From AGT to knots

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THANKS for your support

Motivation for the study of "knots"

From AGT to knots

Knots can well be the next hot topic
in mathematical physics

Why?

Interesting

What is most exciting?

There are formulas

Many formulas

Complicated formulas

Mysterious formulas

Strongly interrelated formulas

Intimately related to all what we studied before:

matrix models,
integrable systems,
topological theories,
CFT,
SW theory, LMNS and Nekrasov fns, AGT relations

Related to most different parts of each story:

Unitary rather than Hermitian matrix models,
Almost no integral formulas [but: CS models],
only AMM/EO topological recursion [R.Dijkgraaf & H.Fuji]

Deviation from ordinary integrability:
OV pf $\sum_R H_R \chi_R\{\bar{p}_k\}$ never ordinary τ-functions,
$\sum_R \mathcal{H}_R\{p_k\}\chi_R\{\bar{p}_k\}$ are, but only sometime

quantum R-matrices and non-trivial reps of Hecke algebras

BPS spectrum, integrality, wall crossing phenomena

modular transformations

Universality classes are labeled by integrable systems

hep-th/9505035

$$\mathcal{N}=2 \text{ SYM models} \quad \stackrel{\text{AGT}}{\longleftrightarrow} \quad 2d \text{ CFT conformalblocks}$$

$$\updownarrow \qquad \text{dictionary } [1995-97] \qquad \updownarrow$$

$$1d \text{ integrable systems} \quad \stackrel{?}{\longleftrightarrow} \quad \text{DF/Penner matrix model}$$

quantization of integrable systems

Shroedinger-like equations (Fourier tr. of Baxter eqs.)

insertions of degenerate states

SW description through BS integrals

$$\Psi(z) = \exp \int^z \Omega, \qquad \Omega = Pdz$$

$$\partial F/\partial a = \oint_B \Omega, \quad a = \oint_A \Omega$$

NS limit $\epsilon_1 \to 0$, $\beta \to \infty$

AGT relation

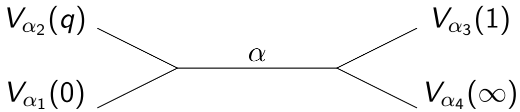
open problems
reformulations
interpretations
proofs

- Dotsenko-Fateev matrix model
- Hubbard-Stratanovich duality
- Relation to integrable systems
- Bohr-Sommerfeld integrals

One particular subject:

Beautiful
Conceptually important
Based on achievements in Osaka

DF/Penner/Selberg matrix model



$$\left\langle e^{\alpha_1\phi(0)} e^{\alpha_2\phi(q)} e^{\alpha_3\phi(1)} e^{\alpha_4\phi(\infty)} \prod_{i=1}^{N_1}\int_0^q e^{b\phi(x_i)} \prod_{j=1}^{N_2}\int_0^1 e^{b\phi(y_j)} \right\rangle$$

$$\alpha_1 + \alpha_2 + bN_1 = \alpha$$

$$\alpha + \alpha_3 + \alpha_4 + bN_2 = 0$$

$$= \int dx_i \int dy_j \, (x_i - x_{i'})^{2\beta} (y_j - y_{j'})^{2\beta} \underline{(x_i - y_j)^{2\beta}} (x_i y_j)^{2\alpha_1 b} \big((q - x_i)(q - y_j)\big)^{2\alpha_2 b} \big((1 - x_i)(1 - y_j)\big)^{2\alpha_3 b}$$

$$= \int_{d\mu(x)} \int_{d\mu(y)} \Big(\text{Mixing term}(x|y)\Big)^2$$

AGT as Hubbard-Stratanovich duality [1012.3137]

$$\approx \int_{d\mu(x)}\int_{d\mu(y)} \exp\left(2\beta\sum_{i,j}\log(1-x_iy_j)\right) =$$

$$= \int_{d\mu(x)}\int_{d\mu(y)} \exp\left(\underline{2}\beta\sum_{k} p_k\bar{p}_k/k\right)$$

