\left(r^2+\mathcal{O}(1)\right)d\theta^2+\frac{dr^2}{r^2+\mathcal{O}(1)}\right] , \\
\phi^a &= u^a+\mathcal{O}(1/r) , \qquad A^i = -i\, e^i(r-\mathcal{O}(1))d\theta , \qquad (1)
\end{aligned}
$$

- Magnetic charges are fluxes on the S^2. Constants v, e^i, u^a fixed to attractor values v_*, e^i_*, u^a_* determined purely in terms of the charges, and set the boundary condition for the functional integral.
- Quantum entropy is purely a function of the charges (q, p).

The functional integral is infrared divergent due to infinite volume of the AdS_2. Holographic renormalization to define the finite part.

Renormalized functional integral

- Put a cutoff at a large $r = r_0$.
- Lagrangian $\mathcal{L}_{bulk}$ is a local functional, hence the action has the form

$$S_{\rm bulk} = C_0 r_0 + C_1 + \mathcal{O}(r_0^{-1}) \ ,$$

  with C_0, C_1 independent of r_0.
- C_0 can be removed by a boundary counter-term (boundary cosmological constant). C_1 is field dependent to be integrated over.

Quantum Entropy gives a proper generalization of Wald entropy to include not only higher-derivative *local* terms but also the effect of *integrating over massless fields*. This is essential for duality invariance and for a systematic comparison since nonlocal effects can contribute to same order.

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To summarize, our most ambitious goal will be twofold.

1. Compute $d(q, p)$ from bound state dynamics of branes.
2. Compute $W(q, p)$ from the bulk for arbitrary *finite* charges by evaluating the functional integral over string fields in AdS_2. Check if the two agree.

The first problem has now been solved in some cases. We now know the exact spectrum of both half and quarter-BPS dyonic black holes for *all charges* at *all points* in the moduli space for certain $\mathcal{N} = 4$ theories. And for a large class of states in $\mathcal{N} = 8$ theories.

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Type II string theory on a T^6

Exact microscopic degeneracy of one-eighth BPS dyonic black holes with charge vector (Q, P) are given in terms of the Fourier coefficients of the following counting function.

$$F(\tau, z) = \frac{\vartheta_1^2(\tau, z)}{\eta^6(\tau)} .$$

where ϑ_1 is the Jacobi theta function and η is the Dedekind function. With $q := e^{2\pi i\tau}$ and $y := e^{2\pi iz}$, they have the product representations

$$\begin{aligned} \vartheta_1(\tau, z) &= q^{\frac{1}{8}}(y^{\frac{1}{2}} - y^{-\frac{1}{2}}) \prod_{n=1}^{\infty}(1 - q^n)(1 - yq^n)(1 - y^{-1}q^n) , \\ \eta(\tau) &= q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n) . \end{aligned}$$

Rademacher expansion

The degeneracy $d(\Delta)$ depends only the U-duality invariant $\Delta = Q^2P^2 - (Q \cdot P)^2$ and admits an *exact* expansion

$$d(\Delta) = \sum_{c=1}^{\infty} d_c(\Delta)$$

with

$$d_c(\Delta) = (-1)^{\Delta+1}\, 2\pi\, \tilde{I}_{\frac{7}{2}}\Big(\frac{\pi\sqrt{\Delta}}{c}\Big)\, \frac{1}{c^{9/2}} K_c(\Delta)\,.$$

where

$$\tilde{I}_\rho(z) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{1}{\sigma^{\rho+1}} e^{\sigma+\frac{z^2}{4\sigma}}\, d\sigma,$$

is a modifield Bessel function and $K_c(\Delta)$ is the Kloosterman sum.

Kloosterman Sum

The Kloosterman sum has intricate number theoretic structure defined by

$$K_c(\Delta) := e^{5\pi i/4} \sum_{\substack{-c \leq d < 0; \\ (d,c)=1}} e^{2\pi i \frac{d}{c}(\Delta/4)} \, M(\gamma_{c,d})^{-1}_{\ell 1} \, e^{2\pi i \frac{a}{c}(-1/4)}$$

with $\ell = \Delta \mod 2$ and $ad = 1 \mod c$,

$$\gamma_{c,d} = \begin{pmatrix} a & (ad-1)/c \\ c & d \end{pmatrix}$$

and M is a particular representation of $SL(2, \mathbb{Z})$.

