BPS States in $\mathcal{N}=4$ Supersymmetric Gauge Theories

Ashoke Sen

Harish-Chandra Research Institute, Allahabad, India

Osaka, April 2012
Goal:

Study the spectrum of BPS states in $\mathcal{N}=4$ supersymmetric SU(N) Yang-Mills theories on the Coulomb branch where the gauge group is broken to $U(1)^{n-1}$.

1. Introduce appropriate indices which are protected from changes under deformations of the moduli and coupling constants except at the walls of marginal stability.

2. Compute these indices.

Review + arXiv:1203.4889
Suppose we have a BPS state that breaks 4n supersymmetries.

→ there will be 4n fermion zero modes (goldstino) on the world-line of the state.

Consider a pair of fermion zero modes $\psi_0, \psi_0^\dagger$ carrying $J_3 = \pm 1/2$ and satisfying

$$\{\psi_0, \psi_0^\dagger\} = 1$$

If $|0\rangle$ is the state annihilated by $\psi_0$ then

$$|0\rangle, \quad \psi_0^\dagger|0\rangle$$

give a degenerate pair of states with $J_3 = \pm 1/4$.

$$\Rightarrow \quad (-1)^F = (-1)^{2J_3} = (-1)^{\pm 1/2} = \pm i$$

$$\Rightarrow \quad \text{Tr}(-1)^F = 0, \quad \text{Tr}(-1)^F(2J_3) = i$$
Lesson: Quantization of the fermion zero modes produces Bose-Fermi degenerate states and makes $\text{Tr}(-1)^F$ vanish.

Remedy: Define

$$B_{2n} = \frac{(-1)^n}{(2n)!} \text{Tr}(-1)^F(2J_3)^{2n} = \frac{(-1)^n}{(2n)!} \text{Tr}(-1)^{2J_3}(2J_3)^{2n}$$

Since there are $2n$ pairs of zero modes,

$$B_{2n} = \frac{(-1)^n}{(2n)!} \text{Tr}_{\text{rest}} \text{Tr}_{\text{zero}}(-1)^{2J_3^{(1)} + \cdots + 2J_3^{(2n)} + 2J_3^{\text{rest}}}$$

$$\times \left(2J_3^{(1)} + \cdots 2J_3^{(2n)} + 2J_3^{\text{rest}}\right)^{2n}$$

$$= (-1)^n \text{Tr}_{\text{rest}} \text{Tr}_{\text{zero}}(-1)^{2J_3^{(1)} + \cdots 2J_3^{(2n)} + 2J_3^{\text{rest}}} \times 2J_3^{(1)} \times \cdots 2J_3^{(2n)}$$

$$= (-1)^n (i)^{2n} \times \text{Tr}_{\text{rest}}(-1)^{2J_3^{\text{rest}}}$$
\[ B_{2n} = \text{Tr}_{\text{rest}}(-1)^{2J_3^{\text{rest}}} \]

Thus \( B_{2n} \) effectively counts \( \text{Tr}_{\text{rest}}(-1)^F \), with the trace taken over modes other than the 4n fermion zero modes associated with broken supersymmetry.

**Note:** \( B_{2n} \) does not receive any contribution from non-BPS states which break more than 4n supersymmetries and hence have more than 4n fermion zero modes.

Due to this property \( B_{2n} \) is protected from quantum corrections.
Examples

$\mathcal{N}=4$ SYM in D=4 has 16 supersymmetries.

1/2 BPS states break 8 supersymmetries.

$\Rightarrow$ the relevant index is $B_4$.

1/4 BPS states break 12 supersymmetries.

Thus the relevant index is $B_6$. 
Twisted index

Suppose the theory has a global symmetry $g$ that commutes with some of the unbroken supersymmetries of the BPS state.

Suppose further that there are $4n$ broken $g$-invariant supersymmetries.

In that case

$$B^g_{2n} = \frac{(-1)^n}{(2n)!} \text{Tr} \left[ (-1)^{2J_3} (2J_3)^{2n} g \right]$$

is a protected index that carries information about $g$ quantum number.
We shall mostly focus on SU(N) SYM theories
– can be geometrically realized on the world-volume of N D3-branes.