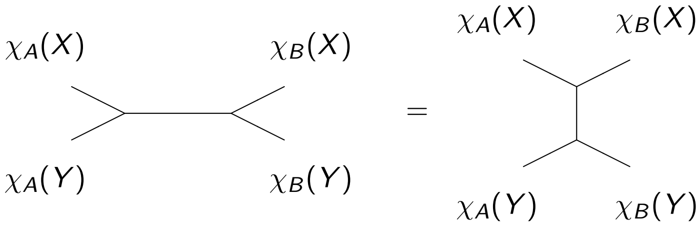
$$= \int_{d\mu(x)}\int_{d\mu(y)} \left(\sum_{A}\chi_A(X)\chi_A(Y)\right)\left(\sum_{B}\chi_B(X)\chi_B(Y)\right)$$

$$= \sum_{A,B}\left(\int_{d\mu(x)}\chi_A(X)\chi_B(X)\right)\left(\int_{d\mu(y)}\chi_A(Y)\chi_B(Y)\right)$$

$p_k = \mathrm{Tr}X^k, \quad \bar{p}_k = \mathrm{Tr}Y^k$ [*H.Itoyama & T.Oota* 1003.2929]

$$\exp\sum_k \frac{[\beta]_{q^k}p_k\bar{p}_k}{k} = \sum_A \frac{C_A}{C_{A'}}M_A(X)M_A(Y)$$

AGT as Hubbard-Stratanovich duality [1012.3137]



$$\sum_{X,Y}\left(\sum_A \chi_A(X)\chi_A(Y)\right)\left(\sum_B \chi_B(X)\chi_B(Y)\right) =$$

$$= \sum_{A,B}\left(\sum_X \chi_A(X)\chi_B(X)\right)\left(\sum_Y \chi_A(Y)\chi_B(Y)\right)$$

$$\text{Conformal block } = \sum_{A,B} N_{A,B}$$

Decomposition problem for $\beta \neq 1$

$$\int_{d\mu(X)} \chi_A(X)\chi_B(X) \int_{d\mu(Y)} \chi_A(Y)\chi_B(Y) \stackrel{?}{=} N_{A,B}$$

TRUE for $\beta = 1$
NOT so simple for $\beta \neq 1$

$$< \chi_{[1]}\ \chi_\bullet >< \chi_{[1]}\ \chi_\bullet > + < \chi_\bullet\ \chi_{[1]} >< \chi_\bullet\ \chi_{[1]} > =$$

$$= \frac{1}{(z-\epsilon)}\frac{1}{(z+\epsilon)} + \frac{1}{(z+\epsilon)}\frac{1}{(z-\epsilon)} =$$

$$= \frac{2}{z^2-\epsilon^2} = \frac{1}{z(z-\epsilon)} + \frac{1}{z(z+\epsilon)} = N_{[1],\bullet} + N_{\bullet,[1]}$$

For $\epsilon \neq 0$ ($\beta \neq 1$) particular Nekrasov functions
have extra poles (at $z = 0$), not present in Kac determinant

Decomposition problem

Instead Nekrasov functions are nicely factorized,
while Selberg correlators for $\beta \neq 1$ are not:

$$< \chi_{[3]}\ \chi_{\bullet} >_{BGW} \sim\ z^2 - (5\epsilon_1 + 8\epsilon_2)z + 6\epsilon_1^2 + 23\epsilon_1\epsilon_2 + 19\epsilon_2^2$$

$$\stackrel{\epsilon_2=-\epsilon_1}{\longrightarrow}\ z^2 + 3\epsilon_1 z + 2\epsilon_1^2 = (z+\epsilon_1)(z+2\epsilon_1)$$

Decomposition problem

Natural quantities, e.g. Selberg correlators
(involved into duality relations)
are linear combinations of
the nicely factorized functions (Nekrasov functions),
which possess extra singularities

AGT: generalizations

more models (quivers, 5d)
Wall crossing
hidden properties:
action of chiral algebra on the moduli spaces
(modular transform and other ingredients of SW theory)

3d AGT

What takes the place of conformal block?

"knot polynomials"

Task

The knot polynomials
should be studied and understood
at least as well as as conformal blocks

Enormous amount of experimental material
katlas.org

Almost no general formulas

Results

Formulas with free parameters,
which characterize either knots or representations,
for:

- HOMFLY pols for generic 3, 4, (5)-strand knots in $R = [1]$

- Colored HOMFLY for generic 3-strand braids and knots in $R = [2], [3], [4], [5]$

- Torus superpolynomials $\mathcal{B} = [m, n]$ in $R = [1]$
with $n \pmod{m} = \pm 1, \pm 2, \pm 3, \pm 4$
(up to $m = 9$, $[9, 13]$; the first unavailable is $[11, 16]$)
Mysterious hidden tropical structure is revealed in HL expansions

