A NOVEL LARGE-N REDUCTION FOR N=4 SYM ON R×S^3 AND THE ADS/CFT CORRESPONDENCE

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Introduction

What we want to do

We would like to formulate N=4 SYM nonperturbatively through a novel large-N reduction and study its strongly coupled regime

- Motivation

1. Testing the AdS/CFT correspondence

2. Nonperturbative formulation of 4D supersymmetric gauge theory as an alternative to lattice theory

3. Emergent curved space in matrix model

cf.) Kawai's talk, Nishimura's talk

AdS/CFT correspondence

Maldacena ('97)

N=4 SU(N) SYM ⟷ Type IIB string theory on $AdS_5 \times S^5$

$$PSU(2,2|4) \supset SO(4,2) \times SO(6)$$

32 supercharges

$$\lambda = g_{YM}^2 N = 2\pi g_s N = R^4/(2\alpha'^2)$$

planar limit ('t Hooft limit) $N \to \infty, \quad \lambda = \text{fixed}$ ⟷ string loop correction suppressed $g_s \to 0$

large $\lambda, \ N \to \infty$ ⟷ α' correction suppressed SUGRA or classical string approximation is good

important (→ large $\lambda, \ N \to \infty$)

tractable (→ α' correction suppressed, SUGRA or classical string approximation is good)

AdS/CFT correspondence (cont'd)

- The conjecture has not been proven, although there is a physical argument based on near horizon limit and open-closed duality
- In order to test the conjecture, one should study the large 't Hooft coupling region in the planar limit on the gauge theory side
- Examples of calculations which have been done so far in such a region

  vev of half-BPS Wilson loop ~ localization works   Pestun

  3-pt function of chiral primary operators ~ non-renormalization

  Spin chains ~ integrable model interpolating weak and strong couplings
- In order to perform a more direct test for quantities not protected by SUSY such as vev of non-BPS Wilson loop and 4-pt function of CPOs, one needs nonperturbative formulation of N=4 SYM like lattice theory

Large-N reduction as an alternative to lattice regularization

- It is impossible to realize full supersymmetry on lattice ⇐ $\{Q, \bar{Q}\} = \not{P}$
  One or two supersymmetries can be realized

  Sugino, Kaplan-Katz-Unsal, Catterall,…..

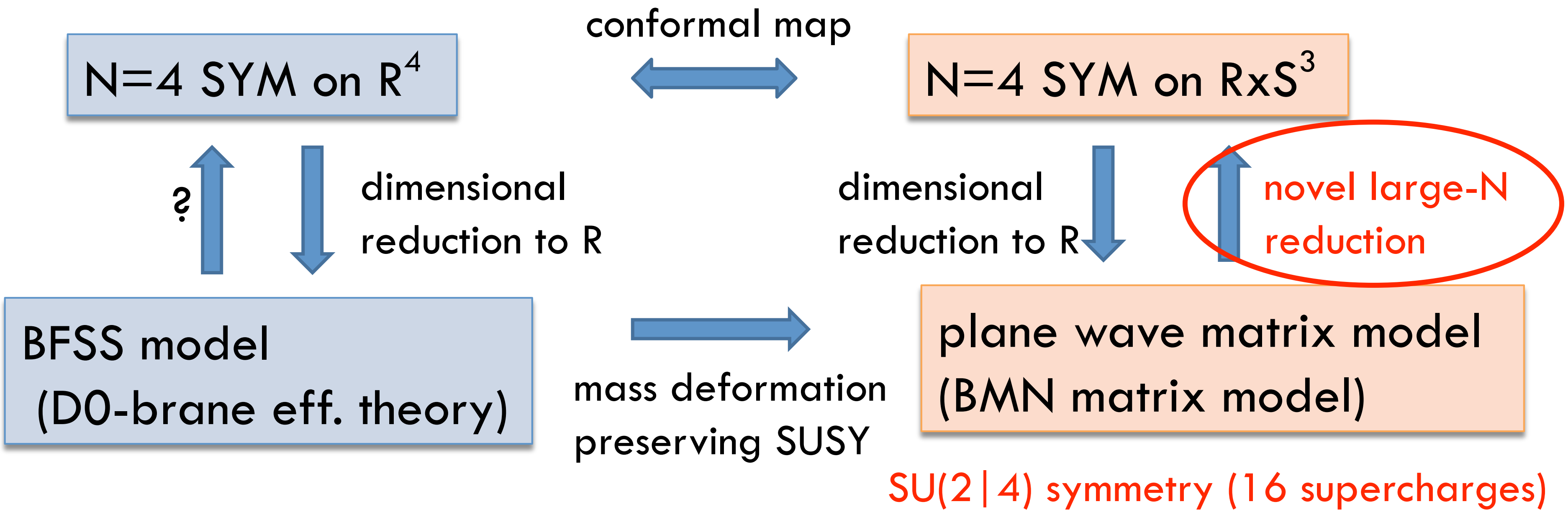
  ⇒ fine-tuning of several parameters is required   cf.) Sugino's talk

- Since we are interested in the planar limit, we may well have a chance to use the idea of the large-N reduction   Eguchi-Kawai ('82)

- Large-N reduction asserts that the planar limit of gauge theory can be described by matrix model obtained by its dimensional reduction, where the matrix size plays the role of UV cutoff
  ~ first example of emergent space-time in matrix model
  ~ but practically one has to overcome the problem of $U(1)^d$ symmetry breaking or background instability

Our strategy

Ishii-Ishiki-Shimasaki-A.T. ('08)



- By expanding PWMM around a particular vacuum which consists of multi fuzzy spheres, we can retrieve the planar limit of N=4 SYM on $R \times S^3$.
- We can resolve the problem of b.g. instability thanks to SUSY and massiveness

Our strategy (cont'd)

- As a regularization, our formulation respects the gauge symmetry and the SU(2|4) symmetry with 16 supercharges, which is included in the superconformal PSU(2,2|4) symmetry
- Our regularization is optimal from the viewpoint of preserving SUSY, because any UV regularization breaks the conformal symmetry
- In the large N limit, the SU(2|4) symmetry is expected to enhance to the PSU(2,2|4) symmetry. We expect to obtain the planar limit of N=4 SYM without fine-tuning
- We put this model on a computer and numerically simulate N=4 SYM in the planar limit to test the AdS/CFT correspondence

  ~ first successful example of numerical simulation for 4D SUSY theory
- Example of emergent curved space in matrix model
- cf.) 2d lattice+ fuzzy sphere   Hanada-Matsuura-Sugino

Plan of the present talk

1. Introduction

2. Large-N reduction

3. N=4 SYM on RxS^3 from PWMM

4. Testing the AdS/CFT correspondence

5. Summary and discussion

Large-N reduction

Large N reduction: Example

- Consider matrix quantum mechanics

$$S = \int d\tau \mathrm{Tr}\left(\frac{1}{2}\left(\frac{d\phi(\tau)}{d\tau}\right)^2 + \frac{m^2}{2}\phi(\tau)^2 + \frac{g^2}{4}\phi(\tau)^4\right)$$

$\phi(\tau)$: NxN hermitian matrix

- Introduce a constant matrix

$$P = \begin{pmatrix} p_1 & & & \\ & p_2 & & \\ & & \ddots & \\ & & & p_N \end{pmatrix}$$

