From AGT to knots

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THANKS for your support
Motivation for the study of "knots"

From AGT to knots
Knots can well be the next hot topic in mathematical physics

Why?
What is most exciting?
There are formulas
Many formulas
Complicated formulas

Mysterious formulas

Strongly interrelated formulas
Intimately related to all what we studied before:

matrix models,
integrable systems,
topological theories,
CFT,
SW theory, LMNS and Nekrasov fns, AGT relations
Related to most different parts of each story:

Unitary rather than Hermitian matrix models,
Almost no integral formulas [but: CS models],
only AMM/OE topological recursion [R.Dijkgraaf & H.Fuji]

Deviation from ordinary integrability:
\[ \text{OV pf } \sum_R H_R \chi_R \{ \bar{p}_k \} \text{ never ordinary } \tau \text{-functions}, \]
\[ \sum_R \mathcal{H}_R \{ p_k \} \chi_R \{ \bar{p}_k \} \text{ are, but only sometime} \]

quantum \( R \)-matrices and non-trivial reps of Hecke algebras

BPS spectrum, integrality, wall crossing phenomena

modular transformations
Universality classes are labeled by integrable systems

\[ \mathcal{N} = 2 \text{ SYM models} \quad \leftrightarrow \quad 2d \text{ CFT conformal blocks} \]

\[ \uparrow \quad \text{dictionary} \quad [1995 - 97] \quad \downarrow \]

1d integrable systems \( \leftrightarrow \) DF/Penner matrix model

quantization of integrable systems
Shroedinger-like equations (Fourier tr. of Baxter eqs.)
insertions of degenerate states
SW description through BS integrals

\[ \Psi(z) = \exp \int^z \Omega, \quad \Omega = Pdz \]
\[ \partial F / \partial a = \oint_B \Omega, \quad a = \oint_A \Omega \]
NS limit \( \epsilon_1 \to 0, \beta \to \infty \)
AGT relation

open problems
reformulations
interpretations
proofs

- Dotsenko-Fateev matrix model
- Hubbard-Stratanovich duality
- Relation to integrable systems
  - Bohr-Sommerfeld integrals
One particular subject:

Beautiful
Conceptually important
Based on achievements in Osaka
\[ V_{\alpha_2}(q) \quad \alpha \quad V_{\alpha_3}(1) \]
\[ V_{\alpha_1}(0) \quad \alpha \quad V_{\alpha_4}(\infty) \]

\[
\left\langle e^{\alpha_1 \phi(0)} e^{\alpha_2 \phi(q)} e^{\alpha_3 \phi(1)} e^{\alpha_4 \phi(\infty)} \prod_{i=1}^{N_1} \int_0^q e^{b\phi(x_i)} \prod_{j=1}^{N_2} \int_0^1 e^{b\phi(y_j)} \right\rangle
\]

\[
\alpha_1 + \alpha_2 + bN_1 = \alpha
\]
\[
\alpha + \alpha_3 + \alpha_4 + bN_2 = 0
\]

\[
= \int dx_i \int dy_j (x_i - x_i')^{2\beta} (y_j - y_j')^{2\beta} (x_i - y_j)^{2\alpha_1 b} ((q-x_i)(q-y_j))^{2\alpha_2 b} ((1-x_i)(1-y_j))^{2\alpha_3 b}
\]

\[
= \int d\mu(x) \int d\mu(y) \left( \text{Mixing term}(x|y) \right)^2
\]
AGT as Hubbard-Stratanovich duality [1012.3137]

\[
\begin{align*}
&\approx \int d\mu(x) \int d\mu(y) \exp \left( 2\beta \sum_{i,j} \log(1 - x_i y_j) \right) \\
&= \int d\mu(x) \int d\mu(y) \exp \left( 2\beta \sum_k p_k \bar{p}_k / k \right) \\
&= \int d\mu(x) \int d\mu(y) \left( \sum_A \chi_A(X) \chi_A(Y) \right) \left( \sum_B \chi_B(X) \chi_B(Y) \right) \\
&= \sum_{A,B} \left( \int d\mu(x) \chi_A(X) \chi_B(X) \right) \left( \int d\mu(y) \chi_A(Y) \chi_B(Y) \right)
\end{align*}
\]

