BPS States in $\mathcal{N}$=4 Supersymmetric Gauge Theories

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Goal:

Study the spectrum of BPS states in $\mathcal{N}$=4 supersymmetric SU(N) Yang-Mills theories on the Coulomb branch where the gauge group is broken to $U(1)^{n-1}$.

1. Introduce appropriate indices which are protected from changes under deformations of the moduli and coupling constants except at the walls of marginal stability.

2. Compute these indices.

Review + arXiv:1203.4889

Helicity trace index / helicity supertrace

Bachas, Kiritsis

Suppose we have a BPS state that breaks 4n supersymmetries.

$\to$ there will be 4n fermion zero modes (goldstino) on the world-line of the state.

Consider a pair of fermion zero modes ψ_0, $\psi_0^\dagger$ carrying $J_3 = \pm 1/2$ and satisfying

$$\{\psi_0, \psi_0^\dagger\} = 1$$

If $|0\rangle$ is the state annihilated by ψ_0 then

$$|0\rangle, \qquad \psi_0^\dagger|0\rangle$$

give a degenerate pair of states with $J_3 = \pm 1/4$.

$$\Rightarrow \quad (-1)^F = (-1)^{2J_3} = (-1)^{\pm 1/2} = \pm i$$

$$\Rightarrow \quad \mathrm{Tr}(-1)^F = 0, \qquad \mathrm{Tr}(-1)^F(2J_3) = i$$

Lesson: Quantization of the fermion zero modes produces Bose-Fermi degenerate states and makes $\mathrm{Tr}(-1)^F$ vanish.

Remedy: Define

$$B_{2n} = \frac{(-1)^n}{(2n)!}\mathrm{Tr}(-1)^F(2J_3)^{2n} = \frac{(-1)^n}{(2n)!}\mathrm{Tr}(-1)^{2J_3}(2J_3)^{2n}$$

Since there are 2n pairs of zero modes,

$$\begin{aligned} B_{2n} &= \frac{(-1)^n}{(2n)!}\mathrm{Tr}_{rest}\mathrm{Tr}_{zero}(-1)^{2J_3^{(1)}+\cdots 2J_3^{(2n)}+2J_3^{rest}} \\ &\quad \times \left(2J_3^{(1)}+\cdots 2J_3^{(2n)}+2J_3^{rest}\right)^{2n} \\ &= (-1)^n\mathrm{Tr}_{rest}\mathrm{Tr}_{zero}(-1)^{2J_3^{(1)}+\cdots 2J_3^{(2n)}+2J_3^{rest}} \times 2J_3^{(1)} \times \cdots 2J_3^{(2n)} \\ &= (-1)^n(i)^{2n} \times \mathrm{Tr}_{rest}(-1)^{2J_3^{rest}} \end{aligned}$$

$$B_{2n} = Tr_{rest}(-1)^{2J_3^{rest}}$$

Thus B_{2n} effectively counts $Tr_{rest}(-1)^F$, with the trace taken over modes other than the 4n fermion zero modes associated with broken supersymmetry.

Note: B_{2n} does not receive any contribution from non-BPS states which break more than 4n supersymmetries and hence have more than 4n fermion zero modes.

Due to this property B_{2n} is protected from quantum corrections.

Examples

$\mathcal{N}$=4 SYM in D=4 has 16 supersymmetries.

1/2 BPS states break 8 supersymmetries.

$\Rightarrow$ the relevant index is B_4.

1/4 BPS states break 12 supersymmetries.

Thus the relevant index is B_6.

Twisted index

Suppose the theory has a global symmetry g that commutes with some of the unbroken supersymmetries of the BPS state.

Suppose further that there are 4n broken g-invariant supersymmetries.

In that case

$$B_{2n}^{g} = \frac{(-1)^{n}}{(2n)!} \mathrm{Tr}\left[(-1)^{2J_3}(2J_3)^{2n} g\right]$$

is a protected index that carries information about g quantum number.