- Our goal now will be to evaluate the formal expression for W(q,p) by doing the functional integral over string fields in AdS_2.
- This is of course highly nontrivial and may seem foolishly ambitious.
- Surprisingly, one can go quite far using localization techniques to reduce the functional integral to finite number of *ordinary integrals*.
- With enough supersymmetry, it seems possible to even evaluate these ordinary integrals all the way under certain assumptions.

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Localization

- Consider a supermanifold $\mathcal{M}$ with an odd (fermionic) vector field Q be such that $Q^2 = H$ for some compact bosonic vector field H.
- To evaluate an integral of a Q-invariant function h with Q-invariant measure we first deform it

$$I := \int_{\mathcal{M}} d\mu\, h\, e^{-\mathcal{S}} \qquad \to \qquad I(\lambda) := \int_{\mathcal{M}} d\mu\, h\, e^{-\mathcal{S}-\lambda QV},$$

where V is a fermionic, H-invariant function.
- Easy to check $I'(\lambda) = 0$, so $I(\lambda)$ is independent of λ. Hence, instead of at $\lambda = 0$, we can evaluate it at $\lambda = \infty$ where semiclassical approximation is exact.

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Choice of the Supercharge

- In this limit, the functional integral *localizes* onto the critical points of the functional $S^Q := QV$. This reduces the functional integral over field space to a this localizing submanifold.
- To apply to our problem, we pick Q which squares to $H = 4(L - J)$. Here L generates rotation of the Euclidean AdS_2 which is a disk and J generates a rotation of S^2, so H is compact.
- Given this choice of Q we choose the localizing action functional to be

$$S^Q = QV; \qquad V = (Q\Psi, \Psi)$$

where Ψ denotes schematically all fermions of the theory.

To apply this directly in string theory is not feasible given the state of string field theory. So we will first solve a simpler problem in supergravity.

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A simpler problem in supergravity

- Consider $\hat{W}(q,p)$ which is the same functional integral but in supergravity coupled to only $n_v + 1$ vector multiplets.
- This is still a complicated functional integral over *spacetime* fields.

We will show using localization that this functional integral reduces to an ordinary integral over $n_v + 1$ real parameters. Huge simplification.

To apply localization inside the functional integral, one requires an *off-shell* formulation. In general, off-shell supergravity is notoriously complicated but for $\mathcal{N} = 2$ vector multiplets an elegant formalism exists that gauges the full superconformal group.

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Multiplets and Off-shell supersymmetry transformations

- *Gravity multiplet*: Vielbein, spin connection, auxiliary fields and fermions
- *Vector multiplet*: vector field A_μ^I, complex scalar X^I and an $SU(2)$ triplet of auxiliary fields Y_{ij}^I and fermions. Here i is $SU(2)$ doublet.

$$\mathbf{X}^I = \left(X^I, \Omega_i^I, A_\mu^I, Y_{ij}^I\right)$$

$$\delta\Omega_i = 2\not{D}X\epsilon_i + \frac{1}{2}\varepsilon_{ij}F_{\mu\nu}\gamma^{\mu\nu}\epsilon^j + Y_{ij}\epsilon^j + 2X\eta_i\,,$$

where ϵ, η are the (superconformal) supersymmetry parameters.

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Importance of being off-shell

- The beauty of off-shell supergravity is that the supersymmetry transformations are specified once and for all and do not depend on the choice of the action. This is crucial for localization.
- Auxiliary fields which are normally eliminated from the physical action acquire nontrivial position dependence for the localizing instanton solutions.

With this setup, the bosonic part $(Q\Omega, Q\Omega)$ of the QV action is a sum of perfect squares. Setting each of these terms to zero gives a set of first-order differential equations. Happily, it turns out they can be solved exactly to obtain an analytic solution.