- Superpolynomials for the figure-eight knot 4_1
in all symmetric and antisymmetric representations $R = [p]$ and $R = [1^p]$

Main ideas

- Switch from knots to braids
- Introduce *extended* polynomials,
depending on infinitely many time-variables
- Study families of braids
- Derive formulas, not only pictures
- Use matrix model techniques,
(Ward identities, integrability, character expansions, integral formulas)
- Search for universalities
- Search for relations

HOMFLY polynomial

$$ {}^{*}\mathcal{H}_R^{\mathcal{K}}(A|\,q)\Big|_{A=q^N} = \left\langle \mathrm{Tr}_R P\exp\left(\oint_{\mathcal{K}} \mathcal{A}\right)\right\rangle_{CS} $$

Alternative definitions:

• from skein relations

• from Khovanov-Rozhansky theory –
as Euler characteristic of the triple-graded complex

$$ P(\mathbf{a}|\mathbf{q}|\mathbf{t}) = \sum_{I,J,K} N_{IJK}\mathbf{a}^I\mathbf{q}^J\mathbf{t}^K $$

$$ \mathcal{H}(A|q) = P(A|q|\mathbf{t}=-1) $$

• from averages of characters $<\chi_R[U]>^{\mathcal{K}}$

Hierarchy of knot polynomials for the $SL(N)$ family

For a given knot K and representation (Young diagram) R

$$\text{Superpolynomial } P_R(A|q|t)$$

$$\swarrow t = q \qquad\qquad \searrow A = 1$$

$$CS \longrightarrow \boxed{\text{HOMFLY } H_R(A|q)} \qquad \text{Heegard} - \text{Floer } HF_R(q|t)$$

$$q = 1 \swarrow \qquad N = 2 \searrow N = 0 \qquad \swarrow t = q$$

$$\text{Special } \sigma_R(A) \quad \text{Jones } J_R(q) \quad \text{Alexander } \mathcal{A}_R(q)$$

Representation dependence

$$\sigma_R(A) = \Big(\sigma_{[1]}(A)\Big)^{|R|}$$

$$\mathcal{A}_R(q) = \mathcal{A}_{[1]}(q^{|R|})$$

HOMFLY and superpolynomials satisfy difference equations
as functions of the representation-variable
Garoufalidis & Le, math/0309214
Fuji, Gukov & Sulkovski, 1203.2182
IMMM, 1203.5978

CS in temporal gauge $A_0 = 0$

$$\kappa \int \epsilon_{ijk} \mathrm{Tr}\Big(A^i \partial_j A^k + \frac{2}{3} A^i A^j A^k \Big) d^3x$$

$$\stackrel{A_0=0}{\longrightarrow} \quad \int \mathrm{Tr}\Big(A_x \dot{A}_y dxdy \Big) dt$$

Quadratic theory with the ultralocal propagator

$$\frac{2\pi i}{\kappa} \mathrm{sign}(t)\delta(x)\delta(y)$$

Average of a Wilson line is given by
projection onto the xy plane
and each intersection contributes

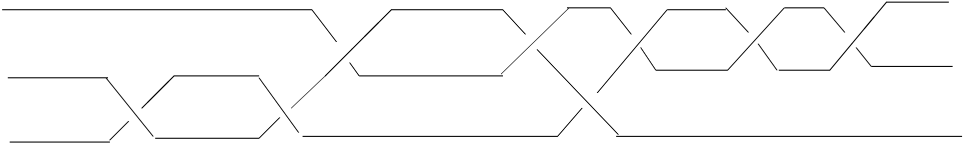
$$q^{\pm T^a \otimes T^a}$$

with $q = e^{2\pi i/(\kappa+N)}$

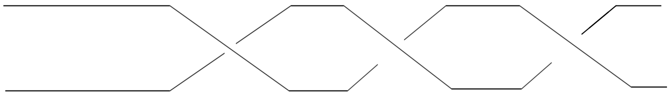
Turaev-Reshetikhin construction

$$\mathcal{H}_R^{\mathcal{B}} = \mathrm{Trace}_{R^{\otimes m}}\ \mathcal{B}$$

$$\mathcal{B} = \prod_s \mathcal{R}^{\pm}_{i(s),i(s)+1}$$



2-strand braids (torus knots $[2, n]$)



$$\mathcal{H}_R^{[2,n]} = \mathrm{Tr}_{R\otimes R}\mathcal{R}^n = \sum_Q \mathrm{Tr}_Q\ \mathcal{R}^n$$

$$R \otimes R = \oplus\ Q$$

In each Q the $\mathcal{R}$-matrix acts as unity: $\mathcal{R}_Q = \lambda_Q \cdot I_Q$

$$\mathcal{H}_R^{[2,n]} = \sum_Q \lambda_Q^n\ \mathrm{Tr}_Q I_Q = \sum_Q \lambda_Q^n\ D_Q$$

