Surprising trace anomaly from freakolography

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Freakonomics: application of economics to what is traditionally beyond its scope, uncovering hidden side of everything

(Levitt, Dubner)
Freakolography:

application of **holography** to what is traditionally beyond its scope, uncovering hidden side of everything

(Nakayama)
Two main questions:

• Scale inv = Conformal inv ?
• Surprising d=4 trace anomaly

Two main freakolographic methods:

• Space-time flipped Horava gravity
• Spontaneous Lorentz (AdS) symmetry breaking and violation of NEC
Scale = Conformal?

• It is not true. Counterexamples in higher dimension (e.g. 5d U(1) Maxwell theory)

• But in d=2, the equivalence was shown by Zamolodchikov-Polchinski under unitarity, causality etc.

• Not known in d=3,4. One of major obstructions for the proof of a-theorem
Conf vs Scale in EM tensor

• Scale invariance

\[ x^\mu \rightarrow \lambda x^\mu \]

→Trace of energy-momentum (EM) tensor is a divergence of a so-called Virial current

\[ T^\mu_\mu = \partial^\mu J_\mu \quad D_\mu = x_\nu T^\nu_\mu - J_\mu \]

• Conformal invariance

\[ x^\mu \rightarrow \frac{x^\mu + a^\mu x^2}{1 + 2a^\mu x_\mu + a^2 x^2} \]

• EM tensor can be improved to be traceless

\[ J_\mu = \partial^\nu L_{\mu\nu} \quad T^\mu_\mu \rightarrow \tilde{T}^\mu_\mu = 0 \quad K_\mu = \nu_\nu \tilde{T}^\nu_\mu \]
Unexpected trace anomaly

• Dimensional analysis (4d):

\[ T^\mu_\mu = a(Euler) - c(Weyl^2) \]
\[ + bR^2 + b' \Box R + e\epsilon^{\rho\sigma\alpha\beta} R_{\rho\sigma\mu\nu} R^{\mu\nu}_{\alpha\beta} + \text{non anomalous terms} \]

• Euler = \[ R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} - 4R^{\mu\nu} R_{\mu\nu} + R^2 \]
  is important in “a-theorem” (“a” decreases along RG flow)

• Hirzebruch-Pontryagin term is CP violating but can appear in CP-violating CFT (in principle)

• \[ R^2 \] is inconsistent(?) for CFTs but it can appear in scale but non-CFT

• We’ll see these unexpected terms from freakolography
Scenario one could imagine

- To get scale (but non-conf) inv, the beta function may not vanish
  \[ T^\mu_{\mu} = \beta^i O_i = \partial^\mu J_\mu \]

- If the virial current is chiral, then we expect gravitational chiral anomaly
  \[ T^\mu_{\mu} - bR^2 - aEuler + c(Weyl)^2 = \beta^i O_i = D^\mu J_\mu - \epsilon^\rho\sigma\alpha\beta R^\mu_{\rho\sigma\mu\nu} R^\nu_{\alpha\beta} \]

- $R^2$ and Hirzebruch-Pontryagin term can both appear (in principle)

- Is it easy if you try?
  - If you break unitarity, it is easy.
  - It is inconsistent with strongest a-theorem (if any)
    \[ \frac{da}{d\log \mu} = -g_{ij} \beta^i \beta^j \]
Holographic / freakolographic computation

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Holography: Start from geometry

d+1 metric with d dim Poincare + scale invariance automatically selects AdS\(_{d+1}\) space

\[ ds^2 = \frac{dz^2}{z^2} + f(z) dx_\mu dx^\mu \]

\[ z \rightarrow \lambda z , \quad t \rightarrow \lambda t , \quad x \rightarrow \lambda x \]

\[ ds^2 = \frac{dz^2 + dx_\mu^2}{z^2} \]

\[ \delta x_\mu = 2(\epsilon^\nu x_\mu)x_\nu - (z^2 + x^\nu x_\nu)\epsilon_\mu , \quad \delta z = 2(\epsilon^\nu x_\nu)z \]
Freakolography: space-time flipped Horava theory

Enhancement of “Isometry” requires $d+1$ diffeomorphism, so Horava theory which only preserves foliation preserving diffeomorphism does not work.

$$ds^2 = \frac{dz^2 + dx_\mu^2}{z^2}$$

$$\delta x_\mu = 2(\epsilon^\nu x_\mu) x_\nu - (z^2 + x^\nu x_\nu) \epsilon_\mu, \quad \delta z = 2(\epsilon^\nu x_\nu) z$$

is not foliation preserving diff

$$\delta N = \partial_r (N f)$$

$$\delta N^{\mu} = \partial_r (N^{\mu} f) + \partial_r \xi^{\mu} + \mathcal{L}_\xi N^{\mu}$$

$$\delta g_{\mu\nu} = f \delta_r g_{\mu\nu} + \mathcal{L}_\xi g_{\mu\nu}.$$
Alternatively Lorentz breaking

Non-trivial vector matter configuration may break AdS isometry spontaneously:
Identify its dual as Virial current.
(Horava gravity and Lorentz breaking are closely related)

Example: non-trivial vector field

\[ A = A_M dx^M = \frac{adz}{z} \]

Not invariant under special conformal

\[ \delta x_\mu = 2(\epsilon^\nu x_\mu) x_\nu - (z^2 + x^\nu x_\nu) \epsilon_\mu, \quad \delta z = 2(\epsilon^\nu x_\nu) z \]

Dual to Virial current \[ T^\mu_\mu = \partial^\mu J_\mu \]
Interpretation: Holographic c-theorem

