Hilbert Series and Its Applications to SUSY Gauge Theories

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Collaborative works

- Based on a series of papers from 2008 with

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What are we studying?

- Look at degenerate ground states of supersymmetric gauge theories
- Such ground states form a continuous space, known as the

moduli space of vacua

- The moduli space is parametrised by VEVs of scalar components of certain gauge invariant quantities
- In general, such gauge invariant quantities are polynomials with the variables being chiral fields

What are we studying?

- Can assign charges of a global $U(1)$ symmetry (e.g. R-charges) or mass dimensions to such gauge invariant quantities
- Such gauge invariants may satisfy certain polynomial relations (or constraints) among themselves
- These polynomial relations are usually homogeneous, i.e. each monomial carries equal $U(1)$ global charge (or mass dimension).
- Mathematically speaking,
  - a moduli space can be viewed as an algebraic variety
  - a coordinate ring of such a variety is graded by $U(1)$-global charges

Hilbert series

- A mathematical tool that can be used to characterise such moduli spaces is known as Hilbert-Poincaré series or Hilbert series

$$H(t_1, t_2, \ldots, t_k) = \sum_{i_1, \ldots, i_k} d_{i_1, \ldots, i_k} t_1^{i_1} \cdots t_k^{i_k}$$

  - $d_{i_1, \ldots, i_k}$ counts chiral objects carrying charges $(i_1, i_2, \ldots, i_k)$ under the global symmetry $U(1)^k$
  - $t_1, \ldots, t_k$ are referred to as the **fugacities** of the k $U(1)$ global symmetries
- Contains information about the generators of the moduli space and their constraints
- Contains certain geometrical information of the moduli space: Dimension, Calabi-Yau, complete intersection, etc.

- A "robust" mathematical tool to study and characterise moduli spaces of SUSY gauge theories
- Can be computed for a large class of SUSY gauge theories using the data from Lagrangians or can be conjectured from other theories belonging to the same class
- Or can even be deduced directly if enough mathematical information is known for a particular moduli space
- For a theory with a known Langrangian, the Hilbert series is sensitive to details of gauge groups, matter content and the superpotential

Let's jump to the conclusions

- Can be used to study various dualities in field and string theories
  - Seiberg duality for 4d $\mathcal{N} = 1$ SQCD *(Pouliot '98; Römelsberger '05; Work in progress)*
  - Toric duality (a class of Seiberg duality) between theories dual to $\mathrm{AdS}_5 \times \mathrm{SE}^5$ and $\mathrm{AdS}_4 \times \mathrm{SE}^7$ *(Hanany, Zaffaroni, et. al. from '06)*
  - Dualities between theories on M5 branes wrapping Riemann surfaces (Gaiotto's theories) *(Benvenuti, Hanany, N.M. '10)*
- Matrix descriptions & Integrability of the moduli spaces of 4d $\mathcal{N} = 1$ SQCD

  (Basor, Chen, N.M. '11)
- Moduli space in various asymptotic limits: Thermodynamics properties

  (Basor, Chen, N.M. '11 ; Jokela, Järvinen, Keski-Vakkuri '11; Work in progress)
- Relations to Hall-Littlewood indices *(Gadde, Rastelli, Razamat, Yan '11)*
  - Hilbert series for multi-instantons in various groups *(Gaiotto, Razamat '12)*

Example 1: A theory of n free chiral fields

A theory of n free chiral fields $\phi_1, \ldots, \phi_n$

- The chiral operators are $\phi_1^{k_1} \ldots \phi_n^{k_n}$ (where $k_1, \ldots, k_n = 0, 1, 2, \ldots$)
- There is a $U(n)$ global symmetry
  - There is a collection of maximally commuting $U(1)$'s in $U(n)$.
  - Assign the charge fugacity t_i for each $U(1)$ to the field ϕ_i (with $i = 1, \ldots, n$).
- The Hilbert series is

$$H(t_1, \ldots, t_n) = \sum_{k_1, \ldots, k_n = 0}^{\infty} t_1^{k_1} \ldots t_n^{k_n} = \prod_{i=1}^{n} \frac{1}{1 - t_i}$$

- Rewrite the fugacities t_i as

$$t_1 = t x_1,\ t_2 = t\frac{x_2}{x_1},\ t_3 = t\frac{x_3}{x_2},\ \ldots,\ t_{n-1} = t\frac{x_{n-1}}{x_{n-2}},\ t_n = \frac{t}{x_{n-1}}$$

where $x_1, \ldots, x_{n-1}$ are $SU(n)$ fugacities and t is the $U(1)$ fugacity

A theory with n free chiral fields $\phi_1, \ldots, \phi_n$ (continued)

- Then, we obtain

$$H(t; x_1, x_2, \ldots, x_{n-1}) = 1 + t \underbrace{\left(x_1 + \frac{x_2}{x_1} + \ldots + \frac{x_{n-1}}{x_{n-2}} + \frac{1}{x_{n-1}} \right)}_{\text{the character of the fundamental rep of } SU(n)} + \ldots$$

- Dynkin label $[k_1, \ldots, k_{n-1}]$ denotes the character of an irrep of $SU(n)$
- To all order,

$$H(t; x_1, x_2, \ldots, x_{n-1}) = \sum_{k=0}^{\infty} [k, 0, \ldots, 0] t^k \equiv \mathrm{PE}\left[[1, 0, \ldots, 0] t \right]$$

where $[k, 0, \ldots, 0]$ is the k-th symmetric power of the fundamental rep

The plethystic exponential

- For a function of several variables $f(t_1, \ldots, t_n)$ vanishing at the origin, $f(0, \ldots, 0) = 0$, the plethystic exponential is defined as

$$\mathrm{PE}[f(t_1, \ldots, t_n)] = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} f(t_1^k, \ldots, t_n^k)\right)$$