\[
p_k = \text{Tr} X^k, \quad \bar{p}_k = \text{Tr} Y^k \quad [H.Itoyama & T.Oota 1003.2929]
\]

\[
\exp \sum_k \frac{[\beta]}{k} q_k p_k \bar{p}_k = \sum_A \frac{C_A}{C_A'} M_A(X) M_A(Y)
\]
AGT as Hubbard-Stratanovich duality \[1012.3137\]

\[
\begin{align*}
\chi_A(X) & \quad \chi_B(X) \\
\chi_A(Y) & \quad \chi_B(Y)
\end{align*}
\]

\[
\sum_{X,Y} \left( \sum_A \chi_A(X) \chi_A(Y) \right) \left( \sum_B \chi_B(X) \chi_B(Y) \right) = \\
= \sum_{A,B} \left( \sum_X \chi_A(X) \chi_B(X) \right) \left( \sum_Y \chi_A(Y) \chi_B(Y) \right)
\]

Conformal block \[= \sum_{A,B} N_{A,B} \]
Decomposition problem for $\beta \neq 1$

$$\int d\mu(X) \chi_A(X) \chi_B(X) \int d\mu(Y) \chi_A(Y) \chi_B(Y) \overset{?}{=} N_{A,B}$$

TRUE for $\beta = 1$
NOT so simple for $\beta \neq 1$

$$< \chi[1] \chi\bullet > < \chi[1] \chi\bullet > + < \chi\bullet \chi[1] > < \chi\bullet \chi[1] > =$$

$$= \frac{1}{(z - \epsilon)} \frac{1}{(z + \epsilon)} + \frac{1}{(z + \epsilon)} \frac{1}{(z - \epsilon)} =$$

$$= \frac{2}{z^2 - \epsilon^2} = \frac{1}{z(z - \epsilon)} + \frac{1}{z(z + \epsilon)} = N_{[1],\bullet} + N_{\bullet,[1]}$$

For $\epsilon \neq 0$ ($\beta \neq 1$) particular Nekrasov functions have extra poles (at $z = 0$), not present in Kac determinant
Decomposition problem

Instead Nekrasov functions are nicely factorized, while Selberg correlators for $\beta \neq 1$ are not:

$$< \chi[3] \chi \cdot >_{BGW} \sim z^2 - (5\epsilon_1 + 8\epsilon_2)z + 6\epsilon_1^2 + 23\epsilon_1\epsilon_2 + 19\epsilon_2^2$$

$$\epsilon_2 \overset{\epsilon_2 \equiv -\epsilon_1}{\longrightarrow} z^2 + 3\epsilon_1 z + 2\epsilon_1^2 = (z + \epsilon_1)(z + 2\epsilon_1)$$
Decomposition problem

Natural quantities, e.g. Selberg correlators (involved into duality relations) are linear combinations of the nicely factorized functions (Nekrasov functions), which possess extra singularities.
AGT: generalizations

more models (quivers, 5d)
Wall crossing
hidden properties:
action of chiral algebra on the moduli spaces
(modular transform and other ingredients of SW theory)

3d AGT
What takes the place of conformal block?
"knot polynomials"
The knot polynomials should be studied and understood at least as well as as conformal blocks.