Holographic c-theorem gives:
\[ ds^2 = e^{2A(r)} dx^\mu dx_\mu + dr^2 \]
\[ \frac{da}{dr} \sim \frac{A''}{(A')^d} \sim (T_t^t - T_r^r) = k^M k^N T_{MN} \]

Null energy condition (NEC) leads to strong c-theorem
\[ k^M k^N T_{MN} \geq 0 \quad k^2 = 0 \]

Strict null energy condition leads to strongest c-theorem (= positivity of metric)
\[ T_t^t - T_r^r \sim G^{IJ} \partial_r \Phi_I \partial_r \Phi_J \sim G^{IJ} \beta_I \beta_J \geq 0 \]

Modification of the possibility \( \beta^I O_I = \partial^\mu J_\mu \) \( \rightarrow \) sigma model is gauged.
\( G^{IJ}(\partial_r - A_r) \Phi_I (\partial_r - A_r) \Phi_J \)
\[ \Phi^I \sim \zeta^i a \] corresponds to “cyclic” RG-flow

In unitary gauge, we conclude scale but non-conf
bulk vector condensation
\[ A = A_M dx^M = \frac{adz}{z} \]
Such a non-trivial configuration violates (strict) **Null Energy Condition**

Null energy condition: \( R_{MN} k^M k^N \geq 0 \), \( k^M k_M = 0 \)

\[
L = -\frac{1}{4} F_{MN} F^{MN} + m^2 A_M A^M + \lambda (A_M A^M)^2
\]

\[
R_{zz} + R_{tt} = (m^2 + 2\lambda a^2) a^2 = 0
\]

- More generically, we need strict NEC to completely exclude the possibility
- Equivalent for strongest holographic c-theorem
- It is true in supergravity compactification
Freakolography and trace anomaly

• Consider space-time flipped Horava gravity whose dual is scale but non-conformal

\[ ds^2 = N^2 dr^2 + G_{\mu\nu} (dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr) \]
\[ S = \int N dr \sqrt{-G} d^d x (K^{\mu\nu} K_{\mu\nu} - \lambda K^2 + R + \Lambda) . \]
\[ K_{\mu\nu} = \frac{1}{2N} (\partial_\tau G_{\mu\nu} - D_\mu N_\nu - D_\nu N_\mu) \]

• Introduce the Graham-Fefferman ansatz

\[ ds^2 = l^2 \left( \frac{d\rho^2}{4\rho^2} + \frac{g_{\mu\nu}(\rho, x) dx^\mu dx^\nu}{\rho} \right) \]
\[ g = g^{(0)} + \rho g^{(2)} + \cdots + \rho^{d/2} g^{(d)} + \rho^{d/2} \log \rho h^{(d)} + \mathcal{O}(\rho^{d/2+1}) \]

• Solve EOM and study the log counterterm

\[ S = \frac{l^3}{4} \log \epsilon \int d^4 x \sqrt{-g_0} \left( R_{\mu\nu}^{(0)} R^{\mu\nu}^{(0)} - \frac{\lambda}{4\lambda - 1} R^{(0)2} \right) . \]
Space-time flipped Horava gravity

- Holographic (freakolographic) trace anomaly is

\[
\langle T^\mu_\mu \rangle = -2c \left( R_{\mu\nu}R^{\mu\nu} - \frac{\lambda}{4\lambda - 1} R^2 \right)
\]

\[
= c \left( (\text{Euler} - \text{Weyl})^2 - \frac{2}{3} \frac{\lambda - 1}{4\lambda - 1} R^2 \right).
\]

- We found \( R^2 \) term! Cannot be conformal!

- One may further add \( \int N\, dr \sqrt{-G} \, d^4 x \, K \epsilon^{\rho\sigma\alpha\beta} R_{\rho\sigma\mu\nu} R^{\mu\nu}_{\alpha\beta} \)
  to generate CP violating trace anomaly

\[
T^\mu_\mu = c \epsilon^{\rho\sigma\alpha\beta} R_{\rho\sigma\mu\nu} R^{\mu\nu}_{\alpha\beta}
\]

- Or, CS-gravity coupling with vector condensation directly gives CP-odd as expected from anomaly in virial current

\[
\int d^5 x \sqrt{g} \epsilon_{MNLPQ} A^M R^N_{\ AB} R^{PQAB} = a \log \epsilon \int d^4 x \sqrt{g} \epsilon^{\rho\sigma\alpha\beta} R_{\mu\nu\rho\sigma} R^{\mu\nu}_{\alpha\beta}
\]
Can/should (strict) NEC kill freakolography?

- In $d=2$, boundary ($d=3$ bulk), we can show strongest c-theorem, and scale $= \text{inv}$, so effectively, strict NEC must be true
- (strict) NEC is related to unitarity?
- NEC gives area non-decreasing theorem for black hole holography
- No information in zero-energy states (\(=\) strict NEC)
- Counterexample in higher dimension? $d = 4 - \epsilon$ result by Grinstein et al?
What we leaned from holography

- **Full space-time diff** is tightly related to the emergence of conformal invariance

- It is possible to construct scale but non-conf geometry at the sacrifice of full space-time diff (spontaneous Lorentz symmetry breaking, Horava-like gravity…)

- Are they good? (violation of **NEC**, unitarity?…)

- No holography? Or Freakoholography?
After the success, Levitt and Dubner wrote the second book `\textit{super freakonomics}`.

Naturally, we expect we’ll hear about `\textit{super freakolography}` next time.

Stay tuned!!