Recent developments of supersymmetric gauge theories on lattice

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Based on the papers

- F. S., JHEP **0401** (2004) 015 [hep-lat/0311021] ,JHEP **0403** (2004) 067 [hep-lat/0401017].
- M. Hanada, S. Matsuura, F. S., Prog. Theor. Phys. **126** (2012) 597 [arXiv:1004.5513 [[hep-lat]], Nucl. Phys. B **857** (2012) 335 [arXiv:1109.6807 [hep-lat]].
- M. Hanada, S. Matsuura, F. S., H. Suzuki, work in progress.

1 Introduction

$\diamondsuit$ I will talk about lattice formulations of SUSY gauge theories.

QFT on lattice:

- UV regularization
- More importantly, nonperturbative construction of QFT
- Nonperturbative study by computer simulation

Lattice cannot keep all the symmetries of the continuum QFT:

$$S^{\rm lat} = S^{\rm cont} + \tilde{S} \qquad \text{(when lattice spacing } a \text{ is small)}$$

$$\uparrow$$

$\mathcal{O}(a)$ irrelevant terms at the classical level.

- But, quantum mechanically, some operators in $\tilde{S}$ may become relevant due to radiative corrections. (They are symmetry-breaking relevant operators.)

$\Rightarrow$ In order to obtain the correct continuum theory, we should subtract the symmetry-breaking relevant operators in advance. (fine-tuning)

Lattice models with no fine-tuning is practically best, (although a few operators to be fine-tuned are perhaps acceptable).

- We consider the case that lattice actions $S^{\rm lat}$ have "good symmetries" that protect from generating symmetry-breaking radiative corrections.
  $\longrightarrow$ fine-tuning free

- For some (lower-dimensional) gauge theories with extended SUSY, "nilpotent" SUSY (not generating translations) can play a role of the "good symmetries". [Kaplan et al, Catterall, F. S.]

  Typical examples)

  – 2d $\mathcal{N} = (2,2), (4,4), (8,8)$ SYM

  – 2d $\mathcal{N} = (2,2)$ SQCD, gauged linear sigma models

◇ Although such lattice theories have only the "nilpotent" part of SUSY, they restore the full SUSY in the continuum limit.

- It is analytically shown for all orders in perturbation theory.
- Nonperturbative check by computer simulation for 2d $\mathcal{N} = (2, 2)$ SYM with $G = SU(2), SU(3), SU(4), SU(5)$.

[Kanamori-Suzuki, Hanada-Kanamori]

However, this approach seems not so good to obtain 4d SUSY gauge theories from the lattice.

$\diamondsuit$ Here, I will explain a method to construct 4d $\mathcal{N} = 4$ SYM with no fine-tuning:

1. a lattice formulation for 2d $\mathcal{N} = (8,8)$ $U(N)$ SYM by plane-wave like mass deformation in Sections 2, 3.
2. a procedure to obtain 4d $\mathcal{N} = 4$ $U(k)$ SYM from fuzzy S^2 background of the 2d theory in Sections 4, 5. (k: arbitrary)

c.f.) Maldacena, Sheikh-Jabbari, van Raamsdonk
Fuzzy S^2 background in BMN Matrix Model (1d) $\Rightarrow$ 3d $\mathcal{N} = 8$ SYM

c.f.) Ishii, Ishiki, Shimasaki, Tsuchiya
BMN Matrix Model (1d) $\Rightarrow$ 4d planar ($k = \infty$) $\mathcal{N} = 4$ SYM on $\mathbf{R} \times S^3$

2 Mass-deformed 2d $\mathcal{N} = (8,8)$ SYM

$$
\begin{aligned}
\text{2d } \mathcal{N} = (8,8) \text{ SYM} &\Rightarrow \text{BTFT form [Dijkgraaf-Moore, Blau-Thompson,...]} \\
A_\mu &\Rightarrow A_\mu \\
X_I \quad (I = 3,\cdots,10) &\Rightarrow \begin{cases} X_i & (i = 3,4) \\ B_{\mathrm{A}} & (\mathrm{A} = 1,2,3) \\ C = 2X_8, \ \phi_\pm = X_9 \pm i X_{10} \end{cases} \\
\Psi &\Rightarrow \begin{cases} \psi_{+\mu}, \ \rho_{+i}, \ \chi_{+\mathrm{A}}, \ \eta_+ \\ \psi_{-\mu}, \ \rho_{-i}, \ \chi_{-\mathrm{A}}, \ \eta_- \end{cases}
\end{aligned}
$$