Enormous amount of experimental material
katlas.org

Almost no general formulas
Results

Formulas with free parameters, which characterize either knots or representations, for:

- HOMFLY pols for generic 3, 4, (5)-strand knots in $R = [1]$
- Colored HOMFLY for generic 3-strand braids and knots in $R = [2], [3], [4], [5]$
  - Torus superpolynomials $B = [m, n]$ in $R = [1]$
    with $n \pmod m = \pm 1, \pm 2, \pm 3, \pm 4$
    (up to $m = 9$, [9, 13]; the first unavailable is [11, 16])
  Mysterious hidden tropical structure is revealed in HL expansions
- Superpolynomials for the figure-eight knot $4_1$
  in all symmetric and antisymmetric representations $R = [p]$ and $R = [1^p]$
Main ideas

- Switch from knots to braids
- Introduce extended polynomials, depending on infinitely many time-variables
- Study families of braids
- Derive formulas, not only pictures
- Use matrix model techniques, (Ward identities, integrability, character expansions, integral formulas)
- Search for universalities
- Search for relations
**HOMFLY polynomial**

\[ *\mathcal{H}_R^K(A | q) \big|_{A=q^N} = \left\langle \text{Tr}_R P \exp \left( \oint_{\mathcal{K}} A \right) \right\rangle_{CS} \]

Alternative definitions:

- from skein relations
- from Khovanov-Rozhansky theory – as Euler characteristic of the triple-graded complex

\[ P(a|q|t) = \sum_{I,J,K} N_{IJK} a^I q^J t^K \]

\[ \mathcal{H}(A|q) = P(A|q|t = -1) \]

- from averages of characters \( \langle \chi_R[U] \rangle^K \)
Hierarchy of knot polynomials for the $SL(N)$ family

For a given knot $K$ and representation (Young diagram) $R$

Superpolynomial $P_R(A|q|t)$

$\begin{align*}
\checkmark & \quad t = q \\
\downarrow & \quad A = 1
\end{align*}$

$CS \quad \rightarrow \quad \boxed{\text{HOMFLY} \quad H_R(A|q)} \quad \text{Heegard – Floer} \quad HF_R(q|t)$

$q = 1 \checkmark \quad N = 2 \downarrow \quad N = 0 \quad \checkmark \quad t = q$

Special $\sigma_R(A)$ \quad Jones $J_R(q)$ \quad Alexander $A_R(q)$
$\sigma_R(A) = \left(\sigma_{[1]}(A)\right)^{|R|}$

$A_R(q) = A_{[1]}(q^{|R|})$

HOMFLY and superpolynomials satisfy difference equations as functions of the representation-variable

Garoufalidis & Le, math/0309214

Fuji, Gukov & Sulkovski, 1203.2182

IMMM, 1203.5978
CS in temporal gauge $A_0 = 0$

\[ \kappa \int \epsilon_{ijk} \text{Tr} \left( A^i \partial_j A^k + \frac{2}{3} A^i A^j A^k \right) d^3 x \]

\[ \longrightarrow \int \text{Tr} \left( A_x \dot{A}_y dx dy \right) dt \]

Quadratic theory with the ultralocal propagator

\[ \frac{2\pi i}{\kappa} \text{sign}(t) \delta(x) \delta(y) \]

Average of a Wilson line is given by projection onto the $xy$ plane and each intersection contributes

\[ q^{\pm T^a \otimes T^a} \]

with $q = e^{2\pi i/(\kappa+N)}$
\[ \mathcal{H}^B_R = \text{Trace}_{R^\otimes m} B \]

\[ B = \prod_s R^\pm_{i(s), i(s)+1} \]
2-strand braids (torus knots $[2, n]$)

\[ \mathcal{H}_R^{[2,n]} = \text{Tr}_{R \otimes R} \mathcal{R}^n = \sum_{Q} \text{Tr}_Q \mathcal{R}^n \]

\[ R \otimes R = \bigoplus Q \]

In each $Q$ the $\mathcal{R}$-matrix acts as unity: $\mathcal{R}_Q = \lambda_Q \cdot I_Q$

\[ \mathcal{H}_R^{[2,n]} = \sum_{Q} \lambda_Q^n \text{Tr}_Q I_Q = \sum_{Q} \lambda_Q^n D_Q \]