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The bosonic part of the localizing action $(Q\Psi, Q\Psi)$

$$
\begin{aligned}
& \cosh(\eta)\left[K - 2\mathrm{sech}(\eta) H\right]^2 \\
+ \quad & 4\cosh(\eta)\left[H_1 + H\tanh(\eta)\right]^2 + 4\cosh(\eta)[H_0^2 + H_2^2 + H_3^2] \\
+ \quad & 2A\left[f_{01}^{-} - J - \frac{1}{A}\left(\sin(\psi) J_3 - \sinh(\eta) J_1\right)\right]^2 \\
+ \quad & 2B\left[f_{01}^{+} + J - \frac{1}{B}\left(\sin(\psi) J_3 + \sinh(\eta) J_1\right)\right]^2 \\
+ \quad & 2A\left[f_{03}^{-} + \frac{1}{A}\left(\sin(\psi) J_1 + \sinh(\eta) J_3\right)\right]^2 \\
+ \quad & 2B\left[f_{03}^{+} + \frac{1}{B}\left(\sin(\psi) J_1 - \sinh(\eta) J_3\right)\right]^2
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 2A\left[f_{02}^{-} + \frac{1}{A}\left(\sin(\psi)J_0 + \sinh(\eta)J_2\right)\right]^2 \\
&+ \quad 2B\left[f_{02}^{+} - \frac{1}{B}\left(\sin(\psi)J_0 + \sinh(\eta)J_2\right)\right]^2 \\
&+ \quad \frac{4\cosh(\eta)}{AB}\left[\sinh(\eta)J_0 - \sin(\psi)J_2\right]^2 \\
&+ \quad \frac{4\cosh(\eta)\sinh^2(\eta)}{AB}\left[J_1^2 + J_3^2\right],
\end{aligned}
$$

where

$$
H_a^I := e_a^\mu \partial_\mu H^I\,, \quad J_a^I := e_a^\mu \partial_\mu J^I\,,
$$

$$
A := \cosh(\eta) + \cos(\psi)\,, \qquad B := \cosh(\eta) - \cos(\psi)\,.
$$

It is understood that all squares are summed over the index I.

Localizing instanton Solution

$$X^I = X^I_* + \frac{C^I}{r}\,, \qquad \bar{X}^I = \bar{X}^I_* + \frac{C^I}{r}$$

$$Y_1^{I1} = -Y_2^{I2} = \frac{2C^I}{r^2}\,, \qquad f^I_{\mu\nu} = 0\,.$$

Solves a major piece of the problem by identifying the off-shell field configurations onto which the functional integral localizes. Thus, a functional integral is reduced to a finite dimensional ordinary integral. This instanton is *universal* and does not depend on the physical action.

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We now need to evaluate the physical action on the localizing instantons after proper renormalization to compute $S_{ren}(C, q, p)$

$$
\begin{aligned}
& (-i(X^I\bar{F}_I - F_I\bar{X}^I))\cdot(-\frac{1}{2}R) + [i\nabla_\mu F_I \nabla^\mu \bar{X}^I \\
+ \;& \frac{1}{4}iF_{IJ}(F^{-I}_{ab} - \frac{1}{4}\bar{X}^I T^{ij}_{ab}\varepsilon_{ij})(F^{-abJ} - \frac{1}{4}\bar{X}^J T^{ij}_{ab}\varepsilon_{ij}) \\
- \;& \frac{1}{8}iF_I(F^{+I}_{ab} - \frac{1}{4}X^I T_{abij}\varepsilon^{ij})T^{ij}_{ab}\varepsilon_{ij} - \frac{1}{8}iF_{IJ}Y^I_{ij}Y^{Jij} - \frac{i}{32}F(T_{abij}\varepsilon^{ij})^2 \\
+ \;& \frac{1}{2}iF_{\widehat{A}}\widehat{C} - \frac{1}{8}iF_{\widehat{A}\widehat{A}}(\varepsilon^{ik}\varepsilon^{jl}\widehat{B}_{ij}\widehat{B}_{kl} - 2\widehat{F}^{-}_{ab}\widehat{F}^{-}_{ab}) \\
+ \;& \frac{1}{2}i\widehat{F}^{-ab}F_{\widehat{A}I}(F^{-I}_{ab} - \frac{1}{4}\bar{X}^I T^{ij}_{ab}\varepsilon_{ij}) - \frac{1}{4}i\widehat{B}_{ij}F_{\widehat{A}I}Y^{Iij} + \mathrm{h.c.}] \\
- \;& i(X^I\bar{F}_I - F_I\bar{X}^I)\cdot(\nabla^a V_a - \frac{1}{2}V^a V_a - \frac{1}{4}|M_{ij}|^2 + D^a\Phi^i{}_\alpha D_a\Phi^\alpha{}_i)\,.
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+ \quad & \frac{1}{2} i F_{\widehat{A}} \widehat{C} - \frac{1}{8} i F_{\widehat{A}\widehat{A}} (\varepsilon^{ik} \varepsilon^{jl} \widehat{B}_{ij} \widehat{B}_{kl} - 2 \widehat{F}^-_{ab} \widehat{F}^-_{ab}) \\
+ \quad & \frac{1}{2} i \widehat{F}^{-ab} F_{\widehat{A}I} (F^{-I}_{ab} - \frac{1}{4} \bar{X}^I T^{ij}_{ab} \varepsilon_{ij}) - \frac{1}{4} i \widehat{B}_{ij} F_{\widehat{A}I} Y^{Iij} + \mathrm{h.c.} \big] \\
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Renormalized action