We take appropriate two supercharges Q_+, Q_- to write the action of 2d $\mathcal{N} = (8,8)$ SYM in the Q_+Q_- exact form:

$$
\begin{aligned}
S_0 &= Q_+Q_-\mathcal{F}_0, \\
\mathcal{F}_0 &= \frac{1}{g_{2d}^2}\int d^2x \, \mathrm{tr}\left[-iB_{\mathrm{A}}\Phi_{\mathrm{A}} - \frac{1}{3}\epsilon_{\mathrm{ABC}}B_{\mathrm{A}}[B_{\mathrm{B}}, B_{\mathrm{C}}]\right. \\
&\qquad \left. -\psi_{+\mu}\psi_{-\mu} - \rho_{+i}\rho_{-i} - \chi_{+\mathrm{A}}\chi_{-\mathrm{A}} - \frac{1}{4}\eta_+\eta_-\right]
\end{aligned}
$$

with

$$\Phi_1 = 2(-D_1 X_3 - D_2 X_4),$$
$$\Phi_2 = 2(-D_1 X_4 + D_2 X_3),$$
$$\Phi_3 = 2(-F_{12} + i[X_3, X_4]).$$

$\diamondsuit$ $Q_\pm$ SUSY:

$$Q_\pm A_\mu = \psi_{\pm\mu}, \cdots$$
$$Q_\pm X_i = \rho_{\pm i}, \cdots$$
$$Q_\pm B_{\mathtt{A}} = \chi_{\pm\mathtt{A}}, \cdots$$
$$Q_\pm C = \eta_\pm, \quad Q_\pm \eta_\pm = \pm[\phi_\pm, C], \quad Q_\mp \eta_\pm = \mp[\phi_+, \phi_-],$$
$$Q_\pm \phi_\pm = 0, \quad Q_\mp \phi_\pm = \mp \eta_\pm.$$

$\Rightarrow$ Nilpotency up to gauge transformations

$$\begin{aligned} Q_+^2 &= \text{(infinitesimal gauge transformation by } \phi_+\text{)}, \\ Q_-^2 &= \text{(infinitesimal gauge transformation by } -\phi_-\text{)}, \\ \{Q_+, Q_-\} &= \text{(infinitesimal gauge transformation by } C\text{)}. \end{aligned}$$

$\Rightarrow$ Since $\mathcal{F}_0$ is gauge invariant, S_0 is manifestly invariant under Q_+ and Q_- SUSYs.

◇ $SU(2)_R$ symmetry (J_0, J_{++}, J_{--}) J_0-charge

Doublets : $\begin{pmatrix} \psi_{+\mu} \\ \psi_{-\mu} \end{pmatrix}, \quad \begin{pmatrix} \chi_{+\mathtt{A}} \\ \chi_{-\mathtt{A}} \end{pmatrix}, \quad \begin{pmatrix} \eta_{+} \\ -\eta_{-} \end{pmatrix}, \quad \begin{pmatrix} Q_{+} \\ Q_{-} \end{pmatrix}$ $\begin{matrix} 1 \\ -1 \end{matrix}$

Triplet : $\begin{pmatrix} \phi_{+} \\ C \\ -\phi_{-} \end{pmatrix}$ $\begin{matrix} 2 \\ 0 \\ -2 \end{matrix}$

$$[J_0, J_{\pm\pm}] = \pm 2 J_{\pm\pm}, \qquad [J_{++}, J_{--}] = J_0.$$

$\mathcal{F}_0 : SU(2)_R$-inv. $\Rightarrow S_0 : SU(2)_R$-inv.

◇ Mass deformation of 2d $\mathcal{N} = (8,8)$ SYM

- Mass deformed $Q_\pm$ SUSY by $SU(2)_R$ [Hanada-Matsuura-F.S.]

$$Q_\pm A_\mu = \psi_{\pm\mu}, \cdots$$
$$Q_\pm X_i = \rho_{\pm i}, \cdots$$
$$Q_\pm B_{\mathtt{A}} = \chi_{\pm \mathtt{A}}, \cdots$$
$$Q_\pm C = \eta_\pm, \quad Q_\pm \eta_\pm = \pm[\phi_\pm, C] + \frac{2M}{3}\phi_\pm,$$
$$Q_\mp \eta_\pm = \mp[\phi_+, \phi_-] \pm \frac{M}{3} C, \quad Q_\pm \phi_\pm = 0, \quad Q_\mp \phi_\pm = \mp \eta_\pm.$$

⇒ Nilpotency up to gauge and $SU(2)_R$ transformations

$$Q_+^2 = (\text{infinitesimal gauge transformation by } \phi_+) + \frac{M}{3} J_{++},$$
$$Q_-^2 = (\text{infinitesimal gauge transformation by } -\phi_-) - \frac{M}{3} J_{--},$$
$$\{Q_+, Q_-\} = (\text{infinitesimal gauge transformation by } C) - \frac{M}{3} J_0.$$