Λ : UV cutoff Λ/N : IR cutoff

- Obtain reduced model

$$\phi(\tau) \to \phi, \quad \frac{d}{d\tau} \to i[P,\ ], \quad \int d\tau \to \frac{2\pi}{\Lambda} \quad \Rightarrow \quad S_r = \frac{2\pi}{\Lambda}\mathrm{Tr}\left(-\frac{1}{2}[P,\phi]^2 + \frac{m^2}{2}\phi^2 + \frac{g^2}{4}\phi^4\right)$$

Large N reduction: Example (cont'd)

- The reduced model reproduces the planar limit of the original theory in the limit

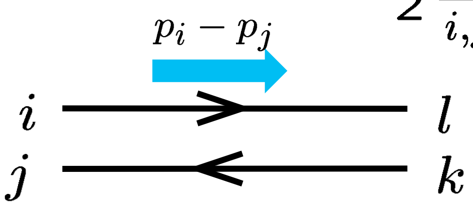
$$N \to \infty, \quad \Lambda \to \infty, \quad \Lambda/N \to 0, \quad g^2 N = \lambda : \text{fixed}$$

UV cutoff IR cutoff 't Hooft coupling

- Feynman rule

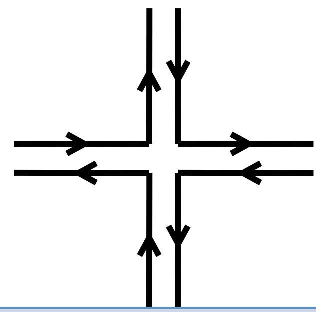
$$S_r = \frac{2\pi}{\Lambda} \mathrm{Tr}\left(-\frac{1}{2}[P,\phi]^2 + \frac{m^2}{2}\phi^2 + \frac{g^2}{4}\phi^4 \right)$$

$$\| \quad \frac{1}{2}\sum_{i,j}(p_i - p_j)^2 |\phi_{ij}|^2$$



$$\frac{\Lambda}{2\pi}\frac{1}{(p_i - p_j)^2 + m^2}\delta_{il}\delta_{jk}$$

momentum of the (i,j) component is $p_i - p_j$



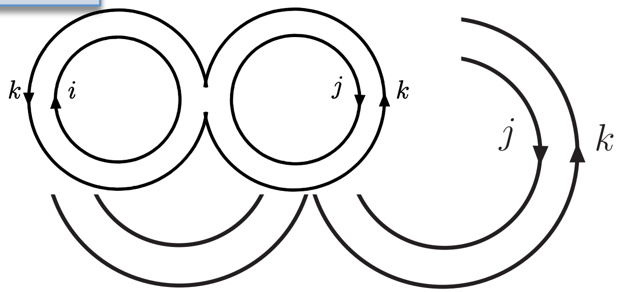
$$\sim \frac{2\pi g^2}{\Lambda}$$

momentum conservation
~matrix product

Large N reduction: Example (cont'd)

- Calculation of free energy

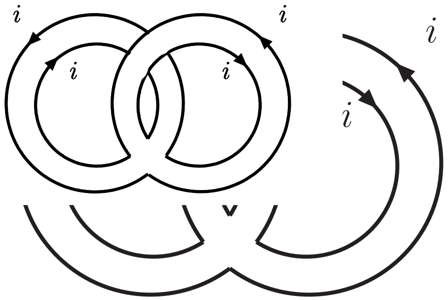
planar



$$= \left(g^2 \frac{2\pi}{\Lambda}\right)\left(\frac{\Lambda}{2\pi}\right)^2 \sum_{i,j,k} \frac{1}{(p_i - p_j)^2 + m^2} \frac{1}{(p_j - p_k)^2 + m^2}$$

$$\rightarrow \frac{2\pi}{\Lambda} N^2 \lambda \int \frac{dk_1}{2\pi}\frac{dk_2}{2\pi} \frac{1}{k_1^2 + m^2} \frac{1}{k_2^2 + m^2}$$

non- planar



$$= \frac{2\pi}{\Lambda}\frac{1}{4} N^2 \lambda \frac{1}{m^4} \times \left(\frac{\Lambda}{2\pi N}\right)^2$$

suppressed

No correspondence between reduced model and original one

$$\frac{F}{N^2 V} = \frac{F_r}{N^2 \frac{2\pi}{\Lambda}}$$

Large-N reduction for YM theory

- Apply the rule to the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \longrightarrow i[X_\mu, X_\nu]$$

$$\partial_\mu \to [iP_\mu, \ ] \qquad X_\mu = P_\mu + A_\mu$$

P_μ is D-dimensional analog of P

- Reduced model of YM theory

$$S_r = -\left(\frac{2\pi}{\Lambda}\right)^D \frac{1}{4g_{YM}^2} \mathrm{Tr}[X_\mu, X_\nu]^2$$

dimensional reduction of YM theory to zero-dimension

P_μ is interpreted as a background of X_μ

The background is unstable due to zero-dimensional massless fields

$\Longrightarrow$ quenching $P_\mu = U_\mu X_\mu U_\mu^\dagger$ fixed Not compatible with SUSY !

Bhanot-Heller-Neuberger ('82) Gross-Kitazawa ('82)

N=4 SYM on $R \times S^3$ from PWMM

Dimensional reduction of N=4 SYM on RxS^3

- S^3 can be identified with SU(2)

The isometry of S^3 is SO(4)=SU(2)xSU(2) corresponding to the left and right translations

E^i : the right invariant 1-forms

$\mathcal{L}_i = -iE_i^\mu \partial_\mu$: Killing vector ~ generator of left translation

$[\mathcal{L}_i, \mathcal{L}_j] = i\mu\epsilon_{ijk}\mathcal{L}_k$ $\quad \frac{2}{\mu}$: radius of S^3

- Dimensional reduction

Expand the gauge field on S^3 as $A = X_i E^i$

$$F = dA + iA \wedge A$$
$$= \frac{1}{2}(i\mathcal{L}_i X_j - i\mathcal{L}_j X_i + \mu\epsilon_{ijk}X_k + i[X_i, X_j])E^i \wedge E^j$$

N=4 SYM on RxS^3 ⇒ PWMM Kim-Klose-Plefka ('03)

Plane wave (BMN) matrix model

$$S = \frac{1}{g^2}\int d\tau \, \mathrm{Tr}\left[\frac{1}{2}(D_\tau X_M)^2 - \frac{1}{4}[X_M, X_N]^2 + \frac{\mu^2}{2}(X_i)^2 + \frac{\mu^2}{8}(X_m)^2 + i\mu\epsilon_{ijk}X_iX_jX_k\right]$$
$$+ \text{(fermion part)}$$

$$(\mu\epsilon_{ijk}X_k + i[X_i, X_j])^2$$

$$1 \le M, N \le 9, \quad 1 \le i, j, k \le 3, \quad 4 \le m \le 9$$
$$D_\tau = \partial_\tau - i[A_\tau, \cdot]$$

Berenstein-Maldacena-Nastase ('02)