- Substituting our localizing instanton solution in the above action we can extract the finite piece after removing the leading divergent piece linear in r_0 by holographic renormalization.
- After a tedious algebra, one obtains a remarkably simple form for the renormalized action S_{ren} as a function of $\{C^I\}$.

$$\mathcal{S}_{ren}(\phi, q, p) = -\pi q_I \phi^I + \mathcal{F}(\phi, p) \tag{2}$$

with $\phi^I := e_*^I + 2iC^I$ and $\mathcal{F}$ given by

$$\mathcal{F}(\phi, p) = -2\pi i \left[F\left(\frac{\phi^I + ip^I}{2}\right) - \bar{F}\left(\frac{\phi^I - ip^I}{2}\right) \right],$$

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- Note that $S_{ren}(\phi, q, p)$ equals precisely the *classical* entropy function $\mathcal{E}(e, q, p)$. In particular, $\exp(S_{ren})$ is the topological string partition function. The physics is however completely different.
- $\mathcal{E}(e, q, p)$ depends on the the attractor values X_* of the scalar fields. $S_{ren}(\phi, q, p)$ depends on the value of the scalar fields at the center of AdS_2. This difference is very important.
- Even though scalar fields are fixed at the boundary, their value at the origin can fluctuate and we can integrate over them for large values.

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We at present lack a useful definition of functional integral over string fields. To apply localization, we proceed in three steps.

Three Steps:

1. Integrate out massive string and Kaluza-Klein modes to obtain a local Wilsonian effective action for the massless supergravity fields.
2. Solve a supergravity problem to evaluate $\hat{W}(q,p)$.
3. Use the results in Step II to evaluate $W(q,p)$ There are $\mathbb{Z}_c$ orbifolds of AdS_2 that have the same boundary conditions and hence contribute to the functional integral. Hence, $W(q,p)$ has the form

$$W(q,p) = \sum_{c=1}^{\infty} W_c(q,p)\,.$$

Sen[09, 10], Pioline & Murthy [10]

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Evaluation $W_c(q,p)$ is related to the problem of evaluation of $\hat{W}(q,p)$ in a simple way, under certain assumptions.

General form of the answer for $c = 1$

$$W_1(n,w) = \int_{\mathcal{M}_Q} e^{S_{ren}(n,w,\phi)} \left|Z_{inst}(\phi,n,w)\right|^2 Z_{det}(\phi)\,[d\phi]_\mu \,.$$

- $[d\phi]_\mu$ is the induced measure on the localizing manifold.
- $|Z_{inst}|^2$ is contribution of instantons localized at north pole of S^2 and anti-instantons localized at the south pole.
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No off-shell formalism with $\mathcal{N} = 4$ with finite number of auxiliary fields. So we will proceed in the $\mathcal{N} = 2$ language.

Assumptions

- Drop two gravitini multiplets. These contain four vector fields but no scalar fields. If the black hole does not couple to these vector fields, it should be reasonable to drop this.
- Drop hyper multiplets. The off-shell supersymmetry transformations of the vector multiplets do not change by adding hypers. So our localizing instantons will continue to exist.
- Drop D-terms. A large class of D-terms are known not contribute to Wald entropy. de Wit, Katmadas, Van Zalk [10]

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Prepotential

For our example with this $\mathcal{N} = 2$ restriction, the prepotential is

$$F(X, \hat{A}) = -\frac{1}{2}\frac{X^1}{X^0}\sum_{a,b=2}^{7} C_{ab}X^aX^b,$$

in the Type-IIA frame. Here C_{ab} is the intersection matrix for the 6 2-cycles of T^6. Choose a simple charge configuration with five charges q_0 of D0-branes, q_2 D2-branes and p^1, p^2, p^3 D4-branes wrapping three independent 4-cycles in T^6.

