COUNTING IN 4,5,6 DIMENSIONS

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Progress in Quantum Field Theory and String Theory
Osaka City University April, 2012

Outline

- Count degrees of freedom on 6d (2,0) SCFT in the Coulomb phase: M5

  Stefano Bolognesi, K.L.

- Index of instantons in SYM on R^{1+4} (Coulomb and symmetric phase): D4

  Hee-Cheol Kim, Seok Kim, Eunkyung Koh, K.L., Sungjay Lee

- Index of SYM on S^3 x S^1 with `t Hooft operator and S-duality: D3

  Dongmin Gang, Eunkyung Koh, K.L.

6d (2,0) Superconformal field theories

- simple-laced A,D,E types of theories
  - A_{N-1} on N M5 branes , D_N on 2N M5 branes+ OM5
  - Type IIB on C^2/Z_N ➔ A, C^2 / dihedral ➔ D,

    C^2/ tetra, octahedral, icosahedral ➔ E [Witten96]

- Fields: $B_{\mu\nu}$,φ_I ,ψ_A : self-dual H=dB, *H=H on single M5
  - Tensionless selfdual strings in the nonabelian SCFT
  - (2,0) supersymmetry + $SO(5)_R$ symmetry

6d (2,0) SCFT (formalism)

- It is not easy to write down explicitly this quantum theory which does not have any weak coupling regime.
  - If one is successful, one would have, say, the local field theory for both electric and magnetic objects in 4-dim after compactification. This is not likely.
- With dimensionful coupling, 5d SYM is naively not expected to UV complete. There is a recent proposal that 5d maximal SYM is sufficient to define the (2,0) theory on R^{1+4} x S^1 . [Douglas; Lambert,Papageorgakis,Schmidt-Sommerfeld]
  - Instantons in 5d SYM are the Kaluza-Klein mode. [Seiberg] (See the second part.)
  - Instantons are dipole-configuration and belong to the adjoint representation of dual group.
  - the perturbative effect + non perturbative effect may complete the theory.
  - The role of magnetic monopole string is not clear.

6d (2,0) SCFT (d.o.f.)

- Degrees of freedom on N M5 branes:
  - Extremal black hole solutions : N^3 [Klebanov,Tseytlin]
  - Weyl anomaly : N^3 [Henningson,Skenderis]
  - Anomaly from tangent and normal bundle of M5 branes [Witten: Harvey,Minasian,Moore:Yi: Intriligator]

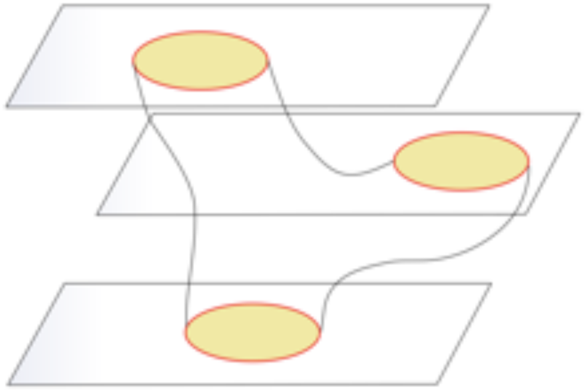
- Anomaly polynomial

$$I_8[G] = r_G I_8[1] + \frac{1}{24} c_G p_2(NW)$$

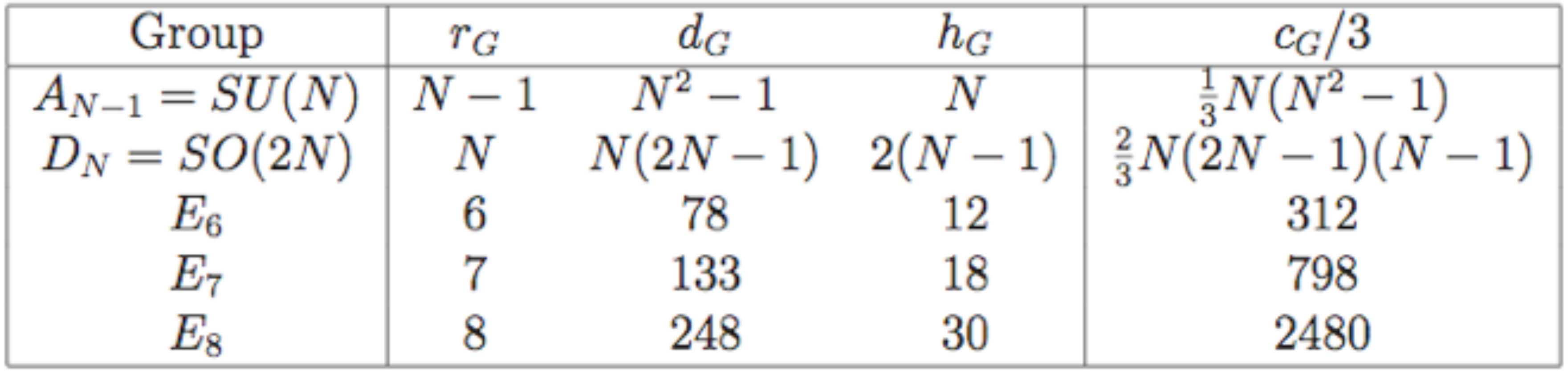
  - rank r_G, dim d_G,
  - Coxeter number $h_G = (d_G - r_G)/r_G$ (=> $d_G = (h_G+1) r_G$)
  - anomaly coefficient $c_G = d_G h_G$

6d (2,0) SCFT

- For $A_{N-1}=SU(N)$, $r_G=N-1$, $d_G=N^2-1$, $h_G=N$, $c_G= N(N^2-1)$
  - for large N, $c_G \sim N^3$
  - Pant diagram?
  - Dyonic Instantons?
  - Instanton Partons?
  - Junctions?



- Anomaly Coefficient $c_G=h_Gd_G$

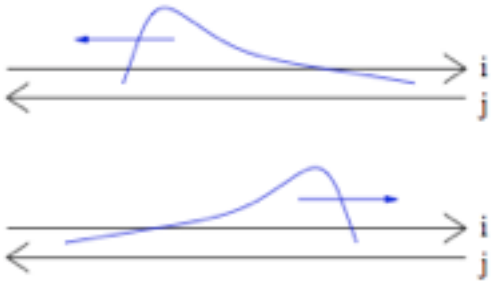


Group	r_G	d_G	h_G	$c_G/3$
$A_{N-1}=SU(N)$	$N-1$	N^2-1	N	$\frac{1}{3}N(N^2-1)$
$D_N=SO(2N)$	N	$N(2N-1)$	$2(N-1)$	$\frac{2}{3}N(2N-1)(N-1)$
E_6	6	78	12	312
E_7	7	133	18	798
E_8	8	248	30	2480

Table I: r_G, d_G, h_G and $c_G/3$ for simple-laced groups ADE

5d SYM (d.o.f.)

- Assuming 5d SYM captures the essence of the (2,0) theory, one would ask how N^3 degrees of freedom is realized in 5d theory. Let us look for the spectrum.
- In the symmetric phase
  - 1/2 BPS elementary particles has N^2-1 dof. (adjoint of SU(N))
  - 1/2 BPS instanton has N dof. (4N zero modes, N instanton partons?)
- In the Coulomb phase,
  - 1/2 BPS monopole strings have (N^2-N)/2 dof. = anti-mono strings
  - 1/4 BPS dyonic instantons have……(The second part of this talk…)



- 1/4 BPS monopole strings with wave: $\Gamma^{1235}\varepsilon=\varepsilon$, $\Gamma^{04}\varepsilon=\pm\varepsilon$

$$F_{12}=D_3\phi_5,\ F_{23}=D_1\phi_5,\ F_{31}=D_2\phi_5$$
$$F_{0i}-F_{4i}=0,\ D_0\phi_5-D_4\phi_5=0$$

5d SYM (Monopole String Junctions)

- Monopole strings can form 1/4 BPS junctions
  - Lock SO(4)$_{\text{rot}}$ with SO(4) of SO(5)$_{\text{R}}$: A_a , Φ_a (a=1,2,3,4)
  - 1/16 BPS equation (dyonic webs of junctions) [Kapustin,Witten: Yee,KL]

$$F_{ab} = \epsilon_{abcd} D_c \Phi_d - i[\Phi_a, \Phi_b] \ , \ \ D_a \Phi_a = 0$$