- mass deformation of BFSS model　　mass^2 ~ curvature of S^3
- SU(2|4) symmetry (16 supercharges)　$\subset$　PSU(2,2|4)
- Vacua　$X_i = \mu L_i$

$$[L_i, L_j] = i\epsilon_{ijk}L_k$$

N-dimensional reducible representation of SU(2) generators

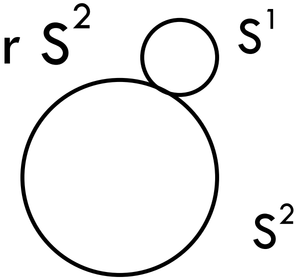
represent multi fuzzy spheres

preserve the SU(2|4) symmetry and are all degenerate

Retrieving N=4 SYM on RxS^3

Ishii-Ishiki-Shimasaki-A.T. ('08)

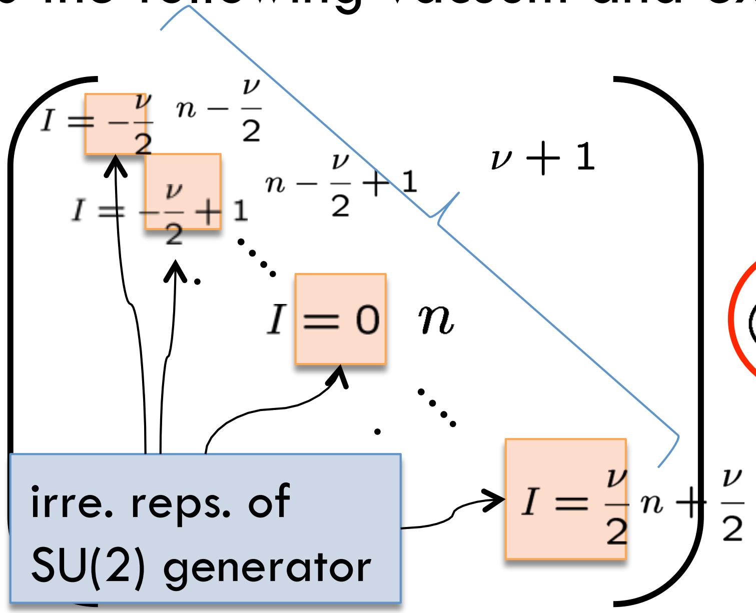
S^3 is locally S^2xS^1, but globally a nontrivial S^1-bundle over S^2



We construct S^2 by continuum limit of fuzzy sphere
and construct S^1 by large-N reduction

We pick up the following vacuum and expand the model around it

$X_i = \mu$



$\otimes 1_k$

$$-\frac{\nu}{2} \leq I \leq \frac{\nu}{2}$$

$\nu \to \infty, \quad n/\nu \to \infty, \quad k \to \infty$

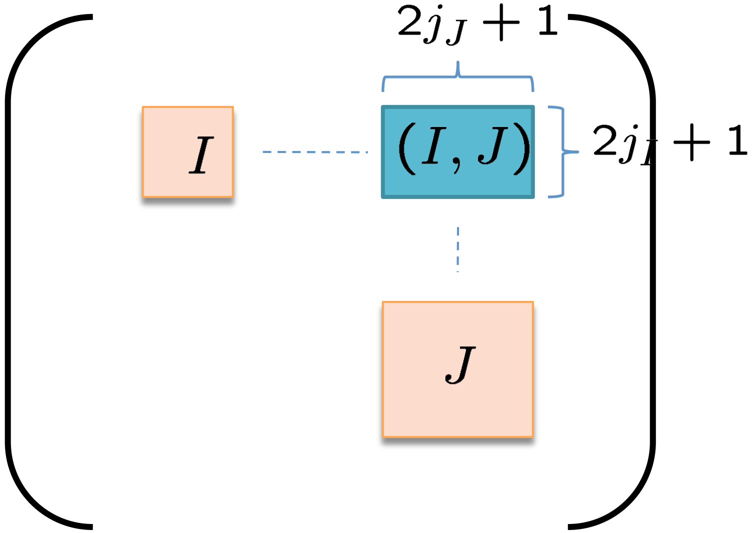
with $\frac{g^2 k}{n} = \frac{\lambda}{V_{S^3}}$ fixed

ν : UV cutoff on S^1

n : UV cutoff on S^2

Retrieving N=4 SYM on RxS^3 (cont'd)

- (I,J) block of the fluctuation around the background



$$2j_I + 1 = n + I$$

- Action of SU(2) on the (I,J) block

$$L_i^{[j_I]}\,(I,J) - (I,J)\,L_i^{[j_J]} \quad \Leftarrow \quad [L_i,\ \ ]$$

irreducible decomposition

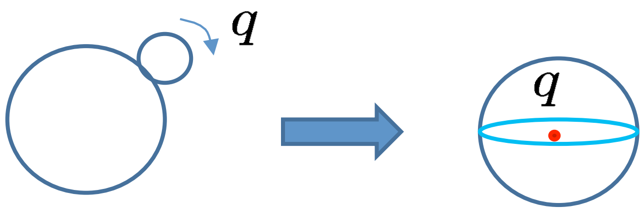
$$\frac{|I-J|}{2} \le j \le n + \frac{I+J}{2} - 1$$

tensor product of spin $j_I = (n+I-1)/2$

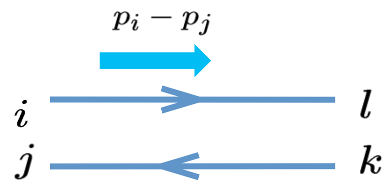
and spin $j_J = (n+J-1)/2$ representations

Retrieving N=4 SYM on $R \times S^3$ (cont'd)

- KK expansion along the fiber S^1 yields KK modes on S^2, whose KK momentum takes integer or half-integer
- The KK mode with the KK momentum q behaves as in the situation where a monopole with monopole charge q exists at the center of S^2



cf.)



- The angular momentum for the KK mode with the momentum q is $|q| \leq j$

  Magnetic field has angular momentum q
- $\frac{|I-J|}{2} \leq j \leq n + \frac{I+J}{2} - 1$ ⇒ the (I,J) block

  = the KK mode with the momentum (I-J)/2
- $-\frac{\nu}{2} \leq I \leq \frac{\nu}{2}$ ⇒ the cutoff for KK momentum = $\nu/2$
- n plays the role of the cutoff for the angular momentum on S^2

Retrieving N=4 SYM on RxS^3 (cont'd)