$D_Q =$ quantum dimension of irreducible representation Q

Symmetric representations

$$[p] \otimes [p] = \oplus_{k=0}^{p} [2p-k, k]$$

$$SL(2):\ \text{spin } p/2 \otimes \text{spin } p/2 = \oplus_{k=0}^{p} \text{spin } (p-k)$$

$$\mathcal{R}_Q = \lambda_Q \cdot I_Q$$

$$\lambda_k = (-)^k q^{\varkappa_{[2p-k,k]}}$$

$$\varkappa_{[2p-k,k]} = 2p^2 - (2k+1)p + k(k-1)$$

Universal formula

This is an example of 1-parametric general formulas, describing all the 2-strand braids at once

For example, $R = [1]$, $[1] \otimes [1] = [2] + [11]$:

$$\mathcal{H}_{[1]}^{[2,n]} = q^n D_{[2]} + (-)^n q^{-n} D_{[11]}$$

Two series n odd (knots) and n even (links):

$$q^n D_{[2]} - q^{-n} D_{[11]}$$

$$q^n D_{[2]} + q^{-n} D_{[11]}$$

Also possible for other representations (multiparametric formula)

W-eigenvalues

This $\varkappa_{[2p-k,k]}$ is an eigenvalue of the cut-and-join operator

$$\hat{W}[2] = \frac{1}{m}\sum_{a,b\geq 1}\left((a+b)p_a p_b \frac{\partial}{\partial p_{a+b}} + abp_{a+b}\frac{\partial^2}{\partial p_a \partial p_b}\right)$$

$$\hat{W}[2]S_Q\{p\} = \varkappa_Q S_Q\{p\}, \qquad \lambda_Q = q^{\varkappa_Q}$$

$$S_1\{p\} = p_1, \quad S_2\{p\} = \frac{1}{2}(p_2 + p_1^2), \quad S_{11}\{p\} = \frac{1}{2}(-p_2 + p_1^2), \quad \ldots$$

$$\varkappa_Q = \sum_i q_i(q_i - 2i + 1) = \nu_Q - \nu_{Q'}$$

$$\nu_Q = \sum_i (i-1)q_i$$

$SL(N)$ characters (Shur fns) $S_Q\{p\}$

are eigenfunctions of cut-and-join operators $\hat{W}(\Delta)$,

$$\hat{W}(\Delta)S_Q = \varphi_Q(\Delta)S_Q$$

$$\hat{W}(\Delta) = :\prod_i \mathrm{tr}\left(X\frac{\partial}{\partial X}\right)^{\delta_i}:$$

$$p_k = \mathrm{tr}X^k = kt_k$$

These operators form a commutative algebra,
which has a non-trivial non-commutative extension
For general theory of cut-and-join operators see [0904.4227]

W-representation of HOMFLY for the torus knot $[2, n]$

$$\mathcal{H}_R^{[2,n]} = \sum_Q \lambda_Q^n \, \mathrm{Tr}_Q I_Q = \sum_Q \lambda_Q^n \, D_Q$$

Extended HOMFLY polynomial:

$$\mathcal{H}_R^{[2,n]}\{p_k\} = \sum_Q \lambda_Q^n \, S_Q\{p_k\} = q^{n\hat{W}} \sum_Q \epsilon_Q^n S_Q\{p_k\}$$

For given series (n odd or even)

$$= q^{n\hat{W}} \sum_Q \epsilon_Q S_Q\{p_k\} = q^{n\hat{W}} S_R\{p_{2k}\}$$

$$= q^{n\hat{W}} \sum_Q S_Q\{p_k\} = q^{n\hat{W}} S_R^2\{p_k\}$$

W-representations

Partition functions can be considered as
a result of "evolution", driven by cut-and-join (W) operators
from very simple "initial conditions" [0902.2627]

$$Z\{p\} = e^{g\hat{W}}\tau_0\{p\}$$

If $W \in UGL(\infty)$, then KP/Toda-integrability is preserved

$$\hat{W}_n = \frac{1}{2}\sum_{a,b}\left((a+b+n)p_a p_b \frac{\partial}{\partial p_{a+b+n}} + ab p_{a+b-n}\frac{\partial^2}{\partial p_a \partial p_b}\right)$$