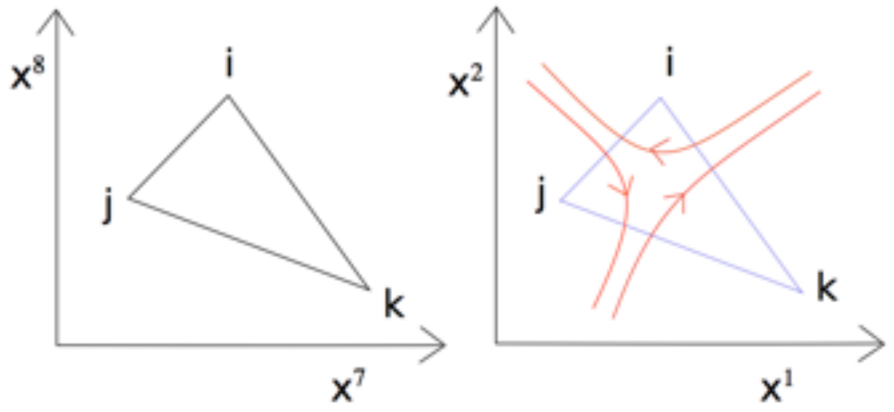
$$D_a^2 \Phi_5 - [\Phi_a, [\Phi_a, \Phi_5]] = 0$$

$$\begin{aligned}
F_{12} + F_{34} + F_{56} + F_{78} &= 0 \\
F_{13} + F_{42} + F_{57} + F_{86} &= 0 \\
F_{14} + F_{23} + F_{76} + F_{85} &= 0 \\
F_{15} + F_{62} + F_{73} + F_{48} &= 0 \\
F_{16} + F_{25} + F_{47} + F_{38} &= 0 \\
F_{17} + F_{35} + F_{64} + F_{82} &= 0 \\
F_{18} + F_{27} + F_{63} + F_{54} &= 0
\end{aligned}$$

- 1/4 BPS junctions on 3 D4 branes: lock 34 & 78 plane:

$$\begin{aligned}
\Gamma^{1238}\epsilon = \epsilon, \Gamma^{1247}\epsilon = \pm\epsilon, \ \phi_1 = \phi_2 = 0, \\
F_{12} = D_3\phi_4 - D_4\phi_3, F_{23} = D_1\phi_4, F_{31} = D_2\phi_4 \\
F_{41} = D_2\phi_3, F_{24} = D_1\phi_3, F_{43} = -i[\phi_4, \phi_3], D_3\phi_3 + D_4\phi_4 = 0
\end{aligned}$$

- mutually susy & tension balance:

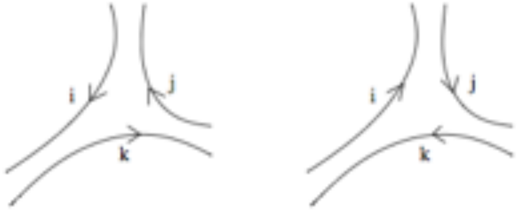


6d (2,0) SCFT (d.o.f.)

- in the Coulomb phase of G-type (G=A,D,E simple-laced)
  - 1/2 BPS massless (2,0) tensor multiplets: r_G
  - 1/2 BPS selfdual strings = anti-strings: $(d_G-r_G)/2 \equiv r_G h_G/2$

$>$ not N^3

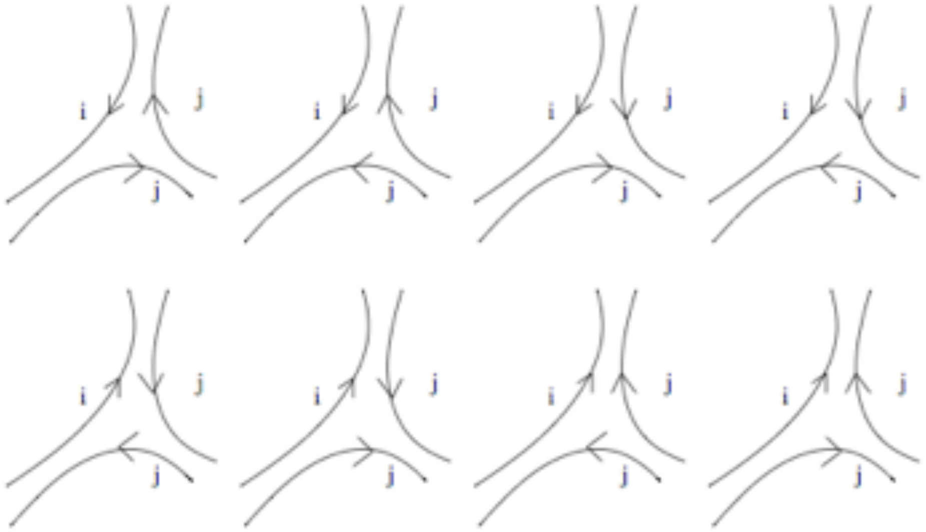
- 1/4 BPS massless waves on selfdual strings & anti-objects: $d_G-r_G = r_G h_G$
- 1/4 BPS junctions and anti-junctions: counting?
- 1/4 BPS objects in A_{N-1} case : roots $\alpha = e_i - e_j$



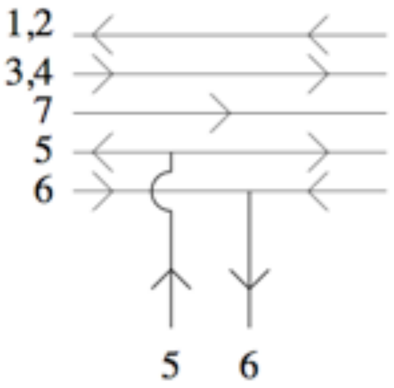
  - waves on string: $A = 2 \times N(N-1)/2 = N(N-1) = d_G - r_G$
  - junctions and anti-junctions: three selfdual strings for roots: e_i-e_j, e_j-e_k, e_k-e_i
  - $B=2 \times N(N-2)(N-2)/6=N(N-1)(N-2)/6$
  - total:

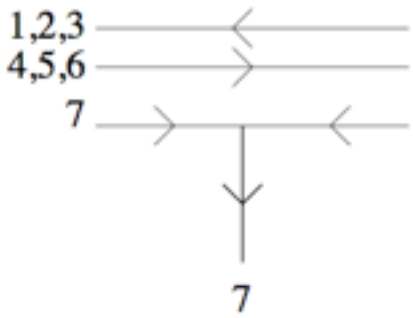
$$A + B = \frac{1}{3}N(N^2-1) = \frac{1}{3}r_{A_{N-1}}d_{A_{N-1}} = \frac{1}{3}c_{A_{N-1}}$$

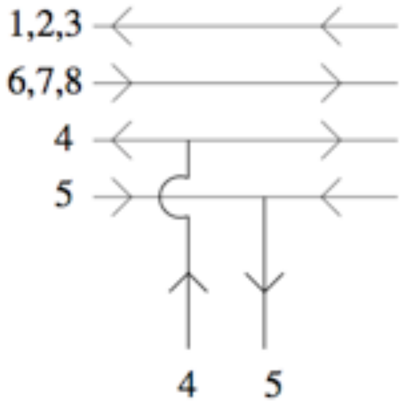
6d (2,0) SCFT (D_N & $E_{6,7,8}$ types)

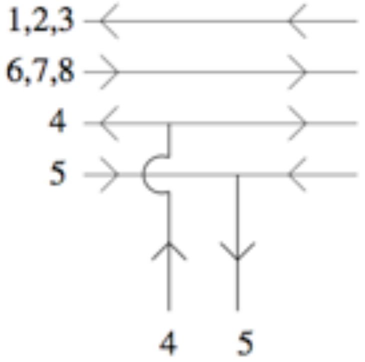


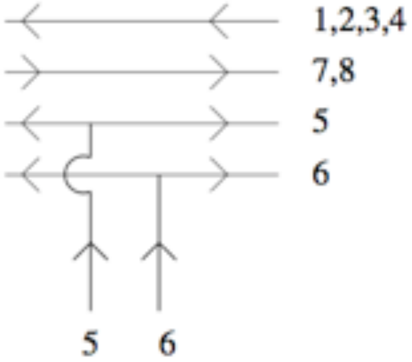
- D_N roots : { $e_i \pm e_j$ } (i,j=1,...N, i ≠ j)
  - A= 4*N(N-1)/2= 2N(N-1)
  - B= 8*N(N-1)(N-2)/6= 4N(N-1)(N-2)/3
  - A+B=2N(N-1)(2N-1)/3 = $C_{DN}/3$
- $E_{6,7,8}$











Periodic Table of (2,0) Theories?

High temperature in the Coulomb phase

- Start with a generic Coulomb phase with string tension ***u***
  - For the free string, there is a Hegadorn temperature

$$T_H \sim \sqrt{u} \qquad E = T_H S$$

  - Imagine the heating the local region beyond the Hegardorn temperature.

$$T >> T_{Hegadorn} = \sqrt{u}$$

- High energy is dominated by the webs of junctions in 5-dim space

Junctions as fundamental objects

- Junctions are more fundamental than selfdual strings as junction+ anti-junction can form a selfdual string after a partial annihilation.
- Junctions are like atoms on M5 branes
- via AGT relation, 2-d Toda central charge has N^3 growth, which is related to N^3 on M5 branes. Better understanding of Toda model is needed.
- Excitation of Junctions

- In the second talk, we have something to say about the degenerated junctions where all the strings are parallel.