$D_Q = $ quantum dimension of irreducible representation $Q$
Symmetric representations

\[ [p] \otimes [p] = \bigoplus_{k=0}^{p} [2p - k, k] \]

\( SL(2): \) spin \( \frac{p}{2} \otimes \text{spin} \, \frac{p}{2} = \bigoplus_{k=0}^{p} \text{spin} \, (p - k) \)

\[ \mathcal{R}_Q = \lambda_Q \cdot I_Q \]

\[ \lambda_k = (-)^k q^{\varepsilon[2p-k,k]} \]

\[ \varepsilon[2p-k,k] = 2p^2 - (2k + 1)p + k(k - 1) \]
Universal formula

This is an example of 1-parametric general formulas, describing all the 2-strand braids at once

For example, \( R = [1], [1] \otimes [1] = [2] + [11] \):

\[
\mathcal{H}_{[1]}^{[2,n]} = q^n D_{[2]} + (-)^n q^{-n} D_{[11]}
\]

Two series \( n \) odd (knots) and \( n \) even (links):

\[
q^n D_{[2]} - q^{-n} D_{[11]}
\]

\[
q^n D_{[2]} + q^{-n} D_{[11]}
\]

Also possible for other representations (multiparametric formula)
This $\kappa_{[2p-k,k]}$ is an eigenvalue of the cut-and-join operator

$$\hat{W}[2] = \frac{1}{m} \sum_{a,b \geq 1} \left( (a+b)p_a p_b \frac{\partial}{\partial p_{a+b}} + ab p_{a+b} \frac{\partial^2}{\partial p_a \partial p_b} \right)$$

$$\hat{W}[2] S_Q\{p\} = \kappa_Q S_Q\{p\}, \quad \lambda_Q = q^{\kappa_Q}$$

$$S_1\{p\} = p_1, \quad S_2\{p\} = \frac{1}{2} (p_2 + p_1^2), \quad S_{11}\{p\} = \frac{1}{2} (-p_2 + p_1^2), \ldots$$

$$\kappa_Q = \sum_i q_i (q_i - 2i + 1) = \nu_Q - \nu_Q'$$

$$\nu_Q = \sum_i (i - 1) q_i$$
$SL(N)$ characters (Shur fns) $S_Q\{p\}$

are eigenfunctions of cut-and-join operators $\hat{W}(\Delta)$,

$$\hat{W}(\Delta)S_Q = \varphi_Q(\Delta)S_Q$$

$$\hat{W}(\Delta) = : \prod_i \text{tr} \left( X \frac{\partial}{\partial X} \right)^{\delta_i} :$$

$$p_k = \text{tr} X^k = kt_k$$

These operators form a commutative algebra, which has a non-trivial non-commutative extension.

For general theory of cut-and-join operators see [0904.4227]
W-representation of HOMFLY for the torus knot \([2, n]\)

\[
\mathcal{H}^{[2,n]}_R = \sum_Q \lambda_Q^n \ \text{Tr}_Q I_Q = \sum_Q \lambda_Q^n \ D_Q
\]

Extended HOMFLY polynomial:

\[
\mathcal{H}^{[2,n]}_R \{p_k\} = \sum_Q \lambda_Q^n \ S_Q \{p_k\} = q^{n\hat{W}} \sum_Q \epsilon_Q^n S_Q \{p_k\}
\]

For given series (\(n\) odd or even)

\[
= q^{n\hat{W}} \sum_Q \epsilon_Q S_Q \{p_k\} = q^{n\hat{W}} S_R \{p_{2k}\}
\]

\[
= q^{n\hat{W}} \sum_Q S_Q \{p_k\} = q^{n\hat{W}} S^2_R \{p_k\}
\]
W-representations

Partition functions can be considered as a result of "evolution", driven by cut-and-join (W) operators from very simple "initial conditions" [0902.2627]

\[ Z\{p\} = e^{g\hat{W}}\tau_0\{p\} \]