Leaving out the 3+1 world-volume directions each D3-brane can be located at any point in the 6 transverse directions.

⇒ 6N dimensional moduli space ⇔ adjoint scalar vev

Of these 6 are associated with the center of mass U(1) SYM and are irrelevant for our problem.

Complex coupling constant:

\[ \tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g_{YM}^2} \]
In this description, BPS states are conveniently represented as string networks ending on D3-branes.

\[(p_1, q_1) \quad (p_2, q_2) \quad (p_3, q_3) \quad (p_4, q_4) \quad (p_5, q_5)\]

\((1,0): D\text{-string}, \quad (0,1): \text{fundamental string}\)

If \(\gcd(p,q) = r\) then \((p,q) \equiv r\) copies of \((p/r, q/r)\).
- carries charges $Q = (q_1, \cdots q_5)$, $P = (p_1, \cdots p_5)$.

Note: $\sum_i q_i = 0 = \sum_i p_i \Rightarrow$ no U(1) charge
Rules for string network

1. The network must be planar.

2. A \((p,q)\) string should lie along \(e^{i\alpha}(p\bar{\tau} + q)\) (class A) or \(e^{i\alpha}(p\tau + q)\) (class B).

\(\alpha\): a constant for a given network.

Schwarz; Aharony, Hanany, Kol; Dasgupta, Mukhi; A.S.
Half BPS states correspond to single \((p, q)\) strings stretched between two D3-branes.

\[ Q = (q, -q, 0, 0, 0), \quad P = (p, -p, 0, 0, 0) \]

Note: Q and P are always parallel.

For these states \(B_4 = 1\) if \(\gcd(q, p) = 1\) and 0 otherwise.

- follows from S-duality invariance and known spectrum of \((0, q)\) states.

Olive, Witten; Osborn; A.S.; Segal, Selby
Quarter BPS states

The relevant index is $B_6$.

A configuration with strings ending on four or more D-branes is non-planar at a generic point in the moduli space and hence non-supersymmetric.

$\Rightarrow$ has $B_6 = 0$. 
Thus $B_6$ receives contribution only from 3-string junctions.

We would like to compute $B_6$ for an arbitrary 3-string junction.
Although the string junction picture is useful in determining the region of existence of 1/4 BPS states, it is not very useful in finding the index $B_6$.

This can be done in special cases by representing them as bound states of multiple monopoles in gauge theories and solving the associated supersymmetric quantum mechanics problem. $B_6 = (-1)^s + 1$.

For this configuration $B_6 = (-1)^s + 1$.

Bak, Lee, Yi; Gauntlett, Kim, Park, Yi; Stern, Yi
A three string junction exists in a certain region of the moduli space bounded by walls of marginal stability.

Along each wall the 3-string junction becomes marginally unstable against decay into a pair of half BPS states.

Beyond these walls the state stops existing.

⇒ \( B_6 = \Delta B_6 \) across the wall
Relevant decays are to a pair of half BPS states.

Across a wall along which \((Q,P)\) decays into a pair of half BPS states with **primitive charges** \(Q_1, P_1\) and \(Q_2, P_2\) the jump in the index \(B_6\) is

\[
(-1)^{Q_1 \cdot P_2 - P_1 \cdot Q_2 + 1} |Q_1 \cdot P_2 - Q_2 \cdot P_1| B_4(Q_1, P_1) B_4(Q_2, P_2)
\]
Consider the wall along which the \((0,s)\) string shrinks to zero-size.

– decays to \((1,k)\) string connecting a pair of D3-branes and a \((-1, -k - s)\) string connecting a pair of D3-branes, each with \(B_4 = 1\).

The jump in \(B_6\) across this wall is given by \((-1)^{s+1}s\) in agreement with direct index computation.

Dabholkar, Nampuri, Narayan
Can we calculate the $B_6$ of a general 3-string junction?

In general the decays are not primitive, e.g. for

none of the decays are primitive.
We need to use general wall crossing formula for non-primitive decays in $\mathcal{N}=4$ supersymmetric string / field theories.