2nd topic: Index of Instantons in 5d SYM

- Calculate the instanton index with all chemical potentials turned on.
- index of dyonic instantons in the Coulomb phase

  S-dual of monopole strings with momentum up on further circle compactification
- index of instantons in the symmetric phase

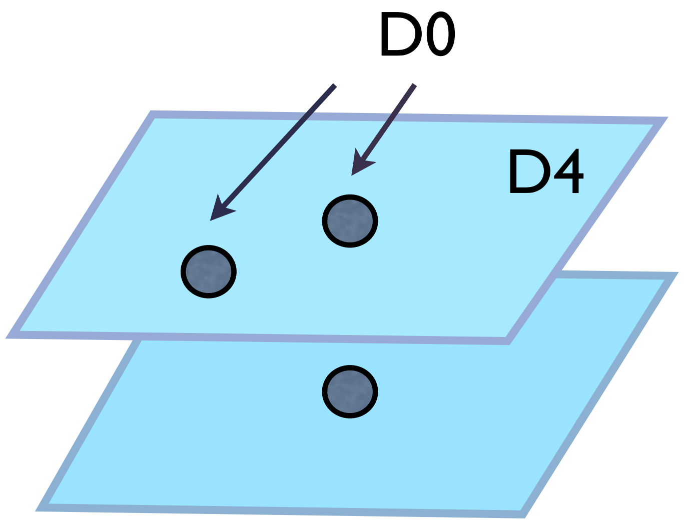
  nonrelativistic superconformal index of DLCD of (2,0) theories.
- References:
  - [Nekrasov(02); Nekrasov,Okoukov(03); Pestun; Okuda,Pestun(10); Dorey.Hollowood,Khoze,Mattis(02); Bruzzo,Fucito,Morales,Tanzini(03); Iqbal,Kozcaz,Shbabir(10)]
  - [Kinney,Maldacena,Minwalla,Raju(07); Dolan,Osborn(03)]
  - [Aharony,Berkooz,Seiberg (98)]

5d Maximal SYM on D4

- Rotational symmetry: $SO(4)_{rot} = SU(2)_{1L} \times SU(2)_{1R}$
- Instantons are KK modes for (2,0) SCFT on $R^{1+4} \times S^1$
    - instantons are massive tensor multiplet with spin (1,0), (1/2,0) , (0,0)
    - anti-instantons are similar with spin (0,1), (0,1/2), (0,0)
- Here we study the dyonic instanton index on $R^4 \times S^1$
- Here we have something new to say from it.
    - unique threshold bound state: support the 5d SYM being (2,0) theory
    - S-duality tells something about monopole strings with momentum
    - it tells something about the degenerate parallel monopole string junctions
    - it tells something about the DLCQ index of (2,0) theory in the symmetric phase

Instanton and anti-instanton
pair can annihilate with
orbital angular momentum

On D4 branes in Coulomb Phase



$\langle \varphi_i^9 \rangle = v_i$

- On D4s in Coulomb phase the symmetry:

$$SO(4)_1 \times SO(4)_2$$

$x^{1\sim4}$ $x^{5\sim8}$

- D4 brane preserve 16 SUSY :

$SO(4)_2$

$$Q^i_\alpha \,,\ \bar{Q}^i_{\dot\alpha}$$

$SU(2)_{1L} \times SU(2)_{1R} \subset SO(4)_1$

- D0-D4 or instantons preserve 8 SUSY :

$SU(2)_{2L} \times SU(2)_{2R} \subset SO(4)_2$

$$\bar{Q}^i_{\dot\alpha} \Rightarrow \bar{Q}^a_{\dot\alpha} \,,\ \bar{Q}^{\dot a}_{\dot\alpha}$$

- F1-D0-D4 or dyonic instantons preserve 4 SUSY :

$$\bar{Q}^{\dot a}_{\dot\alpha} \quad \begin{matrix} \leftarrow SU(2)_{2R} \\ \leftarrow SU(2)_{1R} \end{matrix} \qquad\qquad Q^{\dot +}_{\dot +}$$

Index of Instantons from the D0-dynamics (ADHM)

- Index for BPS states with nonzero instantons $Q = Q^{\dot{+}}_{+}$ $\left.\begin{matrix} SU(2)_{2R} \\ SU(2)_{1R} \end{matrix}\right\} \Rightarrow SU(2)_R$

$$I_k(\mu^i,\gamma_1,\gamma_2,\gamma_3) = \mathrm{Tr}_k\left[(-1)^F e^{-\beta Q^2} e^{-\mu^i \Pi_i} e^{-i\gamma_1(2J_{1L})-i\gamma_2(2J_{2L})-i\gamma_R(2J_R)}\right]$$

adjoint hyper flavor

  - μ_i : chemical potential for $U(1)^N \subset U(N)_{\text{color}}$
  - $\gamma_I, \gamma_2, \gamma_R$: chemical potential for $SU(2)_{1L}$, $SU(2)_{2L}$, $SU(2)_R$
- calculate the index by the localization: $I(q,\mu^i,\gamma_{1,2,3}) = \sum_{k=0}^{\infty} q^k I_k$
- 5d *N=2** instanton partition function on $R^4 \times S^1$: $t \sim t+\beta$
- In $\beta \to 0$ and small chemical potential limit, the index becomes 4d Nekrasov instanton partition function :
- $$a_i = \frac{\mu_i}{2} \quad -\epsilon_1 = i\frac{\gamma_1-\gamma_R}{2} \quad \epsilon_2 = i\frac{\gamma_1+\gamma_R}{2}, \ m = i\frac{\gamma_2}{2} \quad q = e^{2\pi i\tau}$$

instanton fugacity

Scalar Vev

Omega deformation parameter

Adj hypermultiplet mass

D0-dynamics & ADHM formalsm

- k D0 dynamics on N D4: dim. reduction of 10d SYM with U(k) gauge group and N fundamental hypermultiplets.
- turn on the scalar expectation value $\Phi_5 = (v_1, v_2, \ldots\ v_N)$ + FI term
- change of variables for hyper scalars so that there is no time-dependence.
- Choose the twisting

$$Q = \frac{1}{\sqrt{2}}(Q^{\dot{+}}_{\dot{+}} + Q^{\dot{-}}_{\dot{-}})$$

$$\phi = -i(A_\tau + i\varphi_5)$$

- Euclidean time with topological and cohomological setting + chemical potentials
- In the limit β→0 , ζ and ω → ∞, the path integral comes from 1-loop around the saddle points

$$[\phi, B_1] = \frac{i(\gamma_R - \gamma_1)}{\beta} B_1, \ [\phi, B_2] = \frac{i(\gamma_R + \gamma_1)}{\beta} B_2,$$

$$[B_1, B_2] + \bar{x}^{\dot{-}} x_{\dot{+}} = 0, \ [B_1^\dagger, B_1] + [B_2^\dagger, B_2] + \bar{x}^{\dot{+}} x_{\dot{+}} - \bar{x}^{\dot{-}} x_{\dot{-}} = \zeta$$

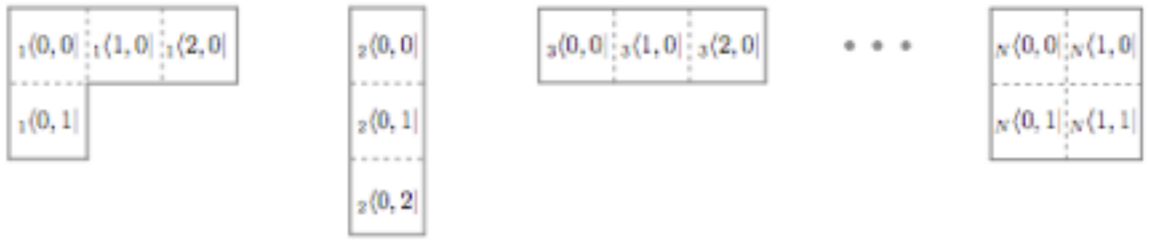
$$x_\pm \phi - \frac{\mu \pm i\gamma_R}{\beta} x_\pm = 0$$

- N-colored Young diagram:

N-colored Young diagram & index

- $$\sum_{i=1}^{N} k_i = k, \;\; k_i = \#\text{of boxes in the i} - \text{th Young diagram}$$

- For example



- Evaluate the Gaussian integral and obtain the instanton index

$$I_{\{Y_1,Y_2,\cdots,Y_N\}} = \prod_{i,j=1}^{N} \prod_{s\in Y_i} \frac{\sinh\frac{E_{ij}-i(\gamma_2+\gamma_R)}{2}\sinh\frac{E_{ij}+i(\gamma_2-\gamma_R)}{2}}{\sinh\frac{E_{ij}}{2}\sinh\frac{E_{ij}-2i\gamma_R}{2}}$$

$$E_{ij} = \mu_i - \mu_j + i(\gamma_1-\gamma_R)h_i(s) + i(\gamma_1+\gamma_R)(v_j(s)+1)$$

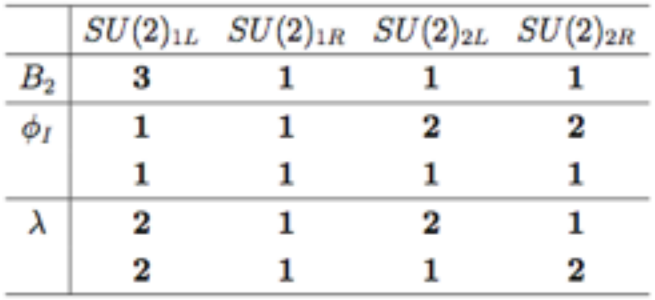
- The index for the instanton

$$I = \sum_{k=0}^{\infty} q^k \sum_{k_i} \sum_{Y_i(k_i)} I_{\{Y_1,Y_2,\cdots Y_N\}}$$