If \( W \in UGL(\infty) \), then KP/Toda-integrability is preserved

\[ \hat{W}_n = \frac{1}{2} \sum_{a,b} \left( (a + b + n)p_ap_b \frac{\partial}{\partial p_{a+b+n}} + abp_{a+b-n} \frac{\partial^2}{\partial p_a \partial p_b} \right) \]
W-representation. Examples

- **Hermitian matrix model**
  \[ Z_N = \int dX e^{\sum_k \frac{p_k}{k} \text{Tr} X^k} \]
  \[ Z_N = e^{\hat{W} - 2} e^{Np_0} \]

- **Kontsevich model**
  \[ Z = \int dX e^{\text{Tr}\left(\frac{1}{3}X^3 - L^2X\right)}, \quad p_k = \text{Tr}L^{-k} \]
  \[ Z = e^{\hat{W}_K^K} \cdot 1 \]
  \[ \hat{W}_K^K = \frac{2}{3} \sum \left(k + \frac{1}{2}\right) \tau_k L^K_{k-1} \quad [\text{A.Alexandrov, 1009.4887}] \]

- **Hurwitz model**
  [V.Bouchard & M.Marino, 0708.1458]
  \[ Z = e^{t\hat{W}_0} e^{p_1} \]

- **Torus knots and links**
  \[ Z = q^{\frac{2n}{m}} \hat{W}_0 \prod_{\text{link comps}} \tilde{\chi}_R \]
Topological invariance (homotopical equivalence) is restored on the topological locus in the space of time variables:

\[ p_k = p^*_k = \frac{A^k - A^{-k}}{q^k - q^{-k}} \]

topological invariants \(\leftarrow\) braid invariants

\[ D_Q \leftarrow S_Q\{p_k\}: \quad S_Q\{p^*_k\}|_{A=q^N} = D_Q \]

\[ D_{[1]} = [N]_q = \frac{q^N - q^{-N}}{q - q^{-1}}, \quad D_{[2]} = \frac{[N][N+1]}{[2]}, \quad D_{[11]} = \frac{[N][N-1]}{[2]}, \ldots \]
\[ \mathcal{H}_{[1]}^{[2,n]} = q^{n^*}S_{[2]} \pm q^{-n^*}S_{[11]} \]

\[ \mathcal{P}_{[1]}^{[2,n]} = q^n * M_{[2]} \pm \left( C_{[11]} \gamma_{[11]} \right) t^{-n} * M_{[11]} \]

\[ M_{[1]}\{p_{2k}\} = p_2 = M_{[2]}\{p_k\} - C_{[11]} M_{[11]}\{p_k\} \]

\[ \gamma_{[11]} = \frac{1 + q^2}{1 + t^2} \]
HL expansions

\[ \mathcal{H}_{[1,\ldots,1]}^{[m,n]} \{ q \mid p_k \} \sim L_{[m,\ldots,m]}^{[q^n]} \{ q^n \mid p_k \} \]

[1203.0667]

\[ L_Q(t) = M_Q(q, t)_{|q=0} \]

\[ P_{[1]}^{[m,r]} \sim \sum_{\frac{Q-m}{l(Q)\leq r}} h_Q^{(m,r)} L_Q(t) \]

\[ r = 1, 2: \quad h_Q = 1 \]
\[ r = 3: \quad h_Q = 1 + t + (q - t)[\min(Q_1 - Q_2, Q_2 - Q_3)]_q \]