W-representation. Examples

- Hermitian matrix model $\quad Z_N = \int dX e^{\sum_k \frac{p_k}{k} \mathrm{Tr} X^k}$

$$Z_N = e^{\hat{W}_{-2}} e^{N p_0}$$

- Kontsevich model $\quad Z = \int dX e^{\mathrm{Tr}(\frac{1}{3}X^3 - L^2 X)}, \; p_k = \mathrm{Tr} L^{-k}$

$$Z = e^{\hat{W}^K_{-1}} \cdot 1$$

$\hat{W}^K_{-1} = \frac{2}{3} \sum \left(k + \frac{1}{2}\right) \tau_k L^K_{k-1}$ [A.Alexandrov, 1009.4887]

- Hurwitz model [V.Bouchard & M.Marino, 0708.1458]

$$Z = e^{t \hat{W}_0} e^{p_1}$$

- Torus knots and links

$$Z = q^{\frac{2n}{m} \hat{W}_0} \prod_{\text{link comps}} \tilde{\chi}_R$$

Back from braids to knots

Topological invariance (homotopical equivalence)
is restored on the *topological locus*
in the space of time variables:

$$p_k = p_k^* = \frac{A^k - A^{-k}}{q^k - q^{-k}}$$

topological invariants $\longleftarrow$ braid invariants

$$D_Q \ \longleftarrow \ S_Q\{p_k\}: \qquad S_Q\{p_k^*\}\big|_{A=q^N} = D_Q$$

$$D_{[1]} = [N]_q = \frac{q^N - q^{-N}}{q - q^{-1}},\ D_{[2]} = \frac{[N][N+1]}{[2]},\ D_{[11]} = \frac{[N][N-1]}{[2]},\ \ldots$$

From HOMFLY to superpolynomials

$$\mathcal{H}_{[1]}^{[2,n]} = q^{n*} S_{[2]} \pm q^{-n*} S_{[11]}$$

$$\mathcal{P}_{[1]}^{[2,n]} = q^{n\ *} M_{[2]} \pm \Big(C_{[11]} \gamma_{[11]} \Big) t^{-n\ *} M_{[11]}$$

$$M_{[1]}\{p_{2k}\} = p_2 = M_{[2]}\{p_k\} - C_{[11]} M_{[11]}\{p_k\}$$

$$\gamma_{[11]} = \frac{1+q^2}{1+t^2}$$

HL expansions

$$\mathcal{H}_{[1,\dots,1]}^{[m,n]}\{q|p_k\} \sim L_{[m,\dots,m]}\{q^n|p_k\}$$

[1203.0667]

$$L_Q(t) = M_Q(q,t)\big|_{q=0}$$

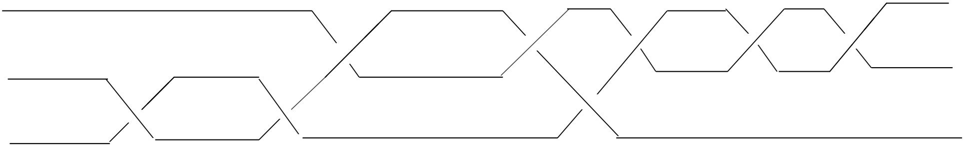
$$\mathcal{P}_{[1]}^{[m,r]} \sim \sum_{\substack{Q\vdash m \\ l(Q)\leq r}} h_Q^{(m,r)} L_Q(t)$$

$$r = 1,2: \quad h_Q = 1$$

$$r = 3: \quad h_Q = 1 + t + (q-t)[\min(Q_1 - Q_2, Q_2 - Q_3)]_q$$

[1201.3339]