U(1) instantons

- D0‘s on a single D4
- As instantons are KK modes of (2,0) theory, one expects a unique threshold bound state for each instanton number k.
- U(1) index for k=1

$$I_{cm} = \frac{\sin\left(\frac{\gamma_1+\gamma_2}{2}\right)\sin\left(\frac{\gamma_1-\gamma_2}{2}\right)}{\sin\left(\frac{\gamma_1+\gamma_R}{2}\right)\sin\left(\frac{\gamma_1-\gamma_R}{2}\right)}$$



	$SU(2)_{1L}$	$SU(2)_{1R}$	$SU(2)_{2L}$	$SU(2)_{2R}$
B_2	**3**	**1**	**1**	**1**
ϕ_I	**1**	**1**	**2**	**2**
	1	**1**	**1**	**1**
λ	**2**	**1**	**2**	**1**
	2	**1**	**1**	**2**

- U(1) index [Iqbal-Kozcaz-Shabbir 10]

$$I_{U(1)}(q, e^{\gamma_i}) = PE[\frac{q}{1-q} I_{cm}(e^{\gamma_i})], \quad \text{Plythethytic exponential}$$

- Expand the single particle index in q

$$\sum_{k=0}^{\infty} q^k I_{cm}$$

unique threshold bound state

Dyonic Instantons in the Coulomb phase

- Compute the degeneracy of wrapped selfdual strings with momentum.
- $$Index = \mathbf{PI}[I_{cm} z_{sp}(q,\mu,\gamma)]$$
- expand in z_{sp} in instanton number and electric charge;

$$q, \;\; x = e^{-(\mu_1-\mu_2)}$$

- SU(2): x^1 or 1-W boson+ n instantons

$$\prod_{n=1}^{\infty}\frac{(1-q^n e^{i(\gamma_2+\gamma_R)})(1-q^n e^{i(\gamma_2-\gamma_R)})(1-q^n e^{i(-\gamma_2+\gamma_R)})(1-q^n e^{i(-\gamma_2-\gamma_R)})}{(1-q^n e^{i(\gamma_1+\gamma_R)})(1-q^n e^{i(\gamma_1-\gamma_R)})(1-q^n e^{i(-\gamma_1+\gamma_R)})(1-q^n e^{i(-\gamma_1-\gamma_R)})}$$

- Simplification: $z_{sp}(\gamma_1=\gamma_R=0, \gamma_2=\pi)$

$$x^2 \;:\; 0+16q+288q^2+2880q^3+21056q^4+125280q^5+\cdots = q\frac{d}{dq}\left[\prod_{n=1}^{\infty}\frac{(1+q^n)^8}{(1-q^n)^8}\right]$$

$$x^3 \;:\; 0+24q+1272q^2+26952q^3+360696q^4+3605520q^5+\cdots$$

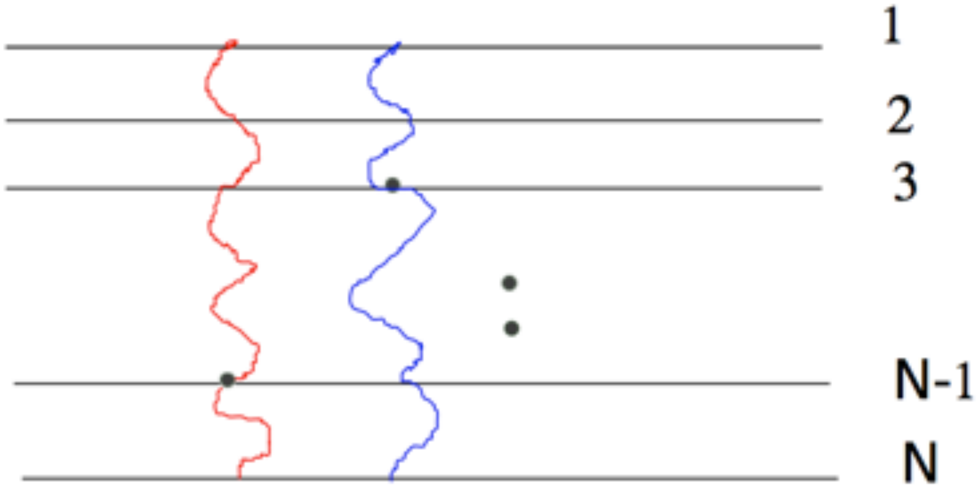
$$x^4 \;:\; 0+32q+4160q^2+169600q^3+3842176q^4+60216000q^5+\cdots$$

$$x^5 \;:\; 0+40q+11080q^2+809760q^3+29471560q^4+692554440q^5+\cdots$$

Find the S-duality calculation:
Study of multiple wrapped
monopole strings + momentum

Dyonic Instantons in U(N) gauge theory

- Single W-boson: $e^{-(\mu_1-\mu_N)}$



$$z_{\rm sp}^{U(3)} = 1+24q+264q^2+2016q^3+12264q^4+63504q^5+290976q^6\cdots$$

$$= \boxed{\prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^4} \times \boxed{\left(1+16q+96q^2+448q^3+1728q^4+5856q^5+18048q^6+\cdots\right)}$$

$= \Psi$

$= \Upsilon$

$$z_{sp}^{U(N)} = \Psi\Upsilon^{N-2}$$

Contribution from degenerated string junctions:
junction points move with speed of light

- 1+1 dim dynamics of monopole strings with momentum are need to produced above results in S-dual version.

Superconformal index

- To get the index in symmetric phase, integrate over $\mu_i = i a_i$ with Haar measure
- DLCQ on null circle: Nonrelativistic superconformal symmetry
  - P_- on the null circle = instanton number [Aharony-Berkooz-Seiberg 97]
  - Superalgebra: $2i\{Q,S\} = iD \mp (4J_{2R} + 2J_{1R}) \ \rightarrow \ iD \geq \pm(4J_{2R} + 2J_{1R})$
- Nonrelativistic superconformal index

$$I_{SC} = \mathrm{Tr}\left[(-1)^F e^{-\beta\{\hat{Q},\hat{S}\}} e^{-2i\gamma_R J_R} e^{-2i\gamma_1 J_{1L} - 2i\gamma_2 J_{2L}} e^{-i\alpha_i \Pi_i}\right]$$

- In the limit β → 0, this superconformal index becomes our index.
- For single instanton with $t = e^{-i\gamma_R}$

$$I_{k=1} = \frac{e^{i\gamma_2} + e^{-i\gamma_2} - e^{i\gamma_1} - e^{-i\gamma_1}}{(1 - te^{i\gamma_1})(1 - te^{-i\gamma_1})}\left[t + \sum_{n=1}^{N-1}(e^{in\gamma_2} + e^{-in\gamma_2})t^{n+1} - \chi_{\frac{N-2}{2}}(\gamma_2)t^{N+1}\right]$$

- Large *N*

$$I_{N\to\infty,k=1} = \frac{e^{i\gamma_2} + e^{-i\gamma_2} - e^{i\gamma_1} - e^{-i\gamma_1}}{(1 - te^{i\gamma_1})(1 - te^{-i\gamma_1})} \frac{t - t^3}{(1 - te^{i\gamma_2})(1 - te^{-i\gamma_2})}$$

AdS7 x S4 calculation confirm it.

4d N=2 SYM on S^3 x S^1 with `t Hooft operators

- Index on S^3 x S^1 [Romelsberger; Kinney,Maldacena,Minwalla,Raju]
- the singular BPS Dirac solutions with nonabelian magnetic charge [`tHooft, E.Weinberg,Kapustin,….]
- include magnetic bubbling (or massless monopoles) [Kapustin,Witten:Weinberg]
- Confirm S-duality: (on S^4, [Gomis,Okuda,Pestun])
- 2d-4d relation [Dimofte,Gaiotto,Gukov;Ito,Okuda,Taki]

‘t Hooft operator on S^3 x S^1 in *N*=4 SYM

$$ds^2_{S^3} = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\varphi^2)$$

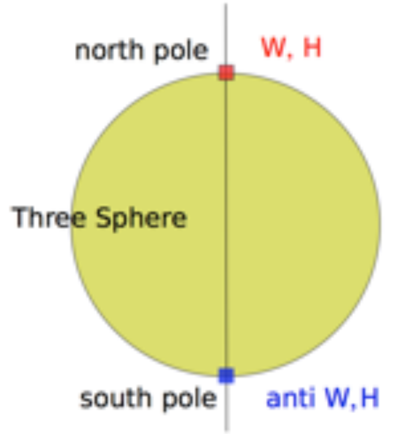
- SO(4)$_{\text{rot}}$ x SU(4)$_{\text{R}}$
- Fields $$A_\mu, X^{AB} \ (\ X^{AB} = \frac{1}{4}\epsilon^{ABCD}(X^{CD})^\dagger), \ \psi^A, \ (A = 1,2,3,4 \ of \ SU(4))$$
- Superconformal: $$Q^{\alpha A}, \ \bar{Q}^{\dot{\alpha}}_A, \ S^\alpha_A, \ \bar{S}^{\dot{\alpha}A}$$

$$\alpha \in (\mathbf{2}, 0), \dot{\alpha} \in (0, \mathbf{2}) \text{ of } SU(2)_L \times SU(2)_R = SO(4)_{rot}$$