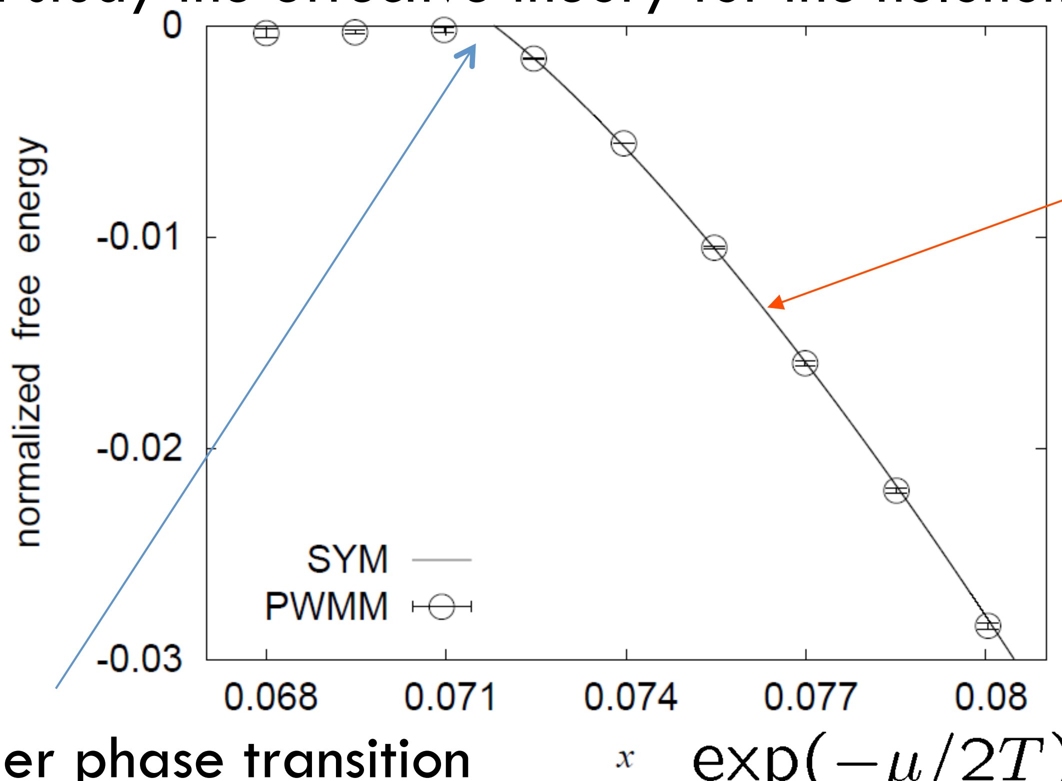
- These two cutoffs preserve the gauge symmetry and SU(2|4) symmetry
- One must extract the planar diagrams in such a way that
  1)the large-N reduction between S^3 and S^2 holds
  2)fuzziness on fuzzy spheres is removed
- For this purpose, the $k \to \infty$ limit should be taken because IR cutoff along the S^1 direction is finite
  cf.) $\Lambda/N \to 0$ in the matrix quantum mechanics
- The model is a massive theory , which has no flat direction.
  The background is classically stable. Furthermore, the background is stable against quantum fluctuations thanks to the SU(2|4) symmetry
- Tunneling to the other vacua through the instanton effects (Yi,…) is suppressed in the $k \to \infty$ limit
- The reduced model reproduces the planar limit of N=4 SYM on RxS^3

Testing the AdS/CFT correspondence

Deconfinement transition at finite T

Ishiki-Kim-Nishimura-A.T. ('08)

In the weak coupling limit at finite temperature, we can integrate out all the massive modes except holonomy around time direction and study the effective theory for the holonomy



$\exp(-\mu/2T)$

known results for N=4 SYM on $R \times S^3$ in the weak coupling limit

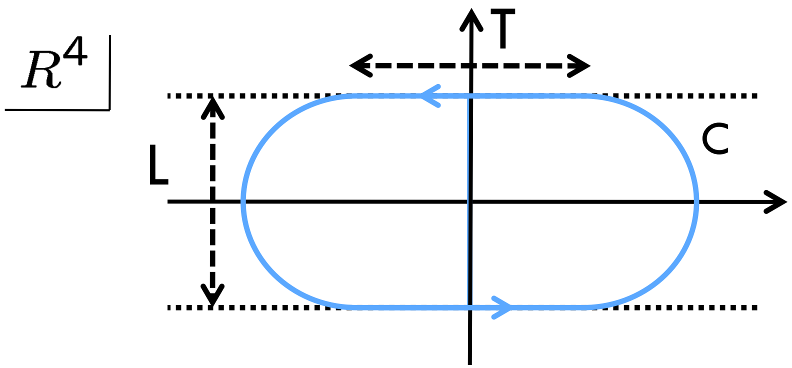
Aharony-Marsano-Minwalla-Papadodimas-Van Raamsdonk ('03))

1st order phase transition

~Hawking-Page transition

Track-shaped Wilson loop

Honda-Ishiki-Nishimura-A.T., work in progress



non-BPS except $T = 0$

1. $T = 0$ circular Wilson loop ~ half-BPS

$$\langle W(C) \rangle = \sqrt{\frac{2}{\lambda}} I_1(\sqrt{2\lambda}) \quad \xrightarrow{\text{large } \lambda} \quad \frac{e^{\sqrt{2\lambda}}}{(\pi/2)^{1/2}(2\lambda)^{3/4}}$$

$e^{\sqrt{2\lambda}}$ agrees with the prediction from the gravity side

Erickson-Semenoff-Zarembo ('00)
Pestun ('07)

2. $T \to \infty$ quark (W-boson) potential

$$V(L) = \lim_{T\to\infty} \frac{-1}{T} \ln W(C) = -\frac{c}{L}$$

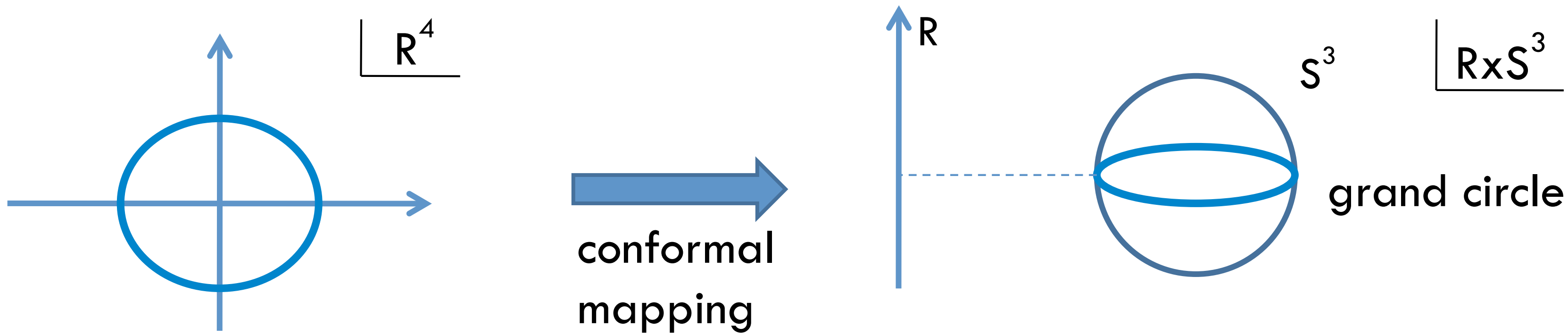
conformal inv.