3-strand knots



$$\mathcal{H}_R^{(a_1,b_1|a_2,b_2|a_3,\ldots)} = \mathrm{Tr}_{R^{\otimes 3}}\left(\mathcal{R}_{12}^{a_1}\mathcal{R}_{23}^{b_1}\mathcal{R}_{12}^{a_2}\mathcal{R}_{23}^{b_2}\mathcal{R}_{12}^{a_3}\ldots\right)$$

$$R \otimes R \otimes R = \oplus\ Q$$

Reduction to the space of intertwiners [1112.2654]

$$[1]\otimes[1]\otimes[1] = ([2]+[11])\otimes[1] = [3]+[21]+[21]+[111]$$

$$[1]\otimes[1]\otimes[1] = [1]\otimes([2]+[11]) = [3]+[21]+[21]+[111]$$

$$\mathcal{H}_{[1]}^{(a_1,b_1|a_2,b_2|a_3,\ldots)} = q^{a_1+b_1+a_2+b_2+\ldots}S_{[3]} + (-q)^{a_1+b_1+a_2+b_2+\ldots}S_{[111]} +$$
$$+\mathrm{tr}_{2\times 2}\left(\hat{\mathcal{R}}^{a_1}U\hat{\mathcal{R}}^{b_1}U^\dagger\hat{\mathcal{R}}^{a_2}U\hat{\mathcal{R}}^{b_2}U^\dagger\ldots\right)S_{[21]}$$

$$\hat{\mathcal{R}} = \begin{pmatrix} q & \\ & -q^{-1}\end{pmatrix}, \qquad U = \begin{pmatrix} \frac{1}{[2]} & \frac{\sqrt{[3]}}{[2]} \\ -\frac{\sqrt{[3]}}{[2]} & \frac{1}{[2]}\end{pmatrix}$$

Mixing matrices

Evaluation of HOMFLY pols
is reduced to the study of the mixing matrices

Can they be found in general form?

Fundamental representation $R = [1]$, many strands m

$$m = 3: \quad [1]^{\otimes 3} = [3] + \underline{2} \cdot [21] + [111]$$

$$\hat{\mathcal{R}}_2 = \begin{pmatrix} q & \\ & -\frac{1}{q} \end{pmatrix} \qquad U_2 = \begin{pmatrix} c_2 & s_2 \\ -s_2 & c_2 \end{pmatrix}$$

$$m = 4: \quad [1]^{\otimes 4} = [4] + ???$$

$$\hat{\mathcal{R}}_3 = \begin{pmatrix} q & & \\ & q & \\ & & -\frac{1}{q} \end{pmatrix} \quad U_3 = \begin{pmatrix} 1 & & \\ & c_2 & s_2 \\ & -s_2 & c_2 \end{pmatrix} \quad V_3 = \begin{pmatrix} c_3 & s_3 & \\ -s_3 & c_3 & \\ & & 1 \end{pmatrix}$$

$$c_k = \frac{1}{[k]}, \qquad s_k = \sqrt{1 - c_k^2} = \frac{\sqrt{[k-1]\,[k+1]}}{[k]}$$

$m = 3$ strands, symmetric representations $R = [p]$

$$\begin{pmatrix} \frac{\sqrt{\lambda\mu}}{\lambda+\mu} & \frac{\sqrt{\lambda^2+\lambda\mu+\mu^2}}{\lambda+\mu} \\ -\frac{\sqrt{\lambda^2+\lambda\mu+\mu^2}}{\lambda+\mu} & \frac{\sqrt{\lambda\mu}}{\lambda+\mu} \end{pmatrix}$$

$$\begin{pmatrix} -\frac{\lambda_1(\lambda_2-\mu)}{(\lambda_1-\lambda_2)(\lambda_1+\mu)} & \frac{\sqrt{(\lambda_1\lambda_2+\mu^2)(\lambda_1^2-\mu\lambda_2)}}{\sqrt{(\lambda_1-\lambda_2)(\lambda_2+\mu)}\ (\lambda_1+\mu)} & -\frac{1}{\lambda_1-\lambda_2}\sqrt{\frac{(\mu\lambda_1-\lambda_2^2)(\lambda_1^2-\mu\lambda_2)}{(\lambda_1+\mu)(\lambda_2+\mu)}} \\ -\frac{\sqrt{(\lambda_1\lambda_2+\mu^2)(\lambda_1^2-\mu\lambda_2)}}{\sqrt{(\lambda_1-\lambda_2)(\lambda_2+\mu)}\ (\lambda_1+\mu)} & -\frac{(\lambda_1+\lambda_2)\mu}{(\lambda_1+\mu)(\lambda_2+\mu)} & -\frac{\sqrt{(\lambda_1\lambda_2+\mu^2)(\mu\lambda_1-\lambda_2^2)}}{\sqrt{(\lambda_1-\lambda_2)(\lambda_1+\mu)}\ (\lambda_2+\mu)} \\ -\frac{1}{\lambda_1-\lambda_2}\sqrt{\frac{(\mu\lambda_1-\lambda_2^2)(\lambda_1^2-\mu\lambda_2)}{(\lambda_1+\mu)(\lambda_2+\mu)}} & \frac{\sqrt{(\lambda_1\lambda_2+\mu^2)(\mu\lambda_1-\lambda_2^2)}}{\sqrt{(\lambda_1-\lambda_2)(\lambda_1+\mu)}\ (\lambda_2+\mu)} & \frac{\lambda_2(\lambda_1-\mu)}{(\lambda_1-\lambda_2)(\lambda_2+\mu)} \end{pmatrix}$$

