Dimers, wiring diagrams and integrable systems

Andrei Marshakov
Lebedev Institute & ITEP, Moscow

Progress in quantum field theory and string theory

Osaka, April 6, 2012

Based on joint work with V. Fock

Plan

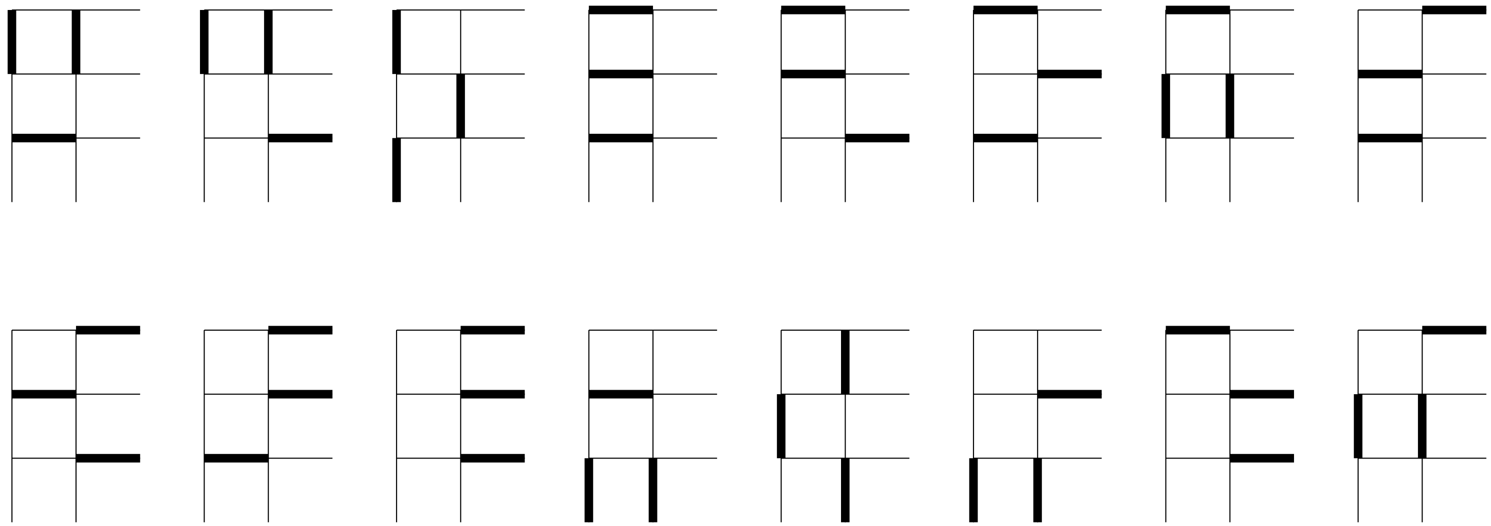
- Dimers on the bipartite graphs on a torus, Poisson quivers;

- Wiring diagrams, (co-extended) affine Weyl groups and loop groups;

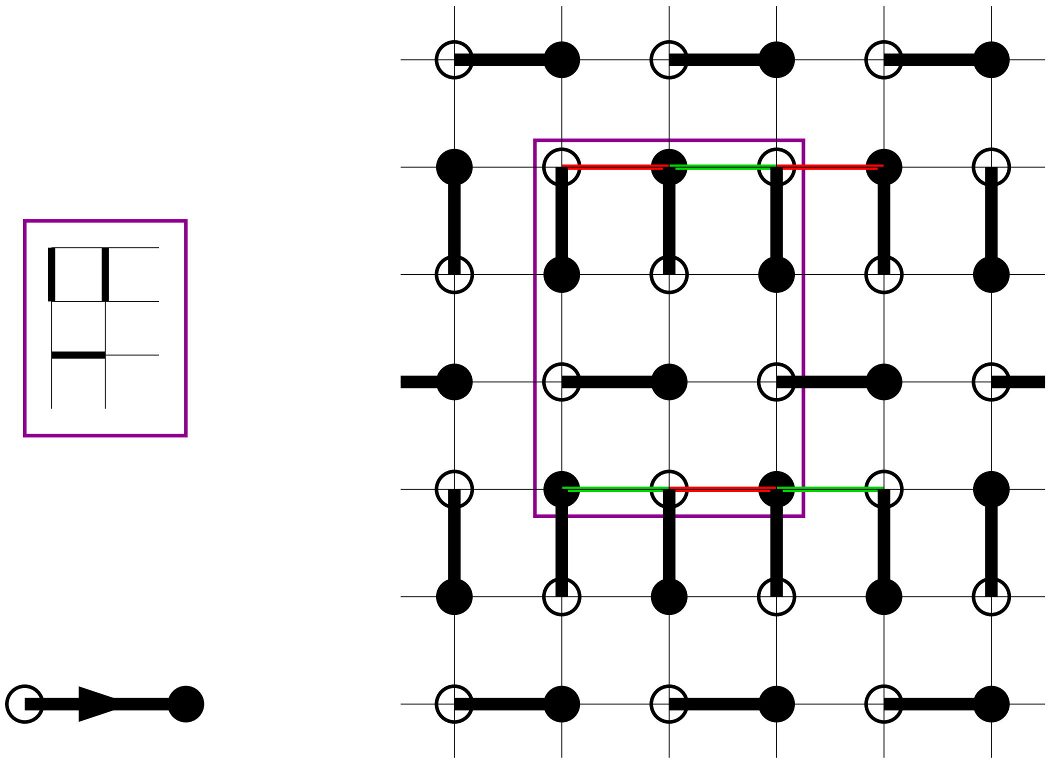
- Integrable systems, Lax maps and spectral curves (Toda type, more?);