- **Example:** For $f(t) = t$, $\mathrm{PE}[f(t)] = \frac{1}{1-t}$
- **Example:** For $f(t) = \sum_k a_k t^k$, $\mathrm{PE}[f(t)] = \prod_k \frac{1}{(1-t^k)^{a_k}}$
- **Example:** $\mathrm{PE}\left[[1, 0, \ldots, 0]t\right] = \sum_{n=0}^{\infty} [n, 0, \ldots, 0] t^n$
- The PE generates symmetrisations.

A theory with n free chiral fields $\phi_1, \ldots, \phi_n$ (continued)

- In fact,

$$H(t; x_1, x_2, \ldots, x_{n-1}) = \mathrm{PE}\left[[1, 0, \ldots, 0]t\right]$$

  is known as the Hilbert series of $\mathbb{C}^n$

- Unrefinement: Setting $x_i = 1$, we obtain

$$H(t; 1, 1, \ldots, 1) = \frac{1}{(1-t)^n}$$

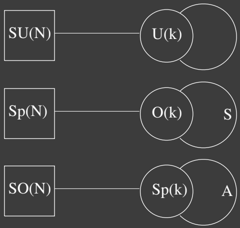
- Dimension of moduli space: In general,

  the order of the pole at $t = 1$ is the complex dim. of the moduli space

Example 2: Moduli spaces of Instantons

ADHM construction

- The moduli space of k $SU(N)$, $SO(N)$, $Sp(N)$ instantons on $\mathbb{R}^4$ can be identified with the Higgs branch of the following 4d $\mathcal{N}=2$ quiver diagrams:



- **D3-D7 system**: The Higgs branch can be realised when D3-branes are on top of D7-branes (possibly with an O-plane)
- The F and D terms give the moment map equations for the hyperKähler quotient of the instanton moduli space *(Douglas, Witten '94-'96)*

Hilbert series of instanton moduli spaces

- The Hilbert series of the Higgs branch can be computed using the data from the quiver diagram, e.g. for k $SU(N)$ instantons:

$$\int \mathrm{d}\mu_{U(k)} \frac{\mathrm{PE}\left[\chi^{\text{bifund}}_{SU(N)\times U(k)} t + \chi^{\text{bifund}}_{U(k)\times SU(N)} t + \chi^{\text{fund}}_{SU(2)_{\mathbb{C}^2}} \chi^{\text{adj}}_{U(k)} t\right]}{\mathrm{PE}\left[\chi^{\text{adj}}_{U(k)} t^2\right]}$$

- Observe the global symmetry $SU(N) \times SU(2)_{\mathbb{C}^2} \times U(1)$
- This has an interpretation as 5 dimensional Nekrasov's partition function
- The integral can be computed using Cauchy's residue theorem. The residues are characterised by partitions of k into N slots.

(Nekrasov '02; Bruzzo, Fucito, Morales, Tanzini '02; Nakajima, Yoshioka '03)

Moduli space of one G-instanton on $\mathbb{R}^4$

- Now focus on the instanton number $k = 1$ and any simple group G
- After the $\mathbb{C}^2$ translational part is factored out:
  - The counting of chiral operators on 1 instanton moduli space was studied by mathematicians. The Hilbert series can be written as

    $$\sum_{m=0}^{\infty} [m, 0, \ldots, 0, m]_G \; t^{2m} \qquad \text{for } [1, 0, \ldots, 0, 1] \text{ the adjoint rep of } G$$

    (Maths papers: Vinberg, Popov '72; Garfinkle '82; Kronheimer '90

    Physics papers: Gaiotto, Neitzke, Tachikawa '08; Benvenuti, Hanany, N.M. '10)
  - This can be computed directly from the ADHM quivers for classical G
  - Also true for exceptional groups G.

Moduli space of one G-instanton on $\mathbb{R}^4$

- Hilbert series:

$$H(t, x_1, \ldots, x_r) = \sum_{m=0}^{\infty} [m, 0, \ldots, 0, m]_G \; t^{2m}$$

- 4d Nekrasov's partition function:

$$\lim_{\beta \to 0} \beta^{2h^\vee - 2} H(e^{\beta(\epsilon_1 + \epsilon_2)}, e^{\beta \vec{a}}) = - \sum_{\vec{\gamma} \in \Delta_l} \frac{1}{(\epsilon_1 + \epsilon_2 + \vec{\gamma} \cdot \vec{a})(\vec{\gamma} \cdot \vec{a}) \prod_{\vec{\gamma}^\vee \cdot \vec{\alpha} = 1, \; \vec{\alpha} \in \Delta} (\vec{\alpha} \cdot \vec{a})}$$

where Δ and Δ_l are the sets of the roots and the long roots.

(Keller, N.M, Song, Tachikawa '11; thanks to A. Bondal and S. Carnahan)

- AGT relation: This is equal to the norm of a certain coherent state of the W-algebra. For non-simply laced G, the coherent state is in the twisted sector of a simply-laced W-algebra

Conclusions

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- Sensitive to details of gauge groups, matter content and the superpotential
- Can be used to study various dualities in field and string theories
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