Prediction from gravity side for large λ

$$c = \frac{4\pi^2\sqrt{2\lambda}}{\Gamma^4(1/4)}$$

Maldacena, Rey-Yee

Circular Wilson loop in the reduced model



1. By summing up all the planar ladder diagrams, we reproduce the exact result, as Erickson-Semenoff-Zarembo did in the continuum theory

Ishiki-Shimasaki-A.T. ('11)

Circular Wilson loop in the reduced model (cont'd)

2. Full Monte Carlo simulation

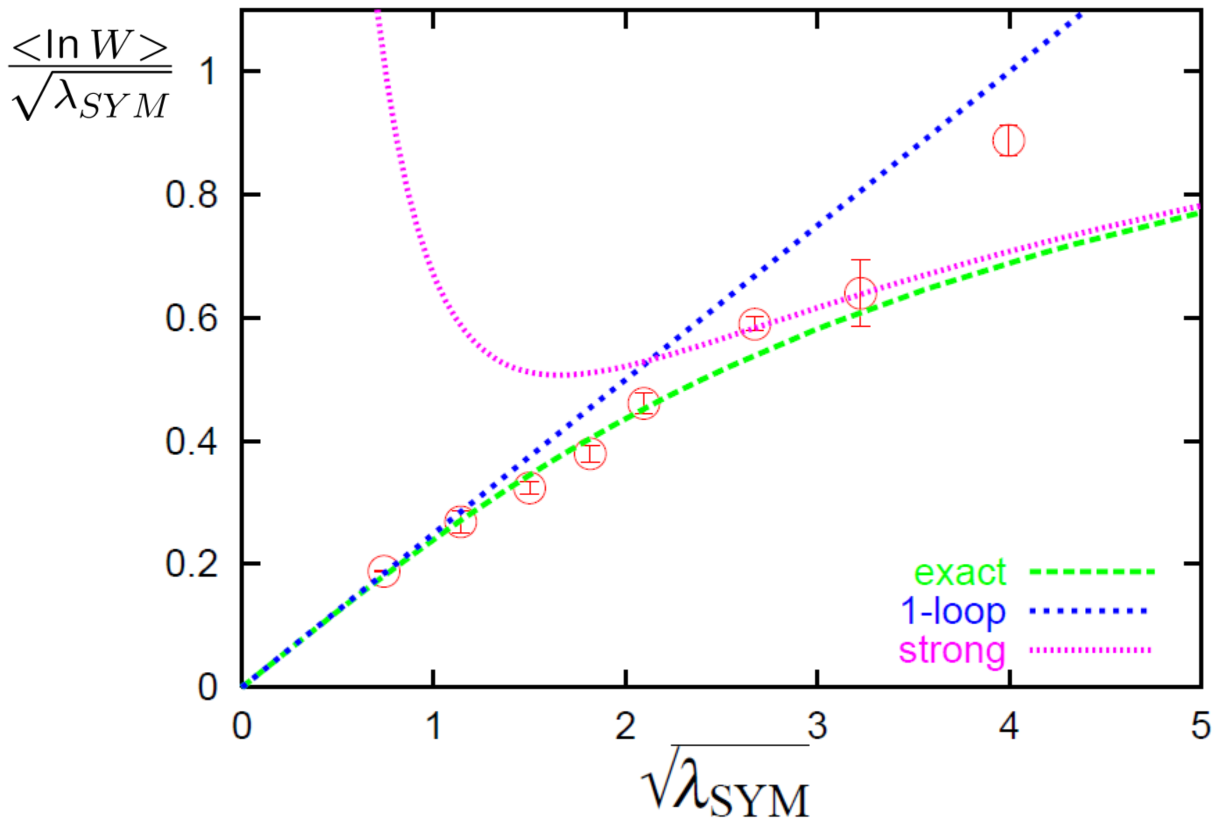
Honda-Ishiki-Nishimura-A.T.

$n = 3/2$

$\nu = 1$

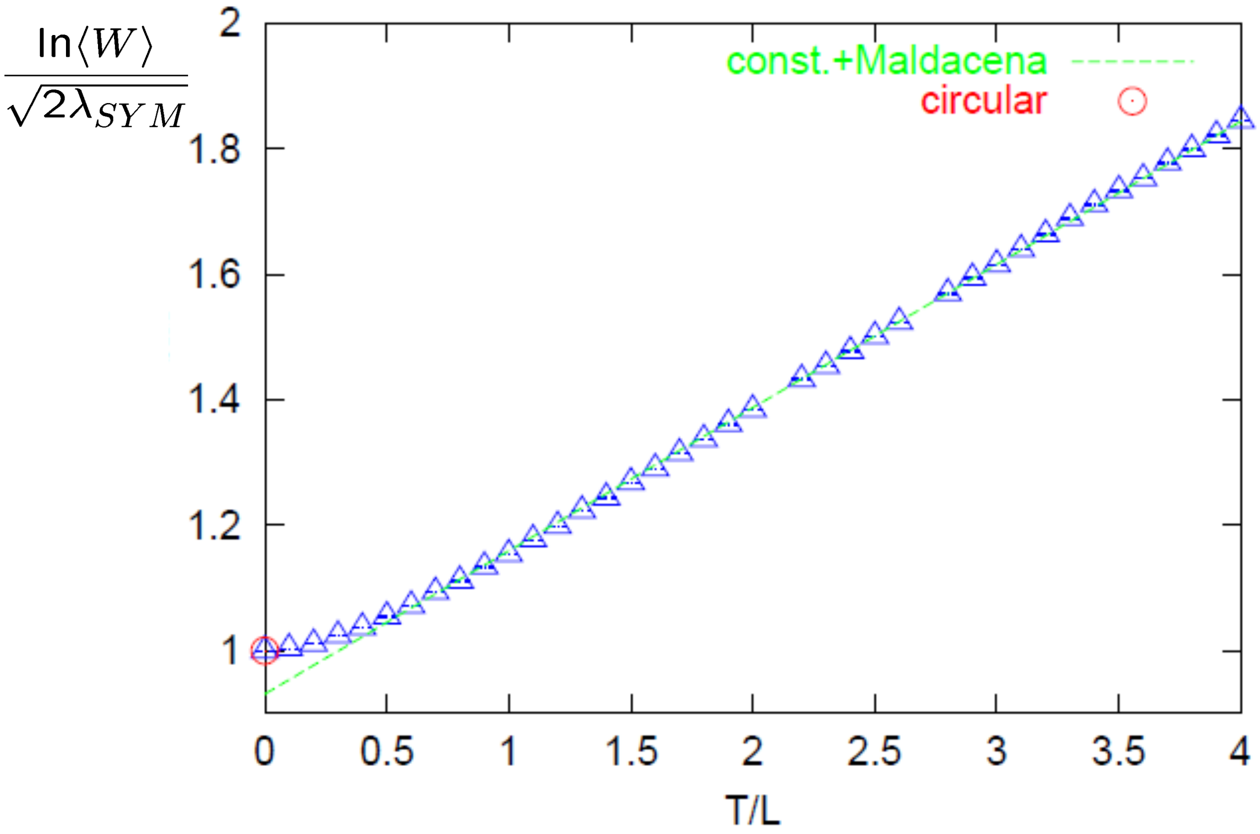
$k = \infty$

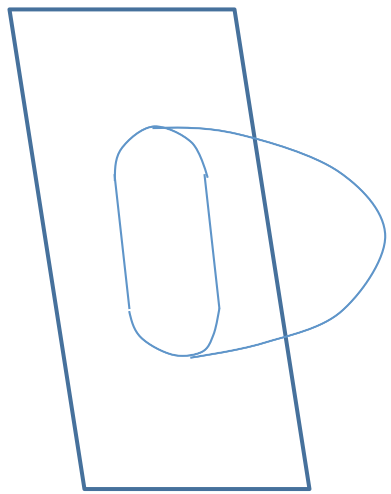
Method
Anagnostopoulos-Hanada-Nishimura-Takeuchi ('07)