- Mutations and discrete flows, Teichmüller space geometry, SUSY gauge theories.

Dimers on a bipartite graph Γ on a torus, example: 3×2 square lattice



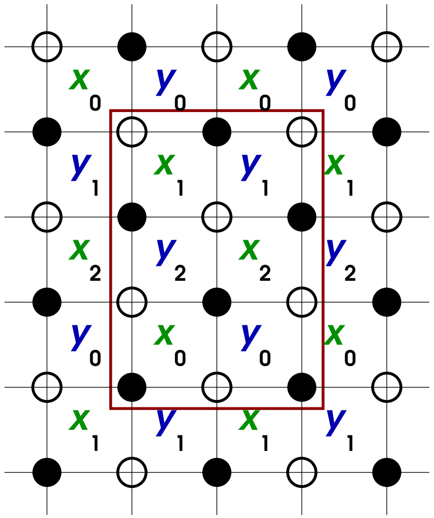
16 possible dimer configurations $D \in \mathcal{D}_\Gamma$: a maximal set of marked edges (with certain weights a_e, $e \in D$) without common vertices.

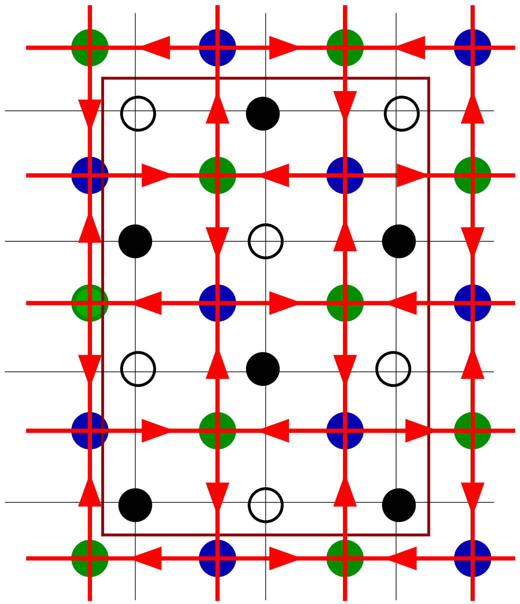


Bipartite: white and black vertices, oriented edges;

$\partial D = \sum_{\mathsf{all}} V_{\bullet} - \sum_{\mathsf{all}} V_{\circ}$, $\forall D \in \mathcal{D}_{\Gamma}$.

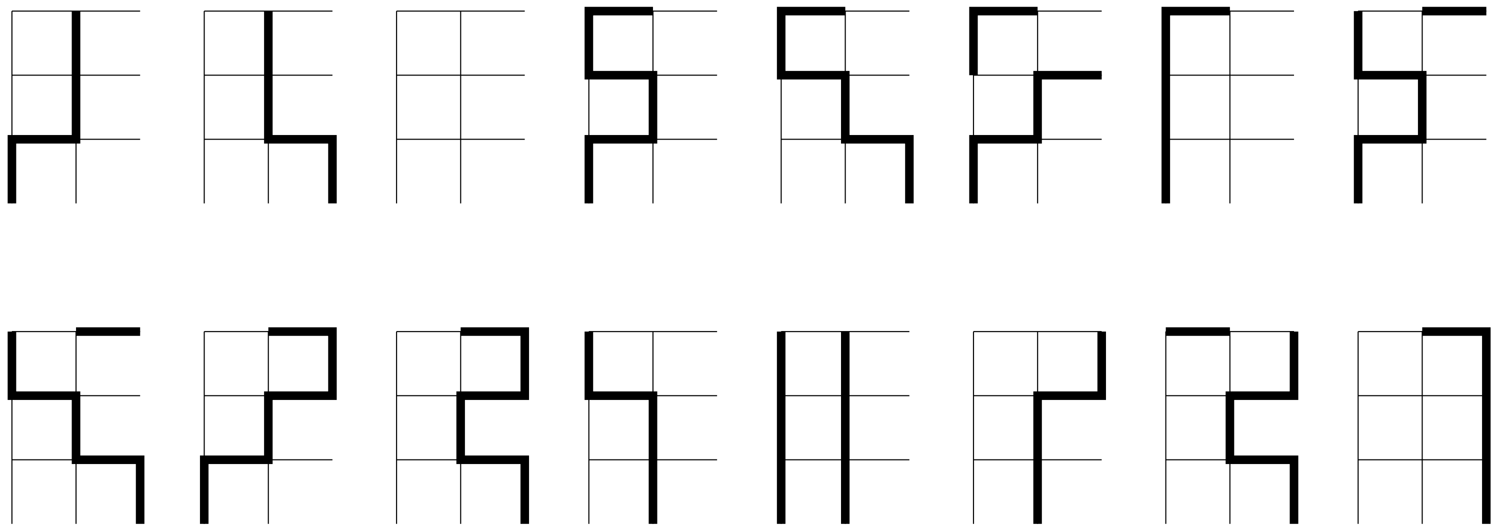
Face variables: dual graph $\Gamma^\vee$ (red): a Poisson quiver for the face variables $z_f = \prod_{e \circlearrowleft f} a_e$,
$\{z_i, z_j\} = \varepsilon_{ij} z_i z_j$, $\varepsilon_{ij} = \#\text{arrows}(i \to j)$