[1201.3339]
3-strand knots

\[ \mathcal{H}_R^{(a_1, b_1 | a_2, b_2 | a_3, \ldots)} = \text{Tr}_{R \otimes 3} \left( R_{12}^{a_1} R_{23}^{b_1} R_{12}^{a_2} R_{23}^{b_2} R_{12}^{a_3} \ldots \right) \]

\[ R \otimes R \otimes R = \bigoplus Q \]
Reduction to the space of intertwiners [1112.2654]


\[ \mathcal{H}(a_1, b_1 \mid a_2, b_2 \mid a_3, \ldots) = q^{a_1+b_1+a_2+b_2+\ldots} S_{[3]} + (-q)^{a_1+b_1+a_2+b_2+\ldots} S_{[111]} + \\
+ \text{tr}_{2 \times 2} \left( \hat{R}^{a_1} U \hat{R}^{b_1} U^\dagger \hat{R}^{a_2} U \hat{R}^{b_2} U^\dagger \ldots \right) S_{[21]} \]

\[ \hat{R} = \begin{pmatrix} q & 0 \\ 0 & -q^{-1} \end{pmatrix}, \quad U = \begin{pmatrix} 1 & \sqrt{3} \\ [2] & [2] \end{pmatrix} \]

\[ -\sqrt{3} \\ [2] \]

\[ \frac{1}{2} \]

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Evaluation of HOMFLY pols is reduced to the study of the mixing matrices.

Can they be found in general form?
Fundamental representation $R = [1]$, many strands $m$

$m = 3 : \quad [1] \otimes^3 = [3] + 2 \cdot [21] + [111]$

$$\hat{R}_2 = \begin{pmatrix} q & -\frac{1}{q} \\ q & -\frac{1}{q} \end{pmatrix} \quad U_2 = \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix}$$

$m = 4 : \quad [1] \otimes^4 = [4] + ???$

$$\hat{R}_3 = \begin{pmatrix} q & & \ & q \\ q & & \ & q \\ & -\frac{1}{q} & \ & \end{pmatrix} \quad U_3 = \begin{pmatrix} 1 & & \ & c_2 & s_2 \\ & c_2 & s_2 & \ & -s_2 & c_2 \end{pmatrix} \quad V_3 = \begin{pmatrix} c_3 & s_3 & \ & -s_3 & c_3 & \ & & 1 \end{pmatrix}$$

$$c_k = \frac{1}{k} \quad \quad s_k = \sqrt{1 - c_k^2} = \sqrt{\frac{[k - 1] [k + 1]}{k}}$$
\[ m = 3 \text{ strands, symmetric representations } R = [\rho] \]

\[
\begin{pmatrix}
\frac{\sqrt{\lambda \mu}}{\lambda + \mu} & \frac{\sqrt{\lambda^2 + \lambda \mu + \mu^2}}{\lambda + \mu} \\
- \frac{\sqrt{\lambda^2 + \lambda \mu + \mu^2}}{\lambda + \mu} & \frac{\sqrt{\lambda \mu}}{\lambda + \mu}
\end{pmatrix}
\]

\[
\begin{pmatrix}
- \frac{\lambda_1 (\lambda_2 - \mu)}{(\lambda_1 - \lambda_2)(\lambda_1 + \mu)} & \frac{\sqrt{(\lambda_1 \lambda_2 + \mu^2) (\lambda_1^2 - \mu \lambda_2)}}{\sqrt{(\lambda_1 - \lambda_2)(\lambda_2 + \mu)(\lambda_1 + \mu)}} & - \frac{1}{\lambda_1 - \lambda_2} \sqrt{\frac{(\mu \lambda_1 - \lambda_2^2) (\lambda_1^2 - \mu \lambda_2)}{(\lambda_1 + \mu)(\lambda_2 + \mu)}} \\
- \frac{\sqrt{(\lambda_1 \lambda_2 + \mu^2) (\lambda_1^2 - \mu \lambda_2)}}{\sqrt{(\lambda_1 - \lambda_2)(\lambda_2 + \mu)(\lambda_1 + \mu)}} & - \frac{(\lambda_1 + \lambda_2) \mu}{(\lambda_1 + \mu)(\lambda_2 + \mu)} & \frac{\sqrt{(\lambda_1 \lambda_2 + \mu^2) (\mu \lambda_1 - \lambda_2^2)}}{\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 + \mu)(\lambda_2 + \mu)}} \\
- \frac{1}{\lambda_1 - \lambda_2} \sqrt{\frac{(\mu \lambda_1 - \lambda_2^2) (\lambda_1^2 - \mu \lambda_2)}{(\lambda_1 + \mu)(\lambda_2 + \mu)}} & \frac{\sqrt{(\lambda_1 \lambda_2 + \mu^2) (\mu \lambda_1 - \lambda_2^2)}}{\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 + \mu)(\lambda_2 + \mu)}} & \frac{\lambda_2 (\lambda_1 - \mu)}{(\lambda_1 - \lambda_2)(\lambda_2 + \mu)}
\end{pmatrix}
\]
Trefoil and the figure eight knot