Track-shaped Wilson loop on the gravity side

We calculate vev of track-shaped Wilson loop on the gravity side by calculating the area of minimal surface numerically   cf.) Satoh' talk





We are now calculating it on the gauge theory side using Monte Carlo simulation

Chiral primary operator

- chiral primary operator (half-BPS)

$$\mathcal{O}_I = T_I^{m_1 m_2 \cdots m_\Delta} \mathrm{tr}(\phi_{m_1}\phi_{m_2}\cdots\phi_{m_\Delta})$$

T_I : traceless symmetric

scaling dimension Δ

- ratio to the result in free theory

$$c_{\Delta_1} = \langle \mathcal{O}_{I_1}(x_1)\mathcal{O}_{I_2}(x_2)\rangle / \langle \mathcal{O}_{I_1}(x_1)\mathcal{O}_{I_2}(x_2)\rangle_{free},$$
$$c_{I_1 I_2 I_3} = \langle \mathcal{O}_{I_1}(x_1)\mathcal{O}_{I_2}(x_2)\mathcal{O}_{I_3}(x_3)\rangle / \langle \mathcal{O}_{I_1}(x_1)\mathcal{O}_{I_2}(x_2)\mathcal{O}_{I_3}(x_3)\rangle_{free}$$

- non-renormalization theorem

Eden-Howe-West ('99),…….

$$c_{\Delta_1} = 1, \qquad c_{I_1 I_2 I_3} = 1$$

- prediction from the gravity side

Lee-Minwalla-Rangamani-Seiberg ('98)

GKP-Witten relation ⇒

$$\left.\frac{c_{I_1 I_2 I_3}}{\sqrt{c_{\Delta_1} c_{\Delta_2} c_{\Delta_3}}}\right|_{N\to\infty, \lambda_{SYM}\to\infty} = 1$$

consistent with non-renormalization theorem

- 4-pt function

renormalized

prediction from the gravity side ⇒ nontrivial test for AdS/CFT

Arutyunov-Frolov ('00)

CPO in reduced model

- correspondence

$$\bar{\mathcal{O}}_I(t) = \int \frac{d\Omega_3}{2\pi^2} T_I^{m_1 m_2 \cdots m_\Delta} \mathrm{tr}(\phi_{m_1}\phi_{m_2}\cdots\phi_{m_\Delta})$$

$$\Updownarrow$$

$$\mathcal{O}_I^{PW}(t) = \frac{1}{n} T^{m_1 m_2 \cdots m_\Delta} \mathrm{Tr}(X_{m_1} X_{m_2} \cdots X_{m_\Delta})$$

- 2-pt function of CPOs in free theory

SYM

$$\int \frac{d\Omega_3}{2\pi^2}\frac{d\Omega_3'}{2\pi^2} \langle \mathrm{tr}((X_a X_b)(t,\Omega_3))\mathrm{tr}((X_a X_b)(t',\Omega_3'))\rangle_{YM} = \frac{\lambda^2}{16\pi^4} e^{\mu(t+t')} \int \frac{d\Omega_3}{2\pi^2}\frac{d\Omega_3'}{2\pi^2} \frac{1}{|x-x'|^4}$$

$$= \boxed{\frac{\lambda^2 \mu^4}{16^2\pi^4} \frac{e^{-\mu(t-t')}}{1-e^{-\mu(t-t')}}}$$

reduced model

$$\frac{1}{n^2\nu} \langle \mathrm{tr}((X_a X_b)(t))\mathrm{tr}((X_a X_b)(t'))\rangle_{PW} = \frac{g^4 k^2}{4n^2\nu\mu^2} \sum_{I,J} \sum_{j=\frac{1}{2}|I-J|}^{n+\frac{1}{2}(I+J)-1} \sum_{m=-j}^{j} \frac{1}{(j+\frac{1}{2})^2} e^{-\mu(2j+1)(t-t')}$$

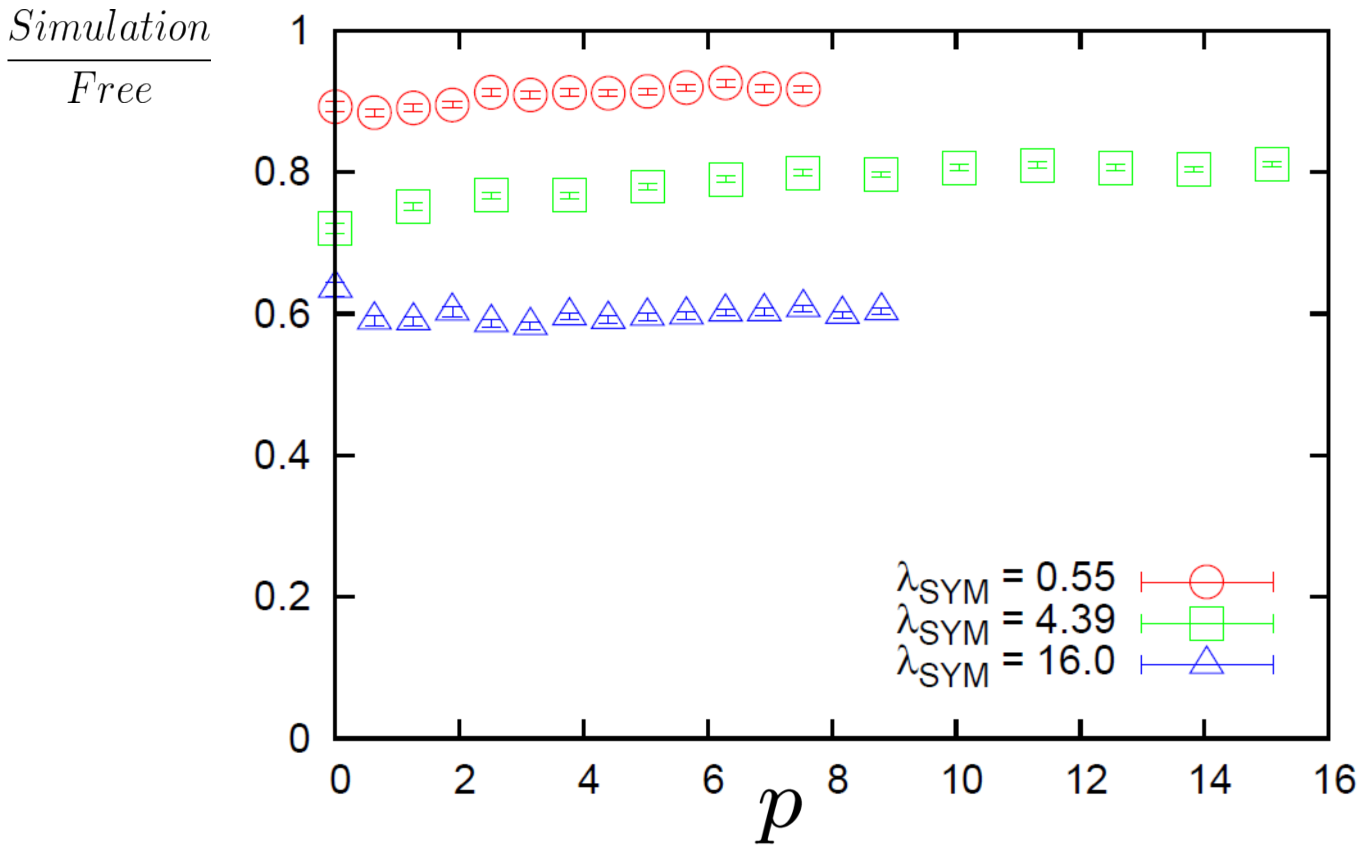
$$= \boxed{\frac{\lambda^2 \mu^4}{16^2\pi^4} \frac{e^{-\mu(t-t')}}{1-e^{-\mu(t-t')}}}$$