E.g. here for the face variables ($x_0 x_1 x_2 y_0 y_1 y_2 = 1$, blue y's and green x's) one gets $\{x_i, x_j\} = \{y_i, y_j\} = 0$ and $\{y_i, x_j\} = \widehat{C}_{ij} y_i x_j$, with the Cartan matrix $\widehat{C}$ of $\mathfrak{g} = \widehat{sl_3}$.

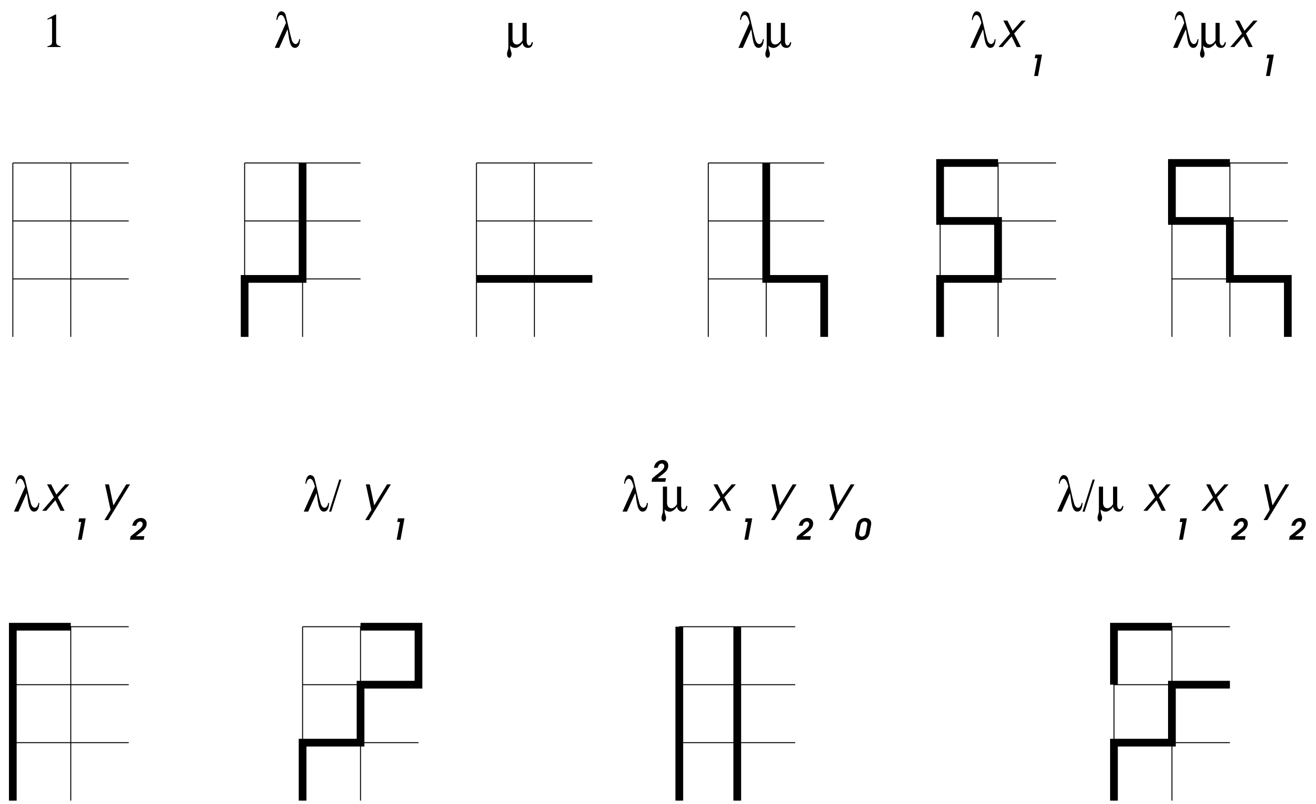
Dimers and faces: fix *any* $D_0 \in \mathcal{D}_\Gamma$, and get a set of



loops, corresponding to $D - D_0$. Since $\partial(D - D_0) = 0$, one has

$$D = D_0 + D_1 + D_2, \qquad D_1 \in H^1, \qquad D_2 = \partial F \tag{1}$$

Assign to each loop $D - D_0$ factor $\lambda^n \mu^k$ with $(n, k) \in H^1$ and the (oriented) product of all from F face variables, examples:



Face partition function:

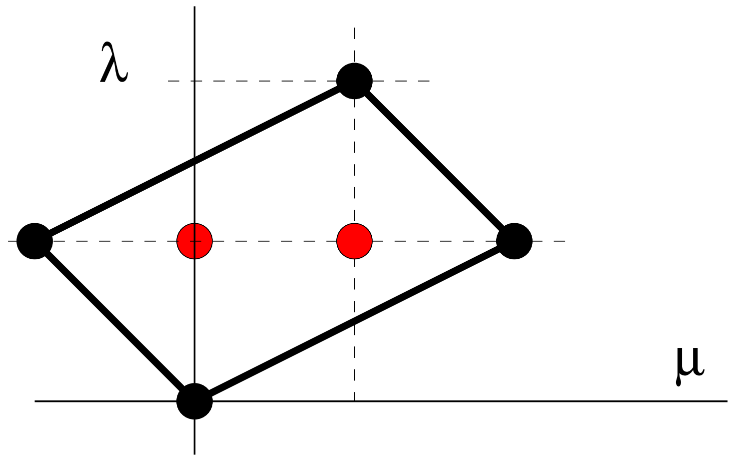
- sum up $S_{\Gamma}(\lambda, \mu | \mathbf{x}, \mathbf{y}) = \sum_{(n,k) \in H^1} \lambda^n \mu^k S_{n,k}(\mathbf{x}, \mathbf{y})$;

- equation $S_{\Gamma}(\lambda, \mu | \mathbf{x}, \mathbf{y}) = 0$ defines a spectral curve in $(\lambda, \mu) \in \mathbb{C}^{\times} \times \mathbb{C}^{\times}$, modulo rescaling of λ and μ (choice a representative in H^1) and total normalisation (choice of D_0);

- Goncharov-Kenyon integrable system for *any bipartite* graph *on a torus*: invariant ratios of $\{S_{n,k}(\mathbf{x}, \mathbf{y})\}$ commute w.r.t. dual Poisson quiver $\Gamma^{\vee}$.

3×2 example: the sum is

$$
\begin{aligned}
S(\lambda,\mu|\mathbf{x},\mathbf{y}) = 1 + \frac{\lambda}{\mu} x_1 x_2 y_2 + \lambda\mu^2 \frac{1}{x_2 y_1 y_2} + \lambda^2 \mu x_1 y_2 y_0 + \\
+\lambda\left(1 + x_1 + x_1 y_2 + \frac{1}{y_1} + x_1 x_2 y_2 y_0 + x_1 x_2 y_2\right) + \\
+\lambda\mu\left(1 + x_1 + \frac{1}{x_2 y_1 y_2} + \frac{1}{y_1} + x_1 y_0 + \frac{1}{x_2 y_1}\right)
\end{aligned} \tag{2}
$$



Casimir ($\{\mathcal{C}, x\} = \{\mathcal{C}, y\} = 0$) - from boundary points

$$\frac{S_{1,2}S_{1,-1}}{S_{2,1}S_{0,0}} = \frac{1}{y_1 y_2 y_0} = x_1 x_2 x_0 = \mathcal{C} \tag{3}$$

and integrals of motion ($\{\mathcal{H}_1, \mathcal{H}_2\} = 0$) - internal points

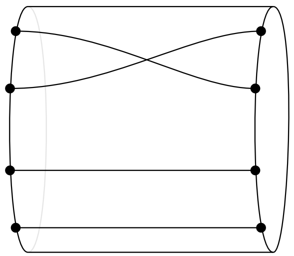
$$\begin{aligned} \mathcal{H}_1 &= \frac{S_{1,0}}{(S_{2,1}S_{1,-1}S_{0,0})^{1/3}\mathcal{C}^{1/3}} \\ &= \frac{1 + y_1 + y_1 x_1 + y_1 x_1 y_2 + y_1 x_1 y_2 x_2 + \mathcal{C}^{-1} x_1 x_2}{(x_1 y_1)^{2/3}(x_2 y_2)^{1/3}} \end{aligned} \tag{4}$$

$$\begin{aligned} \mathcal{H}_2 &= \frac{S_{1,1}}{(S_{2,1}S_{1,2}S_{0,0})^{1/3}\mathcal{C}^{1/3}} \\ &= \frac{1 + y_2 + y_2 x_2 + y_2 x_2 y_1 + y_2 x_2 y_1 x_1 + \mathcal{C}^{-1} x_1 x_2}{(x_1 y_1)^{1/3}(x_2 y_2)^{2/3}} \end{aligned} \tag{5}$$

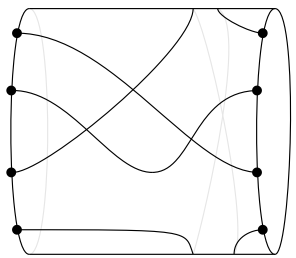
Why do they Poisson commute?