⇒ Mass deformed action

$$\begin{aligned} S_M &= \left(Q_+Q_- - \frac{M}{3}\right)\mathcal{F}_M, \\ \mathcal{F}_M &= \mathcal{F}_0 + \Delta\mathcal{F}, \\ &\qquad \Delta\mathcal{F} = \frac{1}{g_{2d}^2}\int d^2x\, \mathrm{tr}\left[\sum_{\mathtt{A}=1}^{3}\frac{a_{\mathtt{A}}}{2}B_{\mathtt{A}}^2 + \sum_{i=3}^{4}\frac{c_i}{2}X_i^2\right]. \end{aligned}$$

Note

- S_M is $Q_\pm$-invariant: $Q_+S_M = Q_-S_M = 0$, and $SU(2)_R$-invariant.

  ↑

  $[J_{\pm\pm}, Q_\pm] = 0$, $[J_{\pm\pm}, Q_\mp] = Q_\pm$

- When $a_{\mathtt{A}}, c_i \in \left(-\frac{2M}{3}, 0\right)$, scalars $B_{\mathtt{A}}, X_i$ have positive mass terms.
  For convenience, we take $a_1 = a_2 = a_3 = -\frac{2M}{9}$, $c_3 = c_4 = -\frac{4M}{9}$.

- Explicit form of the action

$$S_M = S_0 + \Delta S,$$

where

$$\begin{aligned}\Delta S \; = \; & \frac{1}{g_{2d}^2}\int d^2x\, \mathrm{tr}\Bigg[\frac{2M^2}{81}\left(B_A^2 + X_i^2\right) + \frac{M^2}{9}\left(\frac{C^2}{4} + \phi_+\phi_-\right) \\ & -\frac{M}{2}C[\phi_+, \phi_-] \\ & +\frac{2M}{3}\psi_{+\mu}\psi_{-\mu} + \frac{2M}{9}\rho_{+i}\rho_{-i} + \frac{4M}{9}\chi_{+A}\chi_{-A} - \frac{M}{6}\eta_+\eta_- \\ & -\frac{4iM}{9}B_3\left(F_{12} + i[X_3, X_4]\right)\Bigg].\end{aligned}$$

Flat directions are stabilized by the mass M.

- Fuzzy S^2 configurations satisfying

$$[\phi_+, \phi_-] = \frac{M}{3}C, \quad [C, \phi_\pm] = \pm\frac{2M}{3}\phi_\pm,$$
$$B_{\mathtt{A}} = X_i = 0$$

give $Q_\pm$-SUSY preserving minima ($S_M = 0$).

3 Lattice formulation of mass-deformed 2d $\mathcal{N} = (8, 8)$ SYM

Key aspects: [Hanada-Matsuura-F.S.]

- Lattice gauge fields are on links: $A_\mu(x) \Rightarrow U_\mu(x) = \mathrm{e}^{iaA_\mu(x)}$
  All the other fields are on sites.

$$Q_\pm A_\mu = \psi_{\pm\mu}, \cdots \Rightarrow Q_\pm U_\mu(x) = i\psi_{\pm\mu}(x)U_\mu(x), \cdots$$

  "Nilpotent" mass-deformed $Q_\pm$ SUSY are realized on lattice.

- Gauge configurations are smoothly restricted those satisfying the admissibility condition $||1 - U_{12}(x)|| < \epsilon$ by

$$F_{12} \Rightarrow \frac{i(U_{12}(x) - U_{21}(x))}{1 - \epsilon^{-2}||1 - U_{12}(x)||^2}$$

  with $||A|| = \sqrt{\mathrm{tr}(AA^\dagger)}$, $0 < \epsilon < 2$ for $G = U(N)$.

  - The admissibility condition is necessary to kill unphysical flux vacua.
  - Minimizing the action singles out the gauge configuration $U_{12}(x) = 1$.

- Same as in the continuum theory, flat directions are stabilized, and fuzzy S^2 configurations give $Q_\pm$-SUSY preserving minima.

- The lattice theory is shown to be free from fine-tuning (at least to the all order in the perturbation theory).

- After taking the continuum limit, $M \longrightarrow 0$ limit yields the undeformed theory ($\Rightarrow$ DVV matrix string theory).

4 4d $\mathcal{N}=4$ SYM from 2d lattice [Hanada-Matsuura-F.S.]

Consider the lattice theory around the minimum of k-coincident fuzzy S^2:

$$C = \frac{2M_0}{3} L_3, \qquad \phi_\pm = \frac{M_0}{3}(L_1 \pm i L_2)$$

with $L_a = \underline{L_a^{(n)}} \otimes \mathbb{1}_k$ and $N = nk$.