For the decay

$$(Q, P) \rightarrow (Q_1, P_1) + (Q_2, P_2)$$

$$\Delta B_6 = (-1)^{Q_1 \cdot P_2 - Q_2 \cdot P_1 + 1} |Q_1 \cdot P_2 - Q_2 \cdot P_1|$$

$$\times \sum_{r_1 | Q_1, P_1} B_4(Q_1/r_1, P_1/r_1) \sum_{r_2 | Q_2, P_2} B_4(Q_2/r_2, P_2/r_2)$$

Banerjee, Srivastava, A.S; A.S.

For SYM theories,

$$\sum_{r | Q, P} B_4(Q/r, P/r) = 1$$

$\Rightarrow$ simple formula for $B_6$. 
For

\[ B_6 = (-1)^{p_1q_2 - p_2q_1 + 1} |p_1q_2 - p_2q_1| \]

- symmetric in the three strings.

This solves completely the problem of computing \( B_6 \) for \( \mathcal{N}=4 \) supersymmetric SU(\( N \)) theories.
For other gauge groups we can compute $B_6$ by identifying various SU(3) subgroups and computing $B_6$ for different change vectors in each SU(3).
We shall now return to more general planar string network (possibly with internal loops).

As argued before, $B_6$ and any other index that is defined everywhere in the moduli space vanish for this configuration.
Strategy

Introduce an index that can be defined only in special subspace of the moduli space where all the D3-branes lie in a plane.

⇔ only two of the six adjoint Higgs fields take vev.

Make use of the $\text{SO}(4) \equiv \text{SU}(2)_L \times \text{SU}(2)_R$ rotational symmetry in the four directions transverse to the plane of the 3-branes to introduce a twisted index.
$\text{SU}(2)_L \times \text{SU}(2)_R \times \text{SU}(2)_{\text{rotation}}$ transformation laws of various supersymmetries:

<table>
<thead>
<tr>
<th>State</th>
<th>unbroken susy</th>
<th>broken susy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Half BPS</td>
<td>$(1,2,2) + (2,1,2)$</td>
<td>$(1,2,2) + (2,1,2)$</td>
</tr>
<tr>
<td>Class A quarter BPS</td>
<td>$(1,2,2)$</td>
<td>$(1,2,2) + 2 (2,1,2)$</td>
</tr>
<tr>
<td>Class B quarter BPS</td>
<td>$(2,1,2)$</td>
<td>$2(1,2,2) + (2,1,2)$</td>
</tr>
</tbody>
</table>

Class A states and half BPS states have four broken and four unbroken $\text{SU}(2)_L$ invariant SUSY

Of these two of each are invariant under $I_{3R} + J_3$

$(I_{3L}, I_{3R}, J_3)$: Cartan gen. of $\text{SU}(2)_L \times \text{SU}(2)_R \times \text{SU}(2)_{\text{rotation}}$
Following earlier logic we can now introduce protected index

\[
B_2(z) = -\frac{1}{2!} \text{Tr}\{( -1 )^2 J_3 z^{2 I_3 L} (2 J_3)^2 \}
\]

\[
B_1(y, z) = -\frac{1}{y - y^{-1}} \text{Tr} \left\{ e^{2i\pi J_3} z^{2 I_3 L} y^{2 I_3 R + 2 J_3} (2 J_3) \right\}
\]

Similar index for N=2 SUSY theories are known.

Gaiotto, Moore, Neitzke

One can show that for class A states

\[
B_2(z) = \lim_{y \to 1} B_1(y, z)
\]

\[
B_6 = \lim_{z \to 1} (z + z^{-1} - 2)^{-2} B_2(z)
\]

Thus \(B_1(y, z)\) is the most general index.
Can we compute $B_1(y, z)$ for a general string network?

1. For half BPS states we have all the information and $B_1(y, z)$ is straightforward to compute.

$$B_1(y, z) = (z + z^{-1} - y - y^{-1})(y - y^{-1})$$

for each primitive half BPS state.

2. For collinear configuration of D3-branes only half BPS states contribute to $B_1(y, z)$.

Strategy: Start from this collinear configuration and apply wall crossing formula to find the result elsewhere in the moduli space.