Wiring diagrams: start with

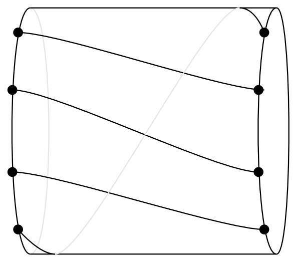
- A Newton polygon (in (λ, μ)-plane H_1) modulo total shift and action of $SL(2,\mathbb{Z})$;
- get an element of the *co-extended* double affine Weyl group $(\widehat{W} \times \widehat{W})^\sharp$, or a (double) wiring diagram on (cut) torus: examples of the elements of $W \subset \widehat{W} \subset \widehat{W}^\sharp$;



$s_1 \in W$

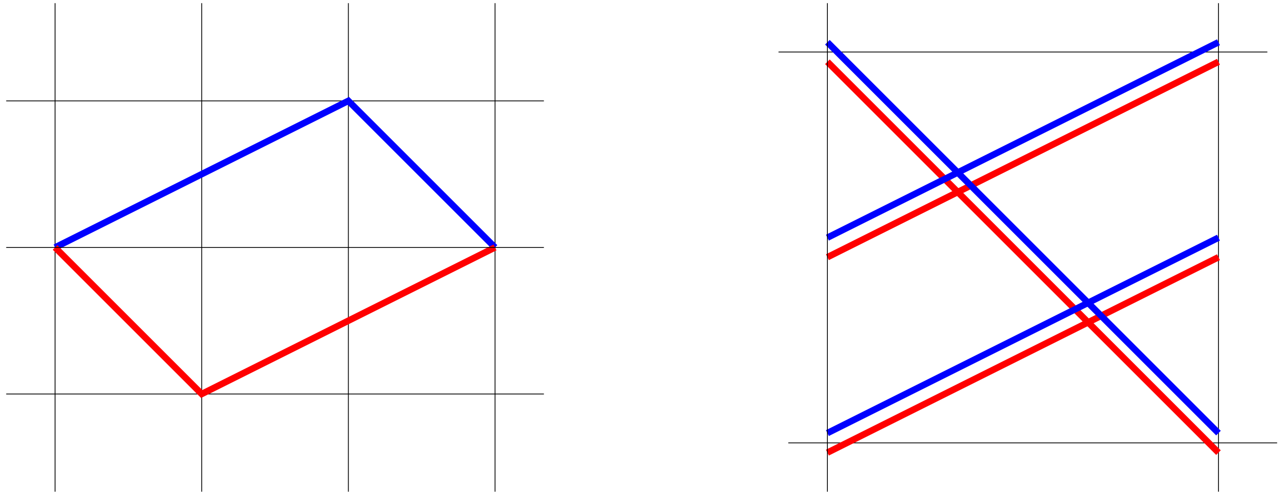


$s_2 s_1 s_2 s_0 \in \widehat{W}$



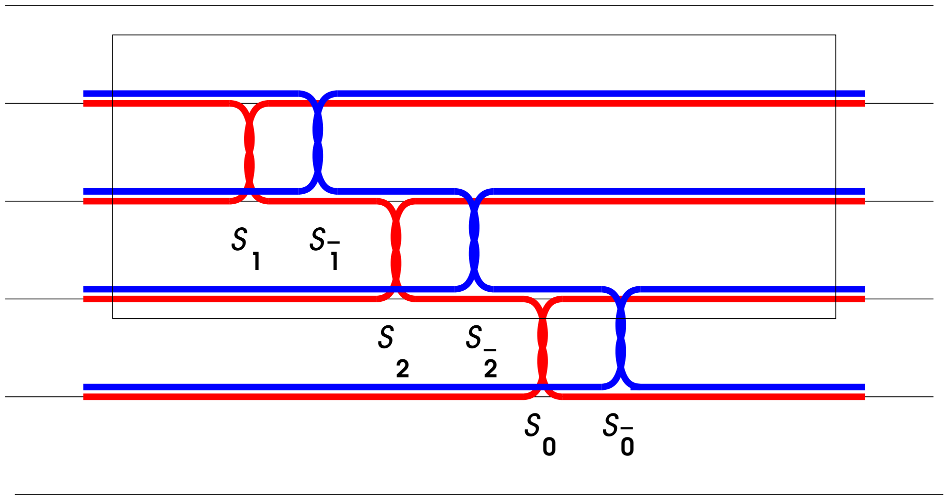
$\Lambda \in \widehat{W}^\sharp$

Example: from Newton polygon to the (double) wiring diagram in the "fundamental domain"



crossing of the red and blue lines - the elements of $(\hat{W} \times \hat{W})^\sharp$.