Honda-Ishiki-Kim-Nishimura-A.T., to appear

Numerical simulation of correlation function of CPOs

- 2-pt function for $\Delta=2$ Honda-Ishiki-Kim-Nishimura-A.T., to appear

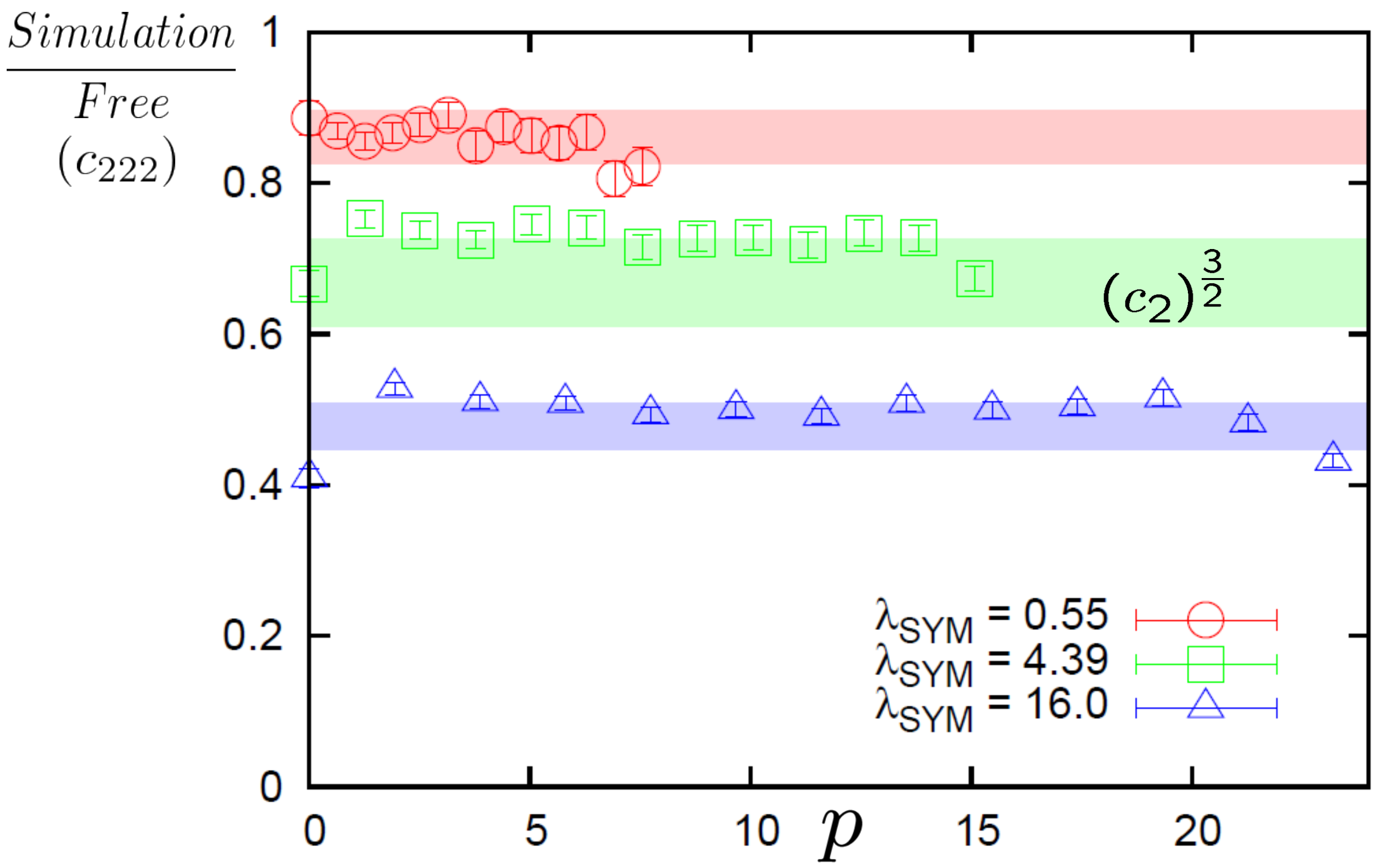
$$n = 3/2, \quad \nu = 1, \quad k = 2$$



extract c_2

Numerical simulation of correlation function of CPOs (cont'd)

- 3-pt function for $\Delta=2$ $G^{(3)}(p,0,-p)=\left(\frac{\mu}{g_{PW}^2 N}\right)^3 \langle tr\left(\widetilde{X_4 X_5}(p)\right) tr\left(\widetilde{X_5 X_6}(0)\right) tr\left(\widetilde{X_6 X_4}(-p)\right)\rangle$



prediction from the gravity side

$c_{222}=(c_2)^{\frac{3}{2}}$

consistent

Summary and discussion

Summary

- We proposed a non-perturbative formulation of planar N=4 SYM through a novel large-N reduction on S^3.
- Our formulation preserves 16 SUSY and the gauge symmetry so that it overcomes difficulties in lattice SUSY and requires no fine-tuning.
- We extended the large-N reduction to general compact semi-simple group manifolds and their coset spaces.
  (no time to discuss) Kawai-Shimasaki-A.T. ('09)
- We provided some consistency check of our formulation at weak t' Hooft coupling:
  vanishing beta fn. (no time to discuss)
  confinine-deconfinement transition at finite temperature
  vev of Wilson loops (all orders)
  correlation fns of chiral primary operators (CPOs).

Summary (cont'd)

- We calculated vev of track-shaped Wilson loops on the gravity side, which can be used for a nontrivial test of the AdS/CFT correspondence
- We showed the result of the numerical simulation for vev of the circular Wilson loop, which is consistent with the exact result
- We showed the result of the numerical simulation for 2-pt and 3-pt fns of CPOs, which is consistent with the AdS/CFT correspondence
- We have done the numerical simulation for 4-pt function of CPOs. We are now analyzing the prediction from the gravity side
- We formulated Chern-Simon theory on S^3 through the novel large-N reduction. We showed that it reproduces the known exact results (no time to discuss) Ishii-Ishiki-Ohta-Shimasaki-A.T.