...

Trefoil and the figure eight knot

trefoil: $\quad 3_1 = [2,3] = [3,2] = (1,1|1,1)$

$$\mathcal{H}^{3_1}_{[1]} = q^3 S_{[2]} - q^{-3} S_{[11]}$$

$$\mathcal{H}^{3_1}_{[1]} = q^4 S_{[3]} + q^{-4} S_{[111]} + \mathrm{tr}_{2\times 2}\left(\hat{\mathcal{R}} U \hat{\mathcal{R}}^{\pm 1} U^\dagger \hat{\mathcal{R}} U \hat{\mathcal{R}}^{\pm 1} U^\dagger \ldots\right) S_{[21]} =$$

$$= \underbrace{\frac{A - A^{-1}}{q - q^{-1}}}_{{}^*S_{[1]}(A|\,q)} \left(q^4 A - (q^2 + q^{-2}) + q^{-4} A^{-4}\right)$$

figure eight knot: $\quad 4_1 = (1,-1|1,-1)$

$$\mathcal{H}^{4_1}_{[1]} = S_{[3]} + S_{[111]} + \mathrm{tr}_{2\times 2}\left(\hat{\mathcal{R}} U \hat{\mathcal{R}}^{-1} U^\dagger \hat{\mathcal{R}} U \hat{\mathcal{R}}^{-1} U^\dagger \ldots\right) S_{[21]}$$

Application to the figure eight knot [1203.5978]

$$\frac{{}^*\mathcal{H}^{4_1}_{[1]}(A|\,q)}{{}^*S_{[1]}(A|\,q)} = A^2 - (q^2 - 1 + q^{-2}) + A^{-2} =$$

$$= 1 + \Big(Aq - (Aq)^{-1}\Big)\Big(Aq^{-1} - A^{-1}q^{-1}\Big) = 1 + \{Aq\}\{Aq^{-1}\}$$

$$\{x\} = x - x^{-1}$$

$$[1]\times[1]\times[1] = [3] + 2\cdot[21] + [111]$$

$$[2]\times[2]\times[2] = [6] + 2\cdot[51] + 3\cdot[42] + [411] + [33] + 2\cdot[321] + [222]$$

$$\frac{{}^*\mathcal{H}^{4_1}_{[2]}(A|\,q)}{{}^*S_{[2]}(A|\,q)} = 1 + [2]_q\{Aq\}\{Aq^{-1}\} + \{Aq^3\}\{Aq^2\}\{A\}\{Aq^{-1}\}$$

[P.Ramadevi and T.Sarkar, hep-th/0009188]

Classical ($q = 1$) case

$$\sigma_R(A) = \lim_{q=1} \frac{{}^*\mathcal{H}_R^{4_1}(A|\,q)}{{}^*S_R(A|\,q)} = \Big(\sigma_{[1]}(A)\Big)^{|R|}$$

$$\sigma_{[1]}(A)^{4_1} = 1 + \{Aq\}\{Aq^{-1}\}\Big|_{q=1} = 1 + \{A\}^2 = 1 + (A - A^{-1})^2$$

$$\sigma_R^{4_1}(A) = \Big(1 + \{A\}\Big)^{|R|} = \sum_{k=0}^{|R|} C_k^{|R|}\{A\}^{2k}$$