- 1/2 BPS Wilson and ‘t Hooft lines:

$$W = \mathrm{Tr}_R \exp i \int dt(A_0 - \phi_9)$$

$$H : F_{ij} = \frac{B}{4}\epsilon_{ijk}\frac{x_k}{|x|^3}, \ \phi_9 = \frac{1}{2}\frac{B}{|x|}$$



$$H \ : \ F = -\frac{B}{2}\sin\theta d\theta \wedge d\varphi$$

$$X_{12} = X^\dagger_{34} = \frac{1}{2}(\phi_6 + i\phi_9) = \frac{iB}{4}\sin\chi$$

$$X^{12} = (X^{34})^\dagger = i\phi_9/2$$

- Preserved supersymmetry: 1/2BPS, choose one supercharge

$$Q = Q^{\alpha=1, A=1} + \bar{Q}^{\dot{\alpha}=1}_{A=2}$$

Locking $SU(2)_R = SU(2)_R$

Index with Line operators

- Supersymmetry

$$Q = Q^{\alpha=1.A=1} + \bar{Q}^{\dot{\alpha}}_{A=2}$$

$$\Delta = \{Q, Q^{\dagger}\} = \epsilon - (j_L + j_R) - r_1$$

  - U(1) charge :

$$r_1 = \mathrm{diag}(1, -1, 0, 0), \quad A = 1, 2, 3, 4$$

  - commuting charge: $[Q, \epsilon + (j_L + j_R)] = 0$

- Index:

$$\mathrm{Index}_{\mathcal{H}_L}(x, \eta_a) = \mathrm{Tr}_{\mathcal{H}_L}(-1)^F x^{\epsilon + j_L + j_R} \prod_a \eta_a^{h_a}$$

  - Chemical potential: N=4 r_2 = diag(0,0,1,-1)
  - trace over the BPS states with $\Delta = \varepsilon - (j_L + j_R) - r_1 = 0$

- Euclidean Path Integral

BPS Fluctuations around `t Hooft line

- Classical corrections vanishes with boundary terms.
- 1-loop: harmonic analysis around ‘t Hooft line:
- taken into account A_0 along S^1

$$F = -\frac{B}{2}\sin\theta d\theta \wedge d\varphi\ ,$$

$$X_{12} := X_{34}^{\dagger} = \frac{1}{2}(X_6 + iX_9) = \frac{B}{4\sin\chi}$$

- X_{13}, X_{14} :

$$M_{q_\alpha}^2 = -\partial_t^2 - \nabla_{S^3} + 1 + \frac{q^2}{\sin^2\chi},\ (q_\alpha = \alpha(B)/2),$$

	ϵ	$j_L + j_R$	r_1	r_2	$e^{i\lambda}$
$X^{13,\alpha}_{n,J,m}$	$n+1$	m	1	1	$e^{i\alpha(\lambda)}$
$X^{14,\alpha}_{n,J,m}$	$n+1$	m	1	-1	$e^{i\alpha(\lambda)}$

$$J = |q|, |q|+1, \ldots,\ |m| \leq J,\ \ n = J, J+1, \ldots$$

- BPS Δ =0 for m=J, n=J=|q|,|q|+1,… and index: $x^{2J+1}=x^{2|q|+1}$, $x^{2|q|+3}$,…
- index with chemical potential η for R-charge r_2 and chemical potential for the gauge group

$$I_{sp;\mathcal{L}_1}(e^{i\lambda_i}, x, \eta) = (\eta + \eta^{-1})\sum_{\alpha}\sum_{n=|q_\alpha|} x^{2n+1}e^{i\alpha(\lambda)} = (\eta+\eta^{-1})\sum_{\alpha}\frac{x\cdot x^{|\alpha(B)|}e^{i\alpha(\lambda)}}{1-x^2}$$

$\{e^{i\lambda_i}\}|_{i=1,\ldots,\mathrm{rank}(G)}$ for Cartan algebra basis $\{H_i\}$

Fermionic Fluctuations around `t Hooft line

- on ψ^1 , ψ^2 $\quad \not{M}_q = \begin{pmatrix} i\not{D}_q & q/\sin\chi \\ q/\sin\chi & -i\not{D}_q \end{pmatrix}$

	ϵ	$j_L + j_R$	r_1	r_2	$e^{i\lambda}$
$\psi^{\alpha;\pm,\kappa}_{n,J,m}$	$n+1$	m	1	0	$e^{i\alpha(\lambda)}$

$J = |q| - \frac{1}{2}$ (exist for $|q| \neq 0), |q| + \frac{1}{2}, |q| + \frac{3}{2},$
$|m| \leq J,\ n = J, J+1 \ldots$

- BPS Δ =0 for
  - m=J, n=J=|q| -1/2 with κ=1 : index $-x^{2J+1}=x^{2|q|}$
  - m=J, n=J=|q| +1/2, |q|+3/2,… with κ=1, 2 : index $-x^{2J+1}=-x^{2|q|+2}$, $-x^{2|q|+4}$,...
- index from ψ^1 , ψ^2

$$I_{sp;\mathcal{L}_2}(e^{i\lambda_i}, x, \eta) = \sum_\alpha [\sum_{J=|q_\alpha|+\frac{1}{2}}^{\infty} (-2x^{2J+1}e^{i\alpha(\lambda)}) - (1-\delta_{q_\alpha,0})x^{2|q_\alpha|}e^{i\alpha(\lambda)}]$$
$$= \sum_\alpha (\frac{-2x^2 \cdot x^{|\alpha(B)|}}{1-x^2} - (1-\delta_{\alpha(B),0})x^{|\alpha(B)|})e^{i\alpha(\lambda)} .$$

- Index from ψ^3 , ψ^4 and A_i, X_{12} : no bps fluctuations and contributions

Index for N=4 theory

- single particle index

$$\tilde{I}_{sp}(e^{i\lambda_i},x,\eta)=\sum_{i=1}^{4} I_{sp;\mathcal{L}_i}$$

$$=\sum_{\alpha}\Big(\frac{(\eta+\eta^{-1})x\cdot x^{|\alpha(B)|}}{1-x^2}-\frac{2x^2\cdot x^{|\alpha(B)|}}{1-x^2}\Big)e^{i\alpha(\lambda)}-\sum_{\alpha(B)\neq 0} e^{i\alpha(\lambda)}x^{|\alpha(B)|}\,.$$

- multi particle index: Plethystic exponetial (P.E): P.E[f(x)]=exp[$\Sigma_{n=1}$ f(x^n)/n]

$$I_{\mathrm{multi}}(e^{i\lambda_i},x,\eta)=\mathrm{P.E}[\tilde{I}_{sp}(e^{i\lambda_i},x,\eta)]$$

- the Haar measure for the unbroken gauge group G_B ={g: g ∈ G and [g,B]=0}
- index for N=4 theory

$$I_B^{1-loop}(x,\eta)=\int [dU]_B Z_B^{1-loop}(e^{i\lambda_i},x,\eta)\ ,\ \text{where}$$

$$[dU]_B\equiv\frac{1}{\mathrm{sym}(B)}\Big(\prod_{i=1}^{\mathrm{rank}(G)}\frac{d\lambda_i}{2\pi}\Big)\prod_{\alpha\neq 0}(1-e^{i\alpha(\lambda)}x^{|\alpha(B)|})\ ,\ \text{and}$$

$$Z_B^{1-loop}(e^{i\lambda_i},x,\eta):=\mathrm{P.E}[I_{sp}(e^{i\lambda_i},x,\eta)]\ ,\ \text{with}$$

$$I_{sp}(e^{i\lambda_i},x,\eta)=\sum_{\alpha=1}^{\dim(G)}\Big(\boxed{\frac{(\eta+\eta^{-1})x\cdot x^{|\alpha(B)|}}{1-x^2}}-\boxed{\frac{2x^2\cdot x^{|\alpha(B)|}}{1-x^2}}\Big)e^{i\alpha(\lambda)}$$

scalar of N=2 adjoint hypermultiplet gluino of N=2 vector multiplet

N=4 U(2) S-duality check

- Wilson line index with out magnetic charge (at the stationary point Φ=0) es)

$$I_{\mathcal{N}=4;B=0}(x)=\int[dU]\,\chi_R(e^{i\lambda})\chi_R(e^{-i\lambda})\exp\left(\sum_\alpha\sum_{n=1}^{\infty}\frac{1}{n}\frac{2x^n}{1+x^n}e^{in\alpha(\lambda)}\right)$$

- fundamental and anti-fundamental at north and south pole (η=1)

$$I_{A_1}^{U(2)}(x)=\frac{1}{2}\int_0^{2\pi}\frac{d\lambda_1 d\lambda_2}{(2\pi)^2}(1-e^{i(\lambda_1-\lambda_2)})(1-e^{-i(\lambda_1-\lambda_2)})(e^{i\lambda_1}+e^{i\lambda_2})(e^{-i\lambda_1}+e^{-i\lambda_2})$$
$$\times\mathrm{P.E}\Big[\frac{2x}{1+x}(2+e^{i(\lambda_1-\lambda_2)}+e^{-i(\lambda_1-\lambda_2)}\Big]$$