Discussion

- Numerical simulation should be continued.
  first successful example of numerical simulation of 4D SUSY theory
- Develop analytic method. Derive integrable structure of N=4 SYM
- N=1 SYM on $R \times S^3$. Gluino condensation.
- ABJM theory

  Hanada-Mannelli-Matsuo Asano-Ishiki-Okada-Shimasaki Honda-Yoshida

  Asano's poster
- Large-N reduction on general curved space-time.
  Description of curved space-time in matrix models

Conformal mapping

$$ds^2_{R^4} = dr^2 + r^2 d\Omega_3^2$$
$$= e^{\mu t}\left(dt^2 + \left(\frac{2}{\mu}\right)^2 d\Omega^2\right)$$
$$= e^{\mu t} ds^2_{R\times S^3}$$

$$r = \frac{2}{\mu} e^{\mu t/2}$$

$\frac{2}{\mu}$: radius of S^3

scalar field $\phi^{R^4} = e^{-\mu\tau/2}\phi^{R\times S^3}$

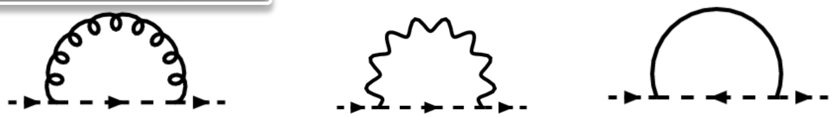
gauge field 1-form $A^{R^4} = A^{R\times S^3}$

fermion field $\psi^{R^4} = e^{-3\mu t/4}\psi^{R\times S^3}$

N=4 SYM on R^4 at a conformal point ⟺ N=4 SYM on $R\times S^3$

Calculation of beta function

fermion self-energy

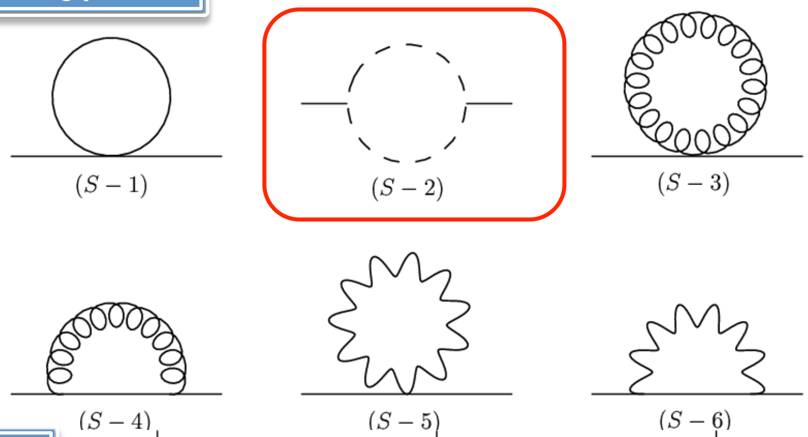


Ishiki-Shimasaki-A.T. ('11)

$$Z_\phi = 1 - \frac{4g^2k}{\mu^3 n}\log\nu$$

no mass renormalization

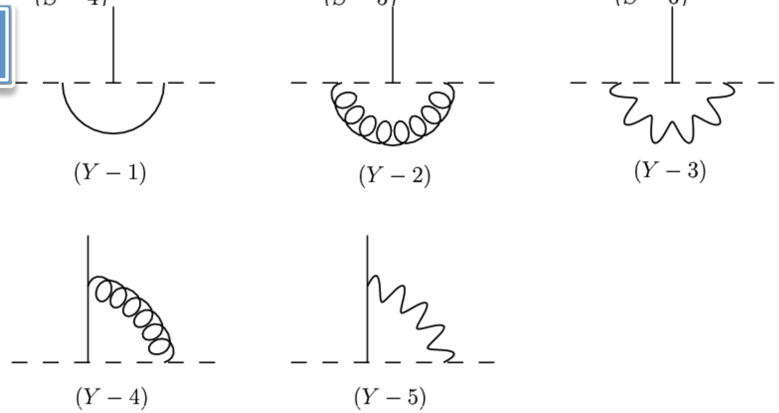
scalar self-energy



~conformal inv.

$$Z_\psi = 1 - \frac{16g^2k}{\mu^3 n}\log\nu$$

Yukawa int.



$$Z_g = 1 - \frac{18g^2k}{\mu^3 n}\log\nu$$

$$Z_g = Z_\psi Z_\phi^{\frac{1}{2}}$$

beta function vanishes !

Calculation of beta function　つづき

example

$$-2k\sum_{J_1m_1\kappa_1}\sum_{J_2m_2\kappa_2}\int\frac{dq}{2\pi}\frac{iq+\kappa_1\omega^{\psi}_{J_1}}{q^2-(\omega^{\psi}_{J_1})^2}\frac{i(p-q)+\kappa_2\omega^{\psi}_{J_2}}{(p-q)^2-(\omega_{J_2}\psi)^2}\hat{\mathcal{F}}^{J_2m_2\kappa_2(j_uj_t)}_{J_1-m_1\kappa_1(j_uj_s)\ Jm(j_sj_t)}\hat{\mathcal{F}}^{J_1-m_1\kappa_1(j_uj_s)}_{J_2m_2\kappa_2(j_uj_t)\ Jm(j_tj_s)}$$

$$=32\mu kn(-1)^{m-(j_s-j_t)}\sum_{R_1=|j_s-j_u|}^{j_s+j_u}\sum_{R_2=|j_t-j_u|}^{j_t+j_u}(2J+1)(2R_1+1)(2R_2+1)$$

$$\times\left[\frac{(R_1+1)(R_2+1)(R_1+R_2+\frac{3}{2})}{p^2+\mu^2(R_1+R_2+\frac{3}{2})^2}\begin{Bmatrix}R_1+\frac{1}{2} & R_1 & \frac{1}{2}\\ R_2+\frac{1}{2} & R_2 & \frac{1}{2}\\ J & J & 0\end{Bmatrix}\right.$$

$$\left.+\frac{R_1R_2(R_1+R_2+\frac{1}{2})}{p^2+\mu^2(R_1+R_2+\frac{1}{2})^2}\begin{Bmatrix}R_1 & R_1+\frac{1}{2} & \frac{1}{2}\\ R_2 & R_2+\frac{1}{2} & \frac{1}{2}\\ J & J & 0\end{Bmatrix}\right]\begin{Bmatrix}R_2 & R_1 & J\\ j_s & j_t & j_u\end{Bmatrix},$$

$$\xrightarrow[J=0\quad s=t=0]{}\quad\frac{N}{\mu^3 n}\sum_{u=-\nu/2}^{nu/2}\sum_{R=|u/2|}^{n-1+u/2}\left\{8\mu^2-\frac{2}{R^2}\left(p^2+\frac{1}{4}\mu^2\right)\right\}$$

Another choice of L_i

Kawai-Shimasaki-A.T. ('09)

$$L_i = \begin{pmatrix} L_i^{[0]} & & & & \\ & L_i^{[1/2]} \otimes 1_2 & & & \\ & & L_i^{[1]} \otimes 1_3 & & \\ & & & \ddots & \\ & & & & L_i^{[\frac{K-1}{2}]} \otimes 1_K \end{pmatrix} \otimes 1_k$$

$$N = k \sum_{d=1}^{K} d^2 = kl$$

regularized regular representation

limit

$$K \to \infty, \quad k \to 0 \quad \text{with} \quad g^2 k = \frac{\lambda}{V_{S^3}} \quad \text{fixed}$$