$\uparrow$

SU(2)-generators of spin-j irre. repre. $(n = 2j+1)$

First, we take continuum limit of the 2d lattice

$\Rightarrow$ Mass-deformed $\mathcal{N}=4$ $U(k)$ SYM on $\mathbf{R}^2 \times$ (Fuzzy S^2)

- 16 SUSYs broken to $Q_\pm$ by M
- Fuzzy S^2: radius $R = \frac{3}{M}$,
  fuzziness $\Theta = \frac{18}{M^2 n}$ $\Rightarrow$ UV cutoff $\Lambda = \frac{M}{3} \cdot 2j$
- $g_{4d}^2 = 2\pi \Theta g_{2d}^2$

Next, we take the two limits:

- Step 1:
  Decompactify fuzzy S^2 to the Moyal plane $\mathbf{R}^2_\Theta$
  $M \longrightarrow 0$ and $n = 2j + 1 \longrightarrow \infty$ with Θ and k fixed

  $\Rightarrow \Lambda \propto n^{1/2} \longrightarrow \infty$ ("continuum limit")

  We expect that M is soft.
  $\Rightarrow$The theory becomes $\mathcal{N} = 4\ U(k)$ SYM on $\mathbf{R}^2 \times \mathbf{R}^2_\Theta$ with 16 SUSYs restored.

- Step 2:
  Commutative limit $\Theta \longrightarrow 0$ with g_{4d} fixed

  The limit is considered to be smooth. [Matusis-Susskind-Toumbas]

Finally, we obtain 4d $\mathcal{N} = 4\ U(k)$ SYM on $\mathbf{R}^4$. k: general
In particular, 4d rotational symmetry is restored.

5 1-loop check of the procedure [Hanada-Matsuura-F.S.-Suzuki]

As a concrete check of the procedure, we carry out 1-loop computation of $\mathcal{N}=4\; U(k)$ SYM on $\mathrm{R}^2\times$ (Fuzzy S^2) and take the limits of Steps 1 and 2.

◇ The final expression of the kinetic terms of X_i $(i=3,4)$ and $B_{\mathtt{A}}$ $(\mathtt{A}=1,2)$ is consistent to the standard 1-loop result of $\mathcal{N}=4$ SYM on R^4:

- The overall $U(1)$ part receives no radiative correction (free).
- The $SU(k)$ part: (q is a 4-momentum, $\mathrm{q}^2=q_1^2+q_2^2+(\frac{M}{3}J)^2$.)

$$\frac{1}{g_{4d}^2}\int\frac{d^4\mathrm{q}}{(2\pi)^4}\mathrm{tr}_k\left[\tilde{x}_i^{SU(k)\,(R)}(-\mathrm{q})\tilde{x}_i^{SU(k)\,(R)}(\mathrm{q})\right]$$

$$\times\mathrm{q}^2\left[1+\frac{g_{4d}^2k}{4\pi^2}\left\{-\frac{1}{2}\ln\frac{\mathrm{q}^2}{\mu_R^2}+1\right\}+\mathcal{O}(g_{4d}^4)\right]+(\text{same form for }\tilde{b}_{\mathtt{A}})$$

after the wave function renormalization (μ_R: renormalization point)

$$\tilde{x}_i^{SU(k)\,(R)}(\mathrm{q})\equiv\left(1+\frac{g_{4d}^2k}{4\pi^2}\ln\frac{\Lambda}{\mu_R}\right)^{1/2}\tilde{x}_i^{SU(k)}(\mathrm{q}),\;\cdots$$

– 4d rotational symmetry is restored!

– Singular behavior due to UV/IR mixing does not survive in the limits.

$\Rightarrow$ The procedure seems to work well.

6 Discussions

◇ We presented a lattice formulation of mass-deformed 2d $\mathcal{N} = (8, 8)$ $U(N)$ SYM preserving two supercharges $Q_\pm$,
and discussed a procedure to obtain 4d $\mathcal{N} = 4$ $U(k)$ SYM from the 2d theory.

◇ We can construct a similar mass-deformed lattice model for 2d $\mathcal{N} = (4, 4)$ $U(N)$ SYM. [Hanada-Matsuura-F.S.]

- Mass-deformed 2d continuum theory preserves full 8 SUSYs
- 4d $\mathcal{N} = 2$ $U(k)$ SYM on $\mathbf{R}^2\times$ NC $\mathbf{R}^2$ is obtained.

◇ Coupled to matter fields, $\mathcal{N} = 1^*, 2^*$ models.
In particular, 4d $\mathcal{N} = 2$ superconformal theories would be interesting.