- `t Hooft operator with magnetic charge B=(1,0)

$$I_{B=(1,0)}^{U(2)}(x)=\int_0^{2\pi}\frac{d\lambda_1 d\lambda_2}{(2\pi)^2}(1-xe^{i(\lambda_1-\lambda_2)})(1-xe^{-i(\lambda_1-\lambda_2)})$$
$$\times\mathrm{P.E}\Big[\frac{2x}{1+x}(2+xe^{i(\lambda_1-\lambda_2)}+xe^{-i(\lambda_1-\lambda_2)}\Big]$$

- expansion in x

$$I_{R=A_1}^{U(2)}=I_{B=(1,0)}^{U(2)}$$
$$=1+2(\eta+\eta^{-1})x+(1+3\eta^2+3\eta^{-2})x^2+4(\eta^3+\eta^{-3})x^3+(1+5\eta^4+5\eta^{-4})x^4$$
$$+(6\eta^{-5}+2\eta^{-1}+2\eta+6\eta^5)x^5+(7\eta^{-6}+\eta^{-2}-1+\eta^2+7\eta^6)x^6+\ldots,$$

N=2 Index

- Hypermultiplets in the representation R_i : scalars, weight $\rho \in R_i$

$$\sum_{\alpha} \frac{(\eta+\eta^{-1})x \cdot x^{|\alpha(B)|}}{1-x^2} e^{i\alpha(\lambda)} \to \sum_{i} \sum_{\rho \in R_i} \frac{x \cdot x^{|\rho(B)|}}{1-x^2} (e^{i\rho(\lambda)} \prod_{a} \eta_a^{h_{i,a}} + e^{-i\rho(\lambda)} \prod_{a} \eta_a^{-h_{i,a}})\,.$$

- index for the \`t Hooft line

$$I_B^{1-loop}(x,\eta_a) = \int [dU]_B Z_B^{1-loop}(e^{i\lambda_i}, x, \eta_a)\,, \text{ where}$$

$$Z_B^{1-loop}(x,\eta_a,e^{i\lambda_i}) = \mathrm{P.E}[I_{sp}(e^{i\lambda_i},x,\eta)]\,,\ I_{sp} = I_{sp}^{vec} + I_{sp}^{hyper} \text{ with}$$

$$I_{sp}^{vec} = -2\sum_{\alpha} \frac{x^2 \cdot x^{|\alpha(B)|}}{1-x^2} e^{i\alpha(\lambda)} \text{ and}$$

$$I_{sp}^{hyper} = \sum_{i} \sum_{\rho \in P_i} \frac{x \cdot x^{|\rho(B)|}}{1-x^2} (e^{i\rho(\lambda)} \prod_{a} \eta_a^{h_{i,a}} + e^{-i\rho(\lambda)} \prod_{a} \eta_a^{-h_{i,a}})\,.$$

- Wilson line

$$I_R = \int [dU]_{B=0} \chi_R(e^{i\lambda}) \chi_{\bar{R}}(e^{i\lambda}) Z_{B=0}^{1-loop}(e^{i\lambda_i}, x, \eta_a)\,.$$

Minuscule Representation

- All weights are related by Weyl reflections: the corresponding B cannot be screened and there is no magnetic bubbling.
- Totally anti-symmetric representation in U(N)
- Index for Wilson lines.

$$I^{U(N)}_{R=A_k}(x,\eta) = \frac{1}{N!}\int \prod_{i=1}^{N}(\frac{d\lambda_i}{2\pi})(\prod_{i\neq j}(1-e^{i(\lambda_i-\lambda_j)}))\mathrm{P.E}[\frac{(\eta+\eta^{-1})x-2x^2}{1-x^2}\sum_{i,j=1}^{N}e^{i(\lambda_i-\lambda_j)}]$$

$$\times\prod_{\pm}(\sum_{1\leq i_1<i_2\ldots<i_k\leq N}e^{\pm i(\lambda_{i_1}+\lambda_{i_2}+\ldots+\lambda_{i_k})})\ . \tag{4.1}$$

- Index for `t Hooft lines

$$I^{U(N)}_{B=(1^k,0^{N-k})}(x,\eta) = \frac{1}{k!(N-k)!}\int\prod_{i=1}^{N}(\frac{d\lambda_i}{2\pi})$$

$$\times\prod_{(i\neq j)=1}^{k}(1-e^{i(\lambda_i-\lambda_j)})\prod_{(i\neq j)=k+1}^{N}(1-e^{i(\lambda_i-\lambda_j)})\prod_{i=1}^{k}\prod_{j=k+1}^{N}\prod_{\pm}(1-xe^{\pm i(\lambda_i-\lambda_j)})$$

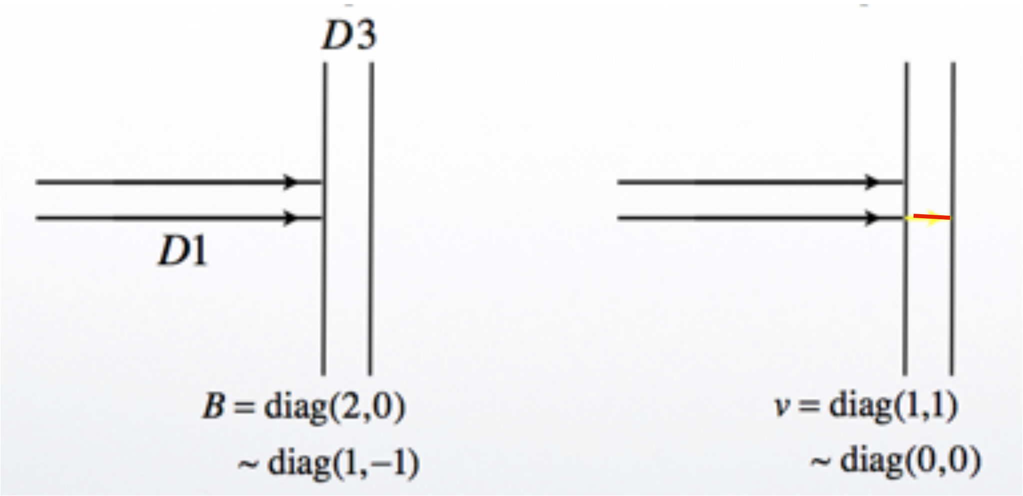
$$\times\mathrm{P.E}\Big[\frac{(\eta+\eta^{-1})x-2x^2}{1-x^2}\Big((\sum_{i,j=1}^{k}+\sum_{i,j=k+1}^{N})e^{i(\lambda_i-\lambda_j)}+\sum_{i=1}^{k}\sum_{j=k+1}^{N}\sum_{\pm}e^{\pm i(\lambda_i-\lambda_j)}x\Big)\Big]\ . \tag{4.2}$$

- Example: B=(1,1,0,0) of U(4)

$$\begin{aligned}
&I^{U(4)}_{R=A_2} = I^{U(4)}_{B=(1,1,0,0)}\\
&= 1+2(\eta+\eta^{-1})+(3+5\eta^{-2}+5\eta^{2})x^2+(8\eta^{-3}+6\eta^{-1}+6\eta+8\eta^{3})x^3\\
&\quad+(14\eta^{-4}+7\eta^{-2}+10+7\eta^{2}+14\eta^{8})x^4+10(2\eta^{-5}+\eta^{-3}+\eta^{-1}+\eta+\eta^{3}+2\eta^{5})x^5+\ldots,
\end{aligned}$$

Magnetic bubbling

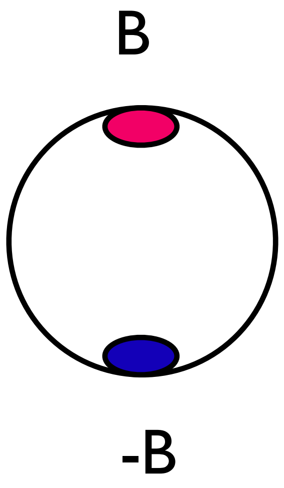
- With unbroken gauge group, nonabelian monopole can be screened by massless monopoles: $B \to \nu$

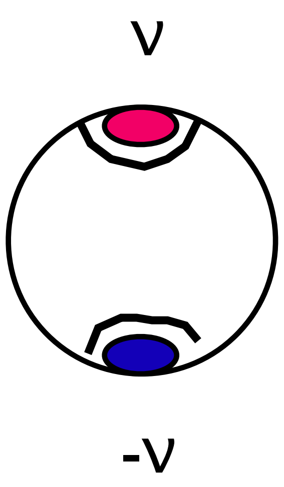


  - Contributions from massless monopoles
  - One could assign the `t Hooft operator of magnetic charge B to the representation of the magnetic group ^{L}G, regarding B as the highest weight.
- But it is not clear how all weights of given magnetic group are realized.
- Also global color problem in the presence of nonabelian magnetic charge

Index with bubbling

- On S3, massless monopoles and anti-monopoles can be created and shield singular and singular anti-monopoles at north and south poles.





$$\int [dU]_B Z_B^{1-loop}(x,e^{i\lambda}) \qquad \int [dU]_\nu Z_{mono}^{S^3}(B,\nu;x,e^{i\lambda}) Z_\nu^{1-loop}(x,e^{i\lambda})$$