Renormalized Action

We would like to evaluate the renormalized action $S_{ren}(\phi, q, p)$ as a functon of the collective coordinates ϕ^I and the charges.

Renormalized action

Since the prepotential corresponds to two-derivative action which has $E_{7,7}(\mathbb{R})$ duality symmetry we expect to be able express the action in terms of the duality invariant Δ.

Duality-invariant variables

It is useful to define new variables

$$\sigma = -\pi p^1 p^2 p^3/\phi^0 \,, \qquad \tau^a = -\frac{1}{\sqrt{\phi^0}}\left(\phi^a + \frac{\phi^1\, p^a}{p^1}\right) \qquad \alpha \sim \phi^1 \,.$$

Renormalized Action

$$S_{ren} = \left(\sigma + \frac{z^2}{4\sigma}\right) - \frac{\pi\,\tau^2}{2} + \frac{\pi\,\alpha^2}{2} \,.$$

with

$$z^2 = \pi^2 \Delta \,.$$

Evaluation of W_1

One-loop determinants

The localizing action is purely quadratic in the fields. Hence the quadtratic fluctuation operator does not depend on the collective coordinates or the charges. Hence the determinant factor is an overall numerical constant.

Instantons

Various branes can contribute to the effective action. The instanton contribution to the prepotential is computed (partially) by the topological string. However, for the $\mathcal{N} = 8$ theory the classical prepotential is quantum exact and there are no corrections.

Induced Measure

Given the measure on space of fields X, the measure on the localizing submanifold parametrized by ϕ can be deduced from the induced metric.

Integration Measure

The line element on ϕ-space is

$$d\Sigma^2 = M_{IJ}\,\delta\phi^I\delta\phi^J$$

with the metric

$$M_{IJ} = K_{IJ} - \frac{1}{4}\frac{\partial K}{\partial\phi^I}\frac{\partial K}{\partial\phi^J}$$

given in terms of the Kähler potential

$$e^{-K} := -i(X^I\bar{F}_I - \bar{X}^I F_I)$$

Measure

$$\int \prod_{I=0}^{7} d\phi^I \sqrt{\det(M)}\,.$$

Final Answer for $W_1(\Delta)$

Conformal compensator

The ϕ^0 variable can be thought of as the conformal compensating field. Conformal factor of the metric has wrong sign kinetic term in Euclidean gravity and hence its contour of integration must be analytically continued to make the functional integral well-defined.

The Gaussian integrals can be readily done to obtain

$$\mathcal{C} \int_{-i\infty}^{+\infty} \frac{dt}{\sigma^{9/2}} \exp[\sigma + \frac{z^2}{4\sigma}] \quad \textit{with} \quad z = \pi\sqrt{\Delta} \ .$$

Up to an overall constant, this is precisely the integral representation of the first term in the Rademacher expansion — $I_{7/2}(\pi\sqrt{\Delta})$!

Comparison of $d(\Delta)$ with $W_1(\Delta)$ and exponential of Wald Entropy

Δ	3	4	7	8	11	12
$d(\Delta)$	8	12	39	56	152	208
$W_1(\Delta)$	7.97	12.20	38.99	55.72	152.04	208.45
$\exp(\pi\sqrt{\Delta})$	230.7	535.4	4071.9	7228.3	33506	53252.3

The area of the horizon goes as $4\pi\sqrt{\Delta}$ in Planck units. Already for $\Delta = 12$ this area would be much larger than one, and one might expect that the Wald entropy would be a good approximation. Not true! The discrepancy between the degeneracy and the exponential of the Wald entropy arises entirely from integration over massless fields.

Index, Degeneracy and Fermions

Table: Some Fourier coefficients

Δ	-1	0	3	4	7	8	11	12	15
$C(\Delta)$	1	-2	8	-12	39	-56	152	-208	513

It is striking the the sign of the Fourier coefficients is alternating. Note that

$$d(\Delta) = (-1)^{\Delta+1} C(\Delta)$$

Since F is the partition function of indexed degeneracy there is no a priori reason why they should they be positive. Surprising from the point of view of the microscopic theory. Holography gives a simple physical explanation of this fact.