Caution: The structure of marginal stability walls is more complicated than before.
$B_1(y, z)$ follows a wall crossing formula similar to the motivic KS formula. 

Kontsevich, Soibelman

There are many physical ‘derivations’ of this formula by now.

Cecotti, Vafa; Gaiotto, Moore, Neitzke; Dimofte, Gukov; Andriyash, Denef, Jafferis, Moore; Lee, Yi; Kim, Park, Wang, Yi; Manschot, Pioline, A.S · · ·

We can derive the wall crossing formula for $B_1(y, z)$ using any of these approaches.
1. Given $\alpha = (Q, P)$ and $\alpha' = (Q', P')$, define

$$\langle \alpha, \alpha' \rangle = Q \cdot P' - Q' \cdot P$$

2. Define

$$\bar{B}_1(\alpha; y, z) \equiv \sum_{m | |\alpha|} m^{-1} \frac{y - y^{-1}}{y^m - y^{-m}} B_1(\alpha/m; y^m, z^m)$$

3. Introduce the algebra:

$$[e_\alpha, e_{\alpha'}] = (-y)^{\langle \alpha, \alpha' \rangle} - (-y)^{-\langle \alpha, \alpha' \rangle} \frac{y - y^{-1}}{y - y^{-1}} e_{\alpha + \alpha'}$$

Then wall crossing formula tells us that:

$$P \left( \prod_{\alpha} \exp \left[ \bar{B}_1(\alpha; y, z)e_\alpha \right] \right)$$

is unchanged across a wall.

P: ordering according to Arg(central charge).
Using the wall crossing formula and the result for half BPS states we can compute $B_1(y, z)$ for any class A quarter BPS states in any chamber in the moduli space.

The corresponding index for class B states can be defined and computed in a similar way by exchanging

$$SU(2)_L \leftrightarrow SU(2)_R$$
Example 1: Consider the network

![Diagram of the network](image)

We can shrink the \((0, s_i)\) strings one by one and apply the primitive wall crossing formula.

Final result for \(B_1(y, z)\):

\[
(-1)^\sum |s_i| + n \left\{ z + z^{-1} - y - y^{-1} \right\}^{n+1} \prod_i \frac{y^{|s_i|} - y^{-|s_i|}}{y - y^{-1}}
\]

We get the same result by shrinking the \((-1, k)\) string and applying semi-primitive wall crossing formula.
\[ B_1(y, z) = (-1)^{\sum |s_i| + n} \left( z + z^{-1} - y - y^{-1} \right)^{n+1} \prod_i \frac{y^{|s_i|} - y^{-|s_i|}}{y - y^{-1}} \]

Consistency check:

1. For \( y = 1, \ z = -1 \) we get \(-4 \prod_i (-1)^{s_i}(4s_i)\) 
   
   – agrees with an appropriate index computed in supersymmetric quantum mechanics of multiple monopoles.

   Stern, Yi

   This has also been derived by applying primitive wall crossing formula.

   Dabholkar, Nampuri, Narayan

2. \( y \)-dependence agrees with quiver quantum mechanics analysis.

   Denef
Example 2:

We can try to compute the index by shrinking any of the external strings to zero size.
Result for $B_1(y, z)$ for cases (a) and (b):

\[ \left\{ z + z^{-1} - y - y^{-1} \right\}^2 \left\{ -y^3 - \frac{1}{y^3} - y - \frac{1}{y} \right\} \]

\[ \left\{ z + z^{-1} - y - y^{-1} \right\}^2 \left\{ -y^3 - \frac{1}{y^3} - 2y - \frac{2}{y} + z + \frac{1}{z} \right\} \]

⇒ there must be marginal stability wall separating (a) and (b).
– can break apart into the (-2,-1) string and the rest.

The jump in $B_1(y, z)$ across this wall precisely accounts for the previous difference.


**Conclusion**

The protected information about the spectrum of BPS states in $\mathcal{N} = 4$ supersymmetric Yang-Mills theories is encoded in the index $B_1(y, z)$.

We now have a complete algorithm for computing this index for any charge vector at any point in the moduli space.