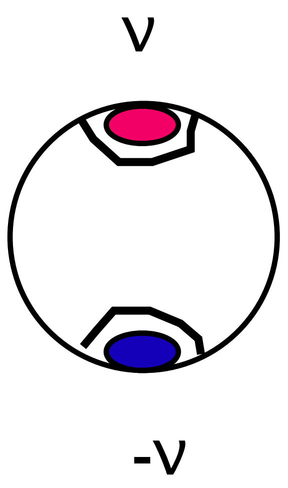
- Final result

$$I_B(x,\eta_a) = \sum_{v\in \mathrm{Rep}(B)} \int [dU]_v Z_{mono}^{S^3}(B,v;e^{i\lambda_i},x,\eta_a) Z_v^{1-loop}(e^{i\lambda_i},x,\eta_a)$$

$$Z_{mono}^{S^3}(B,B;x,e^{i\lambda}) = 1 \text{ (no screening effect)}$$

Magnetic bubbling index

- How to calculate $Z^{S^3}_{mono}(B, \nu; x, e^{i\lambda})$



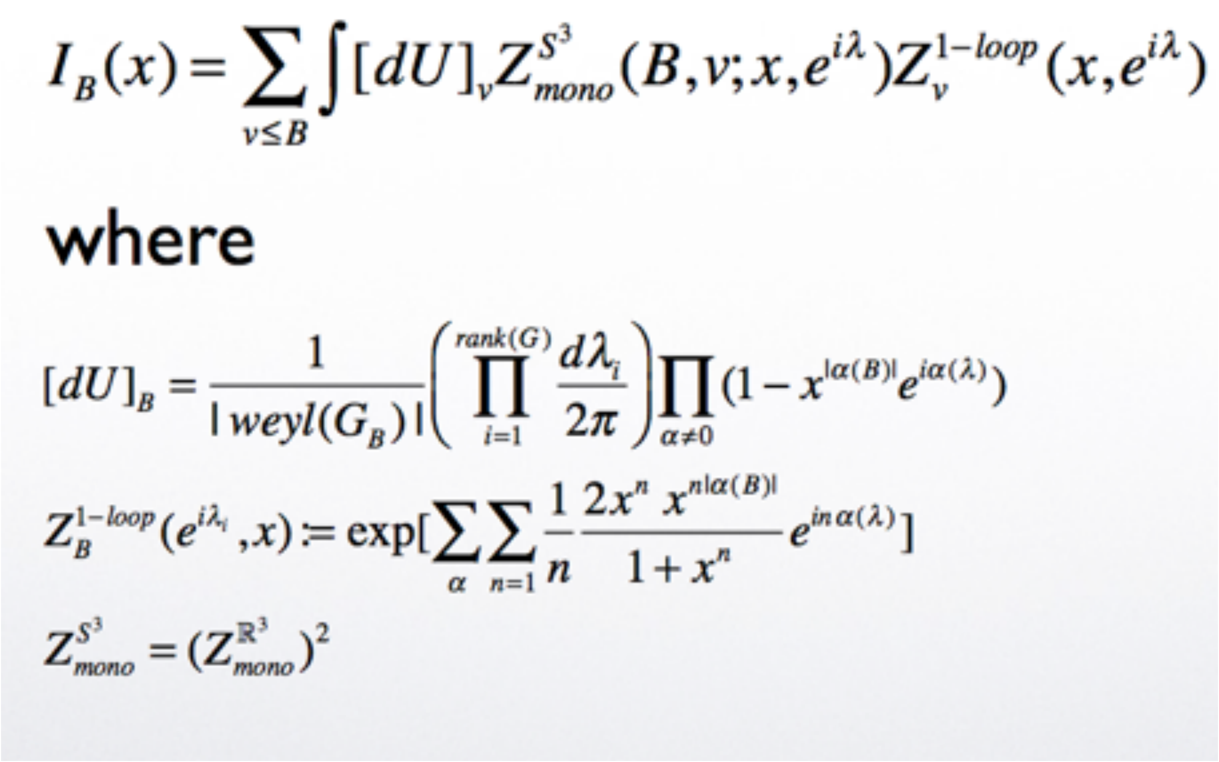
- Note that
  - bubbling happens at two poles
  - locally, $S^3 \sim R^3$ near two poles
- Guess

$$Z^{S^3}_{mono}(B,v) = Z^{S}_{mono}(B,v) Z^{N}_{mono}(B,v) \qquad Z^{(N,S)}_{mono}(B;v) = Z^{\mathbb{R}^3}_{mono}(B,v)$$

- Recently it has been calculated for U(N): [Gomis,Okuda,Pestun;Ito,Okuda,Taki]
  - relating singular and massless monopoles to instantons [Kronheimer]
  - complicated sum over colored Young diagrams
  - S^4 , R^3 x S^1 , S^3 x S^1 are all different.

Magnetic bubbling index

- final index



$$I_B(x) = \sum_{v \le B} \int [dU]_v Z_{mono}^{S^3}(B,v;x,e^{i\lambda}) Z_v^{1-loop}(x,e^{i\lambda})$$

where

$$[dU]_B = \frac{1}{|weyl(G_B)|}\left(\prod_{i=1}^{rank(G)} \frac{d\lambda_i}{2\pi}\right)\prod_{\alpha\neq 0}(1 - x^{|\alpha(B)|}e^{i\alpha(\lambda)})$$

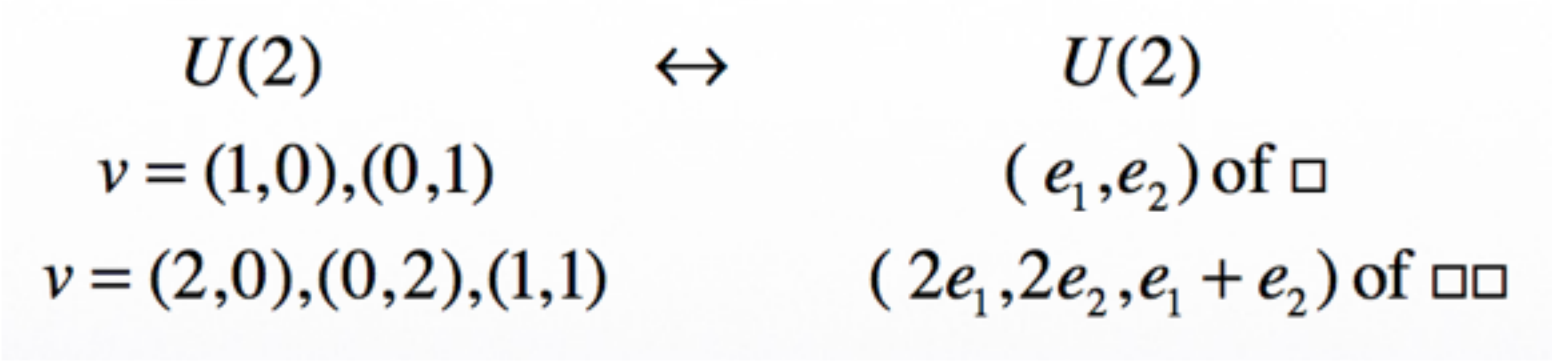
$$Z_B^{1-loop}(e^{i\lambda_i},x) := \exp[\sum_{\alpha}\sum_{n=1}\frac{1}{n}\frac{2x^n\, x^{n|\alpha(B)|}}{1+x^n}e^{in\alpha(\lambda)}]$$

$$Z_{mono}^{S^3} = (Z_{mono}^{\mathbb{R}^3})^2$$

- Meaning of v ≤ B :

G	↔	^{L}G
magnetic charge B		highest weight ^{L}R
v ≤ B		descendants

- example



$U(2)$	↔	$U(2)$
$v = (1,0),(0,1)$		(e_1, e_2) of □
$v = (2,0),(0,2),(1,1)$		$(2e_1, 2e_2, e_1 + e_2)$ of □□

index with bubbling (N=4 case)

- B=(2,0), v=(1,1) in U(2)

$$Z^{S^3;U(2)}_{mono}(B,v) = [\frac{1-2x^2+x^4+(\eta^{-1}+\eta)(x+x^3)-2x^2(e^{-i(\lambda_1-\lambda_2)}+e^{i(\lambda_1-\lambda_2)})}{(1-e^{-i(\lambda_1-\lambda_2)}x^2)(1-e^{i(\lambda_1-\lambda_2)}x^2)}]^2\,,$$

- index with bubbling

$$\begin{aligned} I^{U(2)}_{B=(2,0)}(x,\eta) = &\int_0^{2\pi}\frac{d\lambda_1 d\lambda_2}{(2\pi)^2}(1-e^{i(\lambda_1-\lambda_2)}x^2)(1-e^{-i(\lambda_1-\lambda_2)}x^2) \\ &\times \mathrm{P.E}[\frac{(\eta+\eta^{-1})x-2x^2}{1-x^2}(2+e^{i(\lambda_1-\lambda_2)}x^2+e^{-i(\lambda_1-\lambda_2)}x^2)] \\ &\boxed{\begin{aligned} &+\frac{1}{2}\int_0^{2\pi}\frac{d\lambda_1 d\lambda_2}{(2\pi)^2}Z^{S^3;U(2)}_{mono}(B,v)(1-e^{i(\lambda_1-\lambda_2)})(1-e^{-i(\lambda_1-\lambda_2)}) \\ &\times \mathrm{P.E}[\frac{(\eta+\eta^{-1})x-2x^2}{1-x^2}(2+e^{i(\lambda_1-\lambda_2)}+e^{-i(\lambda_1-\lambda_2)})]\,. \end{aligned}} \end{aligned}$$