Index = Degeneracy

The near-horizon AdS_2 geometry has an $SU(1,1)$ symmetry. With four supersymmetries, closure of the supersymmetry algebra requires that the near horizon symmetry must contain the supergroup $SU(1,1|2)$. This implies that that such a supersymmetric horizon must have $SU(2)$ symmetry which can be identified with spatial rotations. The AdS_2 path integral naturally fixes the charges and not the chemical potentials and hence $J = 0$. Together, this implies

$$\mathrm{Tr}\left(-1^{J}\right) = \mathrm{Tr}\left(1\right)\,, \tag{3}$$

Fermionic boundary conditions

Because the constant mode of the gauge field fluctuates, and since the fermions couple to it, periodic and antiperidic boundary conditions are equivalent.

Nonpertrubative Corrections

There is a family of freely acting supersymmetric $\mathbb{Z}_c$ orbifolds with twists on $AdS_2 \times S^2$ on shifts along S^1. The shift can be effected by modifying the gauge field at infinity. Hence the Wilson line gives a phase both for winding and momentum. Since the orbifold action is freely acting, one obtains the same localizing instanton solution but the renormalized action is divided by c because of the reduced volume.

For each c we then obtain

$$\int \frac{d\sigma}{\sigma^{9/2}} \exp[\sigma + \frac{z^2}{c^2\sigma}] \ .$$

Kloosterman Sum

The Kloosterman sum is a sum over phases. Such phases naturally enter our story from the Wilson lines. Our orbifold is a symmetric twist on $AdS_2 \times S^2$ and a shift in the charge lattice. The shifts can in principle account for the Kloosterman sum.

Summary

Localization of the functional integral in Sugra and String Theory

- We have shown that full functional integral of supergravity coupled to vector fields on AdS_2 localizes onto the submanifold $\mathcal{M}_Q$ of critical points of the functional S^Q where Q is a specific supersymmetry.
- We have obtained exact analytic expression for a family of nontrivial complex instantons as *exact* solutions which are completely *universal* and independent of the form of the physical action.
- In string theory, there are nonperturbative contributions from orbifolds as well as from brane instantons.

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Rademacher expansion from Gravity

The gravitational theory has all the ingredients to exactly reproduce the entire Rademacher expansion.

$$W(\Delta) = \sum_{c=1}^{\infty} K_c(\Delta) \left(\frac{2\pi}{c}\right)^{9/2} \tilde{I}_{7/2}(\frac{\pi\sqrt{\Delta}}{c}) .$$

with

$$\tilde{I}_{7/2}(z) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} \frac{1}{\sigma^{9/2}} e^{\sigma + \frac{z^2}{4\sigma}} dt,$$

Kloosterman sum can possibly follow from the analysis of phases for the Wilson lines but this needs to be done yet.

Some of the assumptions need to be better justified. Requires off-shell realization of at least the charge Q with vector, hypers, D-terms or $\mathcal{N} = 4$ field content. Interesting problem in supergravity.

Comparison with Earlier Work

It seems possible to compute finite size corrections to the entropy of BPS black holes in a systematic way going well beyond Bekenstein-Hawking.

The leading Bessel function was partially derived in Dabholkar [04] and Dabholkar, Denef, Pioline, Moore [05]. However,

- it relied on the OSV conjecture Ooguri, Strominger, Vafa [04];
- it was at best an asymptotic expansion since there was no systematic way of determining the measure or the range of integration since the OSV conjecture applied to a mixed ensemble.
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Our derivation follows from first principles using standard rules of functional integration and localization.

- Functional integral localizes onto a nontrivial localizing instanton solution of the *off-shell theory*. Auxiliary fields play an important role.
- The measure or the range of integration can be determined following usual collective coordinate quantization. Gives the exact Bessel function and not just the asymptotic expansion.
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From Gravity to Number Theory

- The Rademacher expansion is an *exact* expansion. It is rapidly convergent but at no finite order can one assert integrality of the sum. It is a nontrivial fact that all these terms add up to an integer and that we can see the entire expansion from the bulk. To reproduce this completely we need to understand the Kloosterman sum.
- This indicates an underlying integral structure to quantum gravity. If we have two very close but different integers, the bulk theory will be able to distinguish the two. This would be never evident from semiclassical Bekenstein-Hawking formula.
- It is intriguing that two AdS_2 bulk theories which may have very different field content but which yield the same integer $W(q,p)$ are expected to be dual since they will be both dual to the same CFT_1. This is rare but possible.

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