trefoil: $3_1 = [2, 3] = [3, 2] = (1, 1|1, 1)$

$$\mathcal{H}^{3_1} = q^3 S[2] - q^{-3} S[11]$$

$$\mathcal{H}^{3_1} = q^4 S[3] + q^{-4} S[111] + \text{tr}_{2\times 2} \left( \hat{R} U \hat{R}^{\pm 1} U^\dagger \hat{R} U \hat{R}^{\pm 1} U^\dagger \ldots \right) S[21] =$$

$$= \frac{A - A^{-1}}{q - q^{-1}} \left( q^4 A - (q^2 + q^{-2}) + q^{-4} A^{-4} \right)$$

$$\times S[1](A|q)$$

figure eight knot: $4_1 = (1, -1|1, -1)$

$$\mathcal{H}^{4_1} = S[3] + S[111] + \text{tr}_{2\times 2} \left( \hat{R} U \hat{R}^{-1} U^\dagger \hat{R} U \hat{R}^{-1} U^\dagger \ldots \right) S[21]$$
Application to the figure eight knot [1203.5978]

\[ \frac{\mathcal{H}^{41}_[1](A | q)}{\mathcal{S}^{[1]}_1(A | q)} = A^2 - (q^2 - 1 + q^{-2}) + A^{-2} = \]

\[ = 1 + \left( Aq - (Aq)^{-1} \right) \left( Aq^{-1} - A^{-1}q^{-1} \right) = 1 + \{Aq\}\{Aq^{-1}\} \]

\[ \{x\} = x - x^{-1} \]


\[ \frac{\mathcal{H}^{41}_[2](A | q)}{\mathcal{S}^{[2]}_2(A | q)} = 1 + [2]_q\{Aq\}\{Aq^{-1}\} + \{Aq^3\}\{Aq^2\}\{A\}\{Aq^{-1}\} \]

[P.Ramadevi and T.Sarkar, hep-th/0009188]
Classical \((q = 1)\) case

\[
\sigma_R(A) = \lim_{q \to 1} \frac{\mathcal{H}_R^4(A|q)}{\mathcal{S}_R(A|q)} = \left(\sigma_{[1]}(A)\right)^{|R|}
\]

\[
\sigma_{[1]}(A)^{41} = 1 + \{Aq\}{Aq^{-1}}|_{q=1} = 1 + \{A\}^2 = 1 + (A - A^{-1})^2
\]

\[
\sigma_R^{41}(A) = \left(1 + \{A\}\right)^{|R|} = \sum_{k=0}^{|R|} C_k^{|R|}\{A\}^{2k}
\]
Quantization

$$\sigma_R^{41}(A) = \left(1 + \{A\}\right)^{|R|} = \sum_{k=0}^{\lfloor R \rfloor} \frac{|R|!}{k!(|R| - k)!} \{A\}^{2k}$$

$$\frac{\mathcal{H}_{[1]}^{41}(A|q)}{\mathcal{S}_{[1]}(A|q)} = 1 + \{Aq\}\{Aq^{-1}\}$$