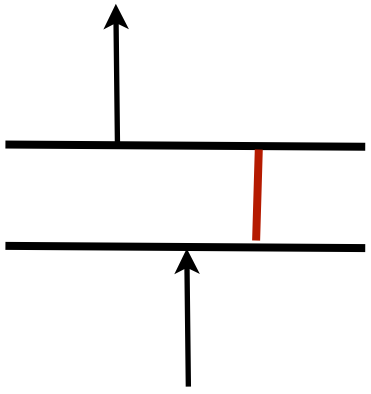
- index for Wilson line

$$\begin{aligned} I^{U(2)}_{R=(1,0)^2} = &\frac{1}{2}\int_0^{2\pi}\frac{d\lambda_1 d\lambda_2}{(2\pi)^2}(1-e^{i(\lambda_1-\lambda_2)})(1-e^{-i(\lambda_1-\lambda_2)})(e^{i\lambda_1}+e^{i\lambda_2})^2(e^{-i\lambda_1}+e^{-i\lambda_2})^2 \\ &\times \mathrm{P.E}[\frac{(\eta+\eta^{-1})x-2x^2}{1-x^2}(2+e^{i(\lambda_1-\lambda_2)}+e^{-i(\lambda_1-\lambda_2)})]\,. \qquad (4.39) \end{aligned}$$

- expansion in x:

$$\begin{aligned} &I^{U(2)}_{B=(2,0)}(x,\eta) = I^{U(2)}_{R=(1,0)^2}(x,\eta) \\ &= 2+5(\eta+\eta^{-1})x+(4+8\eta^2+8\eta^{-2})x^2+(11\eta^3+\eta^{-1}+\eta+11\eta^{-3})x^3 \\ &\quad +(4+14\eta^{-4}+14\eta^4)x^4+(17\eta^{-5}+6\eta^{-1}+6\eta+17\eta^5)x^5+\ldots \end{aligned}$$

N=4 SU(2) gauge group



- SU(2) magnetic charge: B=(p,-p)/2,
- bubbling v= (v,-v)/2 and (B,v)= (p,v), v = p- even integer
- SU(2) bubbling index

$$Z^{N;SU(2)}_{mono}(2,0) = \frac{(1-2x^2+x^4)+(\eta^{-1}+\eta)(x+x^3)-2x^2(e^{-2i\lambda}+e^{2i\lambda})}{(1-e^{-2i\lambda}x^2)(1-e^{2i\lambda}x^2)}$$

$$Z^{N;SU(2)}_{mono}(3,1) = \frac{2(1-x^2-x^4+x^6)+(\eta^{-1}+\eta)(x+x^3+x^5)-3x^3(e^{-2i\lambda}+e^{2i\lambda})}{(1-e^{-2i\lambda}x^3)(1-e^{2i\lambda}x^3)}.$$

- S-duality

$$\begin{aligned}
&I^{SU(2)}_{B=\mathrm{diag}(1,-1)}(x,\eta) = I^{SU(2)}_{R=(1,0)^2}(x,\eta)\\
&= 2+3(\eta^{-1}+\eta)x+3(\eta^{-2}+\eta^2)x^2+(3\eta^{-3}-2\eta^{-1}-2\eta+3\eta^3)x^3\\
&\quad +(3\eta^{-4}-\eta^{-2}+2-\eta^2+3\eta^4)x^4+(3\eta^{-5}+\eta^{-1}+\eta+3\eta^5)x^5+\ldots,\\
&I^{SU(2)}_{B=\mathrm{diag}(\frac{3}{2},-\frac{3}{2})}(x,\eta) = I^{SU(2)}_{R=(1,0)^3}(x,\eta)\\
&= 5+9(\eta^{-1}+\eta)x+(10\eta^{-2}+1+10\eta^2)x^2+(10\eta^{-3}-5\eta^{-1}-5\eta+10\eta^3)x^3\\
&\quad +(10\eta^{-4}-4\eta^{-2}+6-4\eta^2+10\eta^4)x^4+(10\eta^{-5}-\eta^{-3}+3\eta^{-1}+3\eta-\eta^3+10\eta^5)x^5\ldots.
\end{aligned}$$

S-duality in $\mathcal{N}=2$ SU(2) with 4 flavors

- minimal magnetic charge B=(2,-2)/2 is not miniscule.
- bubbling can be obtained by 2d-4d correspondence [iso,okuda,taki]

$$Z_{mono}^{N;SU(2),N_f=4}(2,0) = -\frac{(x^2+\prod_i \eta_i)}{x\prod \eta_i^{1/2}} + \sum_{s=\pm 1} \frac{\prod_{i=1}(xe^{is\lambda}-\eta_i)}{x(1-e^{2is\lambda})(1-x^2e^{2is\lambda})\prod_i \eta_i^{1/2}}$$

- index for the `t Hooft line with B=(2,-2)/2

$$\begin{aligned} I_{B=(1,-1)}^{SU(2);N_f=4}(x,\eta_i) &= \int \frac{d\lambda}{2\pi}(1-e^{2i\lambda}x^2)(1-e^{-2i\lambda}x^2) \\ &\times \mathrm{P.E}\Big[\frac{x(e^{i\lambda}x+e^{-i\lambda}x)}{1-x^2}\sum_i(\eta_i+\eta_i^{-1}) - \frac{2x^2}{1-x^2}(e^{2i\lambda}x^2+e^{-2i\lambda}x^2+1)\Big] \\ &+ \frac{1}{2}\int \frac{d\lambda}{2\pi}(1-e^{2i\lambda})(1-e^{-2i\lambda})\big(Z_{mono}^{N;SU(2),N_f=4}(2,0)\big)^2 \\ &\times \mathrm{P.E}\Big[\frac{x(e^{i\lambda}+e^{-i\lambda})}{1-x^2}\sum_i(\eta_i+\eta_i^{-1}) - \frac{2x^2}{1-x^2}(e^{2i\lambda}+e^{-2i\lambda}+1)\Big] \,. \end{aligned}$$

- index for minimal Wilson line

$$\begin{aligned} I_{R=\mathbf{2}}^{SU(2);N_f=4}(x,\eta_i) &= \frac{1}{2}\int \frac{d\lambda}{2\pi}(1-e^{2i\lambda})(1-e^{-2i\lambda})(e^{i\lambda}+e^{-i\lambda})^2 \\ &\times \mathrm{P.E}\Big[\frac{x(e^{i\lambda}+e^{-i\lambda})}{1-x^2}\sum_i(\eta_i+\eta_i^{-1}) - \frac{2x^2}{1-x^2}(e^{2i\lambda}+e^{-2i\lambda}+1)\Big] \,. \end{aligned}$$

S-duality SU(2) with 4 flavors

- turn off the chemical potential: η_i=1

$$I^{SU(2);N_f=4}_{B=(1,-1)}(x,\eta_i)|_{\eta_i=1} = I^{SU(2);N_f=4}_{R=\mathbf{2}}(x,\eta_i)|_{\eta_i=1}$$
$$= 1 + 62x^2 + 896x^4 + 7868x^6 + 51856x^8 + 281836x^{10} + 1328923x^{12}$$
$$+ 5611146x^{14} + 21671145x^{16} + 77725908x^{18} + 261809269x^{20} + \dots \ .$$

- With chemical potential:

$$I^{SU(2);N_f=4}_{B=(1,-1)}(x,\eta_i) = I^{SU(2);N_f=4}_{R=\mathbf{2}}(x,\tilde{\eta}_i) \ , \ \text{where}$$
$$\tilde{\eta}_1 = \sqrt{\eta_1\eta_2\eta_3\eta_4} \ , \ \tilde{\eta}_2 = \frac{\sqrt{\eta_1\eta_2}}{\sqrt{\eta_3\eta_4}} \ , \ \tilde{\eta}_3 = \frac{\sqrt{\eta_2\eta_4}}{\sqrt{\eta_1\eta_3}} \ , \ \tilde{\eta}_4 = \frac{\sqrt{\eta_1\eta_4}}{\sqrt{\eta_2\eta_3}} \ .$$

- Half-Index and Verlinde Loop Operators:….

Concluding Remarks

- For the N3 d.o.f. on M5 branes, we need to explore further. Maybe the understanding of the central charge in Toda model (via AGT) may help.

- For the instanton calculations, there are more information to be extracted.
    - Study of magnetic monopole strings and confirmation of the instanton calculation
    - Some hints about N3 has appeared.
    - Further exploration along DLCQ of (2,0) is necessary.

- For 4d-index with `t Hooft operators, the magnetic bubbling and Verlinde operators are related and need further study.

- Obviously there are more counting in 4,5,6 dim.