Resolution of the previous wiring diagram



gives the element (word) $u = s_1 s_{\bar{1}} s_2 s_{\bar{2}} s_0 s_{\bar{0}} \in (\hat{W} \times \hat{W})^\sharp$.

Arising of the co-extended affine Weyl group $\rightarrow$ Poisson structure on the co-extended loop group $\widehat{G}^\sharp$.

Poisson submanifolds in $\widehat{G}^\sharp$:

- Definition of $\widehat{G}^\sharp$ (for $\widehat{PGL(N)}^\sharp$): infinite quasiperiodic matrices $A_{I+N}^{J+N} = zA_I^J$, $z \in \mathbb{C}^\times$

$$A_I^J \mapsto \sum_{K\in\mathbb{Z}} A_I^{J+KN}\lambda^K T_z = A(\lambda)T_z, \quad T_z = z^{\partial/\partial\lambda}$$
$$A_1(\lambda)T_{z_1} \cdot A_2(\lambda)T_{z_2} = A_1(\lambda)A_2(z_1\lambda)T_{z_1 z_2} \tag{6}$$

  or $N\times N$ spectral parameter dependent shift operators (the Cartan subalgebra is extended by $h^0 = \partial/\partial\lambda$;

- Co-extended Weyl group $\hat{W}^\sharp = \hat{W} \rtimes \mathbb{Z}/N\mathbb{Z}$: $\{s_i | i \in \mathbb{Z}/N\mathbb{Z}\}$, $s_i^2 = 1$, $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ and additional generator Λ

$$\Lambda s_i = s_{i+1}\Lambda, \quad \Lambda^N = 1 \tag{7}$$

Poisson submanifolds in $\widehat{G}$ (via the co-extension $\widehat{G}^\sharp$): for any cyclically reduced $u = s_{j_1} \dots s_{j_l}$, $s_j \in (\widehat{W} \times \widehat{W})^\sharp$ - the "Lax map"

$$z_1, \dots, z_l \mapsto \mathbf{E}_{j_1}\mathbf{H}_{j_1}(z_1) \cdots \mathbf{E}_{j_l}\mathbf{H}_{j_l}(z_l)$$
$$\mathbf{E}_i = E_i = \exp(e_i), \qquad \mathbf{E}_{\bar{i}} = E_i^{\mathrm{tr}} = \exp(f_i) \tag{8}$$
$$\mathbf{H}_i(z) = H_i(z)T_z, \quad i \neq 0$$

where $H_i(z) = z^{h^i}$, $[h^i, e_j] = [h^i, f_j] = 0$ for $i \neq j$, with extra

$$\mathbf{H}_0(z) = T_z, \quad \mathbf{E}_0 = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \lambda & \cdots & 1 \end{pmatrix}, \quad \mathbf{E}_{\bar{0}} = \begin{pmatrix} 1 & \cdots & \lambda^{-1} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \tag{9}$$

and the final projection $\widehat{G}^\sharp \to \widehat{G}$

$$\prod_j z_j = 1, \qquad \prod_j T_{z_j} = T_{\prod_j z_j} = \mathrm{Id} \tag{10}$$

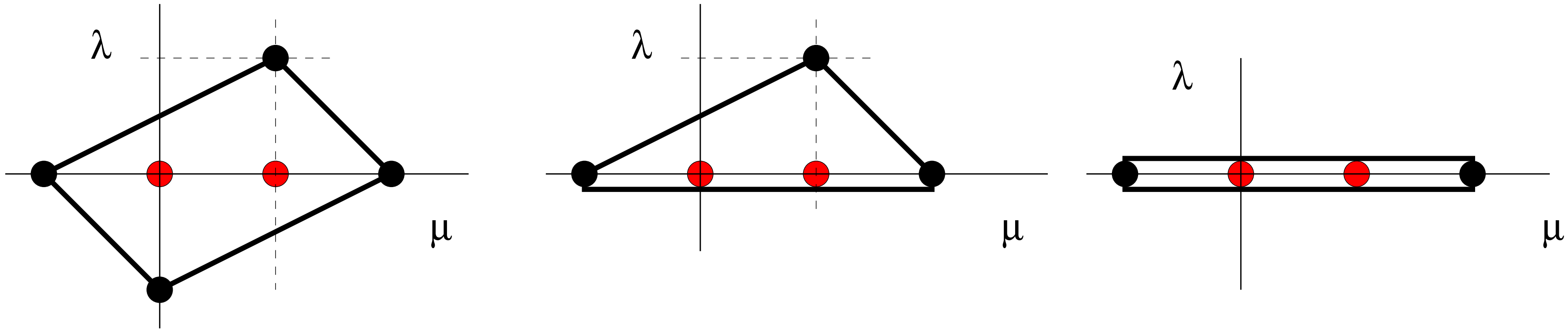
Two promised explanations of the Poisson-commutativity:

- wiring diagram $\to$ bipartite graph Γ $\to$ face dimer partition function $\widehat{G}$ $\to$ integrable system ("Teichmüller formulation");

- wiring diagram $\to$ element $u \in (\widehat{W} \times \widehat{W})^\sharp$ $\to$ Poisson $\Gamma^\vee$-submanifold in $\widehat{G}$ $\to$ "Lax map" $\to$ spectral curve $\to$ integrable system. Non-unique choice for the Weyl group $\widehat{W}$, loop group $\widehat{G}$ and Lax map (dependently on cutting the torus), but always with the same $\Gamma^\vee$;

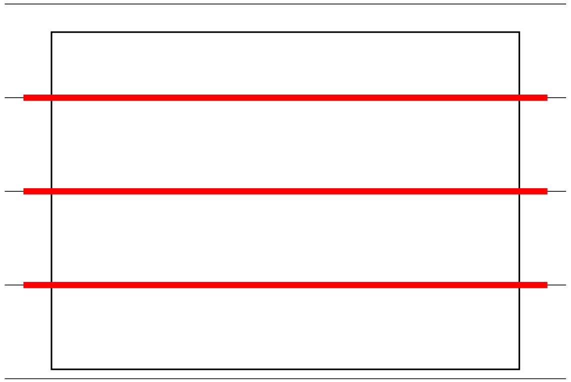
- the second way allows to consider the "degenerate" Newton polygons: the boundary points give rise to the integrals of motion instead of the Casimir functions by self-intersection of the wires.

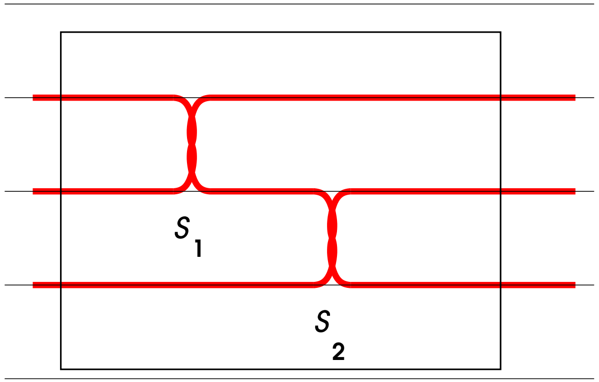
Example with degenerate polygons:



Integer points on the boundary *become* nontrivial integrals of motion after self-twisting the corresponding wires!

Back to the old example: consider maximal self-twisting of the totally degenerate polygon





- can be closed, using just $\Lambda \in \widehat{W}^\sharp$ as on left picture - a "torus knot";

- can be closed with $N-1$ self-intersections, all by the elements $s_i \in W \subset \widehat{W} \subset \widehat{W}^\sharp$, and corresponds to the Poisson submanifold in a simple Lie group $G \subset \widehat{G}^\sharp$.

The corresponding Poisson quiver

$$\Gamma^\vee = {}^{y_1}_{x_1} \Downarrow \overset{\leftarrow}{\rightarrow} \Uparrow {}^{x_2}_{y_2} \overset{\rightarrow}{\leftarrow} \Downarrow^{y_3}_{x_3} \overset{\leftarrow}{\rightarrow} \cdots \overset{\rightarrow}{\leftarrow} \Downarrow^{y_{N-1}}_{x_{N-1}}$$

induces the Poisson bracket on symplectic leaf in $G = SL(N)$: $\{x, x\} = 0$, $\{y, y\} = 0$, and

$$\begin{aligned} \{y_i, x_j\} = C_{ij} y_i x_j, \\ C_{ij} \ = \ 2\delta_{ij} - \delta_{i+1,j} - \delta_{i,j+1}, \quad i, j = 1, \dots, N-1 \end{aligned} \tag{11}$$

with the sl_N-Cartan matrix. The Lax map for the product $u = s_1 s_{\bar{1}} \dots s_{N-1} s_{\overline{N-1}} \in W \times W$

$$\prod_{j=1}^{N-1} E_j H_j(x_j) E_{\bar{j}} H_j(y_j) \sim g_N(\mathbf{x}, \mathbf{y}) \in SL(N) \tag{12}$$

gives integrals of motion $\{\mathcal{H}_i, \mathcal{H}_j\} = 0$ via degenerate spectral curve equation

$$\det\left(g_N(\mathbf{x},\mathbf{y}) + \mu \cdot \mathbf{1}\right) = \sum_{j=0}^{N} \mathcal{H}_j(\mathbf{x},\mathbf{y})\mu^j \tag{13}$$