Quantization

$$\sigma_R^{4_1}(A) = \Big(1 + \{A\}\Big)^{|R|} = \sum_{k=0}^{|R|} \frac{|R|!}{k!(|R|-k)!}\{A\}^{2k}$$

$$\frac{{}^*\mathcal{H}_{[1]}^{4_1}(A|\,q)}{{}^*S_{[1]}(A|\,q)} = 1 + \{Aq\}\{Aq^{-1}\}$$

$$\frac{{}^*\mathcal{H}_{[2]}^{4_1}(A|\,q)}{{}^*S_{[2]}(A|\,q)} = 1 + [2]_q\{Aq\}\{Aq^{-1}\} + \{Aq^3\}\{Aq^2\}\{A\}\{Aq^{-1}\}$$

$$\ldots$$

$$\frac{{}^*\mathcal{H}_{[p]}^{4_1}(A|\,q)}{{}^*S_{[2]}(A|\,q)} = \sum_{k=0}^{p} \frac{[p]!}{[k]![p-k]!} \prod_{i=0}^{p-1} \{Aq^{p+i}\}\{Aq^{i-1}\}$$

Checks

We have seven pieces of evidence:

Our answer

• Reproduces particular examples at $R = [2], [3], [4], [5]$

• For $q \to 1$ reproduces the conjectured special polynomials

• Consistent with the interesting formula, describing the value of ${}^*\mathcal{H}^{4_1}_{[p]}(A|q)$ at the one-dimensional locus $q = e^{\frac{i\pi}{N+p-1}}$, $A = q^N = -e^{\frac{i\pi(1-p)}{N+p-1}}$

• For $A = q^2$ reproduces the known answers for the Jones polynomials

• For $A = 1$ reproduces the Alexander polynomial

• Related antisymmetric HOMFLY polynomial ${}^*\mathcal{H}^{4_1}_{[1^p]}(A|\,q)$ vanishes for $A = q^N$ with $N < p$, i.e. whenever p exceeds the rank of the group by two, and turns its ratio to the unknot turns into unity for $N = p$.

• Consistent with the Ooguri-Vafa conjecture

Superpolynomial for 4_1

$$\sum_{k=0}^{p} \frac{[|R|]!}{[k]![|R|-k]!} \prod_{i=1}^{k} Z_i(A) = \sum_{k=0}^{p} \prod_{i_1 \le \ldots \le i_k} Z_{i_1}(A) Z_{i_2}(Aq) Z_{i_3}(Aq^2) \ldots Z_{i_k}(Aq^{k-1})$$

$$Z_i(A) = \{Aq^{2(p-i)+1}\}\{Aq^{-1}\} \quad \longrightarrow \quad \mathfrak{Z}_i(A) = \{Aq^{2(p-i)+1}\}\{At^{-1}\}$$

$$\frac{{}^*\mathcal{P}^{4_1}_{[p]}(A|\,q,t)}{{}^*M_{[p]}(A|\,q,t)} = \sum_{k=0}^{p} \prod_{1 \le i_1 \le \ldots \le i_k \le p} \mathfrak{Z}_{i_1}(A) \mathfrak{Z}_{i_2}(Aq) \mathfrak{Z}_{i_3}(Aq^2) \ldots \mathfrak{Z}_{i_k}(Aq^{k-1})$$

$$t = \mathbf{q}, \qquad q = -\mathbf{qt}, \qquad A = \mathbf{a}\sqrt{-\mathbf{t}}$$

$$\mathfrak{Z}_i(Aq^s) = \frac{\left(1 + \mathbf{a}^2\mathbf{t}(\mathbf{qt})^{4(p-i)+2+2s}\right)\left(\mathbf{q}^2 + \mathbf{a}^2\mathbf{t}(\mathbf{qt})^{2s}\right)}{\mathbf{a}^2 \cdot (\mathbf{qt})^{2(p-i+s+1)}}$$

Difference equation

$$P_{[p+1]}(A) - P_{[p]}(A) = \{Aq^{2p+1}\}\{At^{-1}\}P_{[p]}(qA)$$

$$P_{[p]} = \frac{{}^{*}\mathcal{P}^{4_1}_{[p]}(A|q,t)}{{}^{*}M_{[p]}(A|q,t)}$$

Generalizations

• from $R = [p]$ to arbitrary R (arbitrary Young diagram $R = \{p_1 \geq p_2 \geq \ldots \geq 0\}$)

• from $\mathcal{K} = (1,-1|1,-1)$ to entire series of 3-strand knots $\mathcal{K} = (1,-1)^n = (1,-1|1,-1|\ldots|1,-1)$, a simple generalization of the torus knot family $\mathcal{K} = [3,n] = (1,1)^n$ (these are knots for n, indivisible by $m = 3$ and 3-component links otherwise)

MANY THANKS FOR YOUR ATTENTION!

THANKS TO THE ORGANIZERS!!!