$$\frac{\mathcal{H}_{[2]}^{41}(A|q)}{\mathcal{S}_{[2]}(A|q)} = 1 + [2]_q \{Aq\}\{Aq^{-1}\} + \{Aq^3\}\{Aq^2\}\{A\}\{Aq^{-1}\} + \cdots$$

$$\frac{\mathcal{H}_{[p]}^{41}(A|q)}{\mathcal{S}_{[2]}(A|q)} = \sum_{k=0}^{\lfloor p \rfloor} \frac{[p]!}{[k]![\lfloor p - k \rfloor]!} \prod_{i=0}^{\lfloor p - 1 \rfloor} \{Aq^{p+i}\}\{Aq^{i-1}\}$$
We have seven pieces of evidence:

Our answer

- Reproduces particular examples at $R = [2], [3], [4], [5]$
- For $q \to 1$ reproduces the conjectured special polynomials
- Consistent with the interesting formula, describing the value of
  \[ *H^4_1[p](A|q) \] at the one-dimensional locus
  \[ q = e^{i\pi N + p - 1}, \quad A = q^N = -e^{i\pi(1-p)(N+p-1)} \]
- For $A = q^2$ reproduces the known answers for the Jones polynomials
  - For $A = 1$ reproduces the Alexander polynomial
- Related antisymmetric HOMFLY polynomial $*H^4_{[1p]}(A|q)$ vanishes for
  $A = q^N$ with $N < p$, i.e. whenever $p$ exceeds the rank of the group by two, and turns its ratio to the unknot turns into unity for $N = p$.
- Consistent with the Ooguri-Vafa conjecture
Superpolynomial for $4_1$

$$\sum_{k=0}^{p} \frac{[|R|]!}{[k]![|R| - k]!} \prod_{i=1}^{k} Z_i(A) = \sum_{k=0}^{p} \prod_{i_1 \leq \ldots \leq i_k} Z_{i_1}(A) Z_{i_2}(Aq) Z_{i_3}(Aq^2) \ldots Z_{i_k}(Aq^{k-1})$$

$$Z_i(A) = \{Aq^{2(p-i)+1}\} \{Aq^{-1}\} \implies 3_i(A) = \{Aq^{2(p-i)+1}\} \{At^{-1}\}$$

$$\star \mathcal{P}_{[p]}^{4_1}(A|q,t) \star \mathcal{M}_{[p]}(A|q,t) = \sum_{k=0}^{p} \prod_{1 \leq i_1 \leq \ldots \leq i_k \leq p} 3_{i_1}(A) 3_{i_2}(Aq) 3_{i_3}(Aq^2) \ldots 3_{i_k}(Aq^{k-1})$$

$$t = q, \quad q = -qt, \quad A = a \sqrt{-t}$$

$$3_i(Aq^s) = \frac{(1 + a^2 t(qt)^{4(p-i)+2+2s}) (q^2 + a^2 t(qt)^{2s})}{a^2 \cdot (qt)^{2(p-i+s+1)}}$$
Difference equation

\[ P_{[p+1]}(A) - P_{[p]}(A) = \{ Aq^{2p+1} \} \{ At^{-1} \} P_{[p]}(qA) \]

\[ P_{[p]} = \frac{\ast P_{[p]}^{41}(A|q, t)}{\ast M_{[p]}(A|q, t)} \]
Generalizations

- from $R = [p]$ to arbitrary $R$ (arbitrary Young diagram $R = \{p_1 \geq p_2 \geq \ldots \geq 0\}$)
- from $\mathcal{K} = (1, -1|1, -1)$ to entire series of 3-strand knots $\mathcal{K} = (1, -1)^n = (1, -1|1, -1|\ldots|1, -1)$, a simple generalization of the torus knot family $\mathcal{K} = [3, n] = (1, 1)^n$ (these are knots for $n$, indivisible by $m = 3$ and 3-component links otherwise)
MANY THANKS FOR YOUR ATTENTION!
THANKS TO THE ORGANIZERS!!!