Well-known integrable model: in Darboux co-ordinates

$$\begin{gathered} x_i = \exp(-\alpha_i \cdot q), \qquad y_i = \exp(\alpha_i \cdot (P+q)) \\ P = p + \frac{\partial}{\partial q}\left(\frac{1}{2}\sum_{k=1}^{N-1} \mathrm{Li}_2\left(-\exp(\alpha_k \cdot q)\right)\right) \end{gathered} \tag{14}$$

so that

$$\begin{gathered} \mathcal{H} = \mathcal{H}_1 + \mathcal{H}_{N-1} = \\ = \sum_{i=1}^{N} \left(e^{p_i} + e^{-p_i}\right)\sqrt{1+e^{q_i - q_{i+1}}}\sqrt{1+e^{q_{i-1}-q_i}} \end{gathered} \tag{15}$$

Some achievements of this approach:

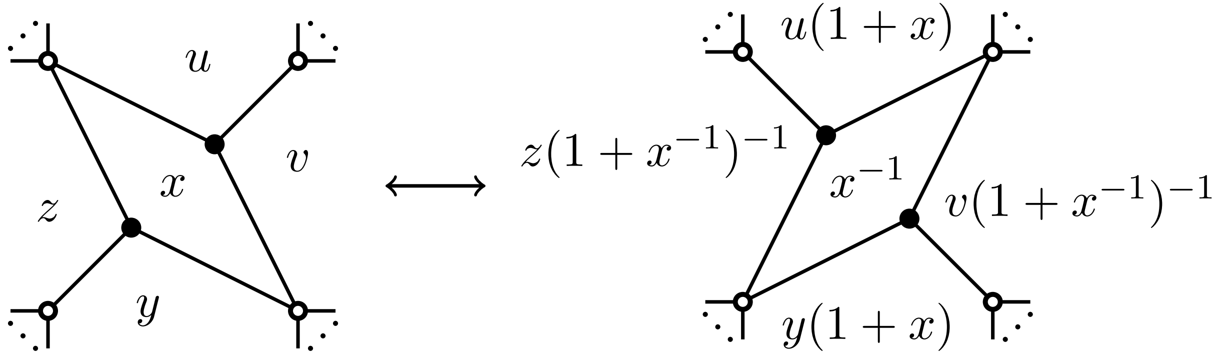
(**A**) New explicit formulas for $\mathcal{H}_j = \prod_k (x_k y_k)^{-C^{-1}_{jk}} \cdot S_j$ and Laurent polynomials

$$S_j(\mathbf{x},\mathbf{y}) = \sum_{0\le m_i\le \epsilon_i}^{m_j\ge m_{j\pm 1}\ge m_{j\pm 2}\ge \dots} \sum_{m_i-1\le n_i\le m_i} \prod_i y_i^{m_i} x_i^{n_i} \qquad (16)$$

with $\epsilon_i = 1,2$ (number of edges entering the i-th vertex of the Dynkin diagram) are related to face dimer partition functions. Examples: Toda, pentagram map, more?

Computation of dimer partition functions via the Lax maps!

(**B**) Poisson quiver $\Gamma^\vee$ defines a structure of a *cluster variety* on the phase space of integrable system. The discrete flows (bilinear Hirota equations) are generated by the cluster mutations (the simplest Y-systems).



Diamond move - mutation (for the graph Γ).

(**C**) Back to geometry: $\Gamma \subset \Sigma_0 =$ torus induces (e.g. from flat connections) a bracket $\{a_{\gamma_1}, a_{\gamma_2}\}_0 = \langle\gamma_1, \gamma_2\rangle_0 a_{\gamma_1} a_{\gamma_2}$, where $\langle\gamma_1, \gamma_2\rangle_0$ - intersection in $H^1(\Sigma_0)$.
$\Gamma \hookrightarrow \Sigma_0$ by gluing the faces by discs; dual (Goncharov-Kenyon) Σ: $\Gamma \hookrightarrow \Sigma$ by gluing discs along the zig-zag paths on Γ.
The dual bracket $\{a_{\gamma_1}, a_{\gamma_2}\} = \langle\gamma_1, \gamma_2\rangle a_{\gamma_1} a_{\gamma_2}$ is given by the Poisson quiver $\Gamma^\vee$.
Σ topologically coincides with the spectral curve of integrable system, and

$$\langle\gamma_1, \gamma_2\rangle_0 \leftrightarrow \frac{d\lambda}{\lambda} \wedge \frac{d\mu}{\mu} \tag{17}$$

i.e. the Seiberg-Witten form, while $\langle\gamma_1, \gamma_2\rangle$ - the BPS charge pairing (intersection form in $H^1(\Sigma)$).

Towards understanding of the wall-crossing formula!

THANK YOU!