We shall mostly focus on SU(N) SYM theories

– can be geometrically realized on the world-volume of N D3-branes.

Leaving out the 3+1 world-volume directions each D3-brane can be located at any point in the 6 transverse directions.

$\Rightarrow$ 6N dimensional moduli space $\Leftrightarrow$ adjoint scalar vev

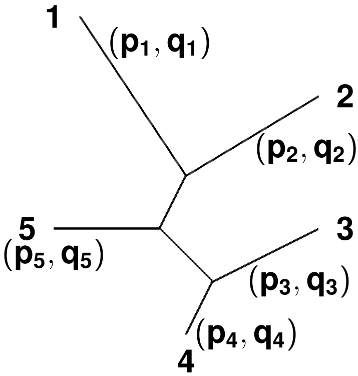
Of these 6 are associated with the center of mass U(1) SYM and are irrelevant for our problem.

Complex coupling constant:

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{YM}^2}$$

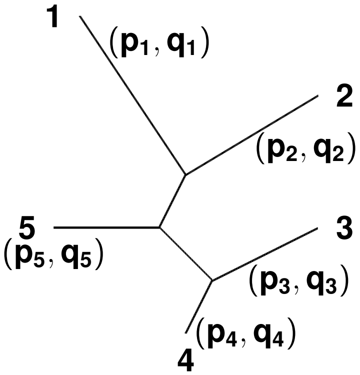
In this description, BPS states are conveniently represented as string networks ending on D3-branes.

Bergman; Bergman, Kol



(1,0): D-string, (0,1): fundamental string

If gcd(p,q) = r then (p,q) $\equiv$ r copies of (p/r, q/r).



– carries charges $Q = (q_1, \cdots q_5)$, $P = (p_1, \cdots p_5)$.

Note: $\sum_i q_i = 0 = \sum_i p_i \Rightarrow$ no U(1) charge

Rules for string network

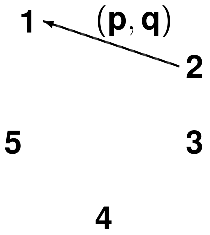
1. The network must be planar.

2. A (p,q) string should lie along $e^{i\alpha}(p\bar{\tau} + q)$ (class A) or $e^{i\alpha}(p\tau + q)$ (class B).

α: a constant for a given network.

Schwarz; Aharony, Hanany, Kol; Dasgupta, Mukhi; A.S.

Half BPS states correspond to single (p, q) strings stretched between two D3-branes.



$$Q = (q, -q, 0, 0, 0), \quad P = (p, -p, 0, 0, 0)$$

Note: Q and P are always parallel.

For these states $B_4 = 1$ if gcd(q,p)=1 and 0 otherwise.

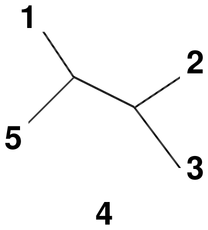
– follows from S-duality invariance and known spectrum of (0,q) states.

Olive, Witten; Osborn; A.S.; Segal, Selby

Quarter BPS states

The relevant index is B_6.

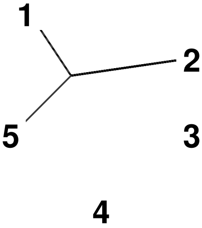
A configuration with strings ending on four or more D-branes is non-planar at a generic point in the moduli space and hence non-supersymmetric.



$\Rightarrow$ has $B_6 = 0$.

Thus B_6 receives contribution only from 3-string junctions.

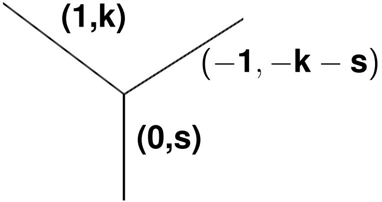
Bergman



We would like to compute B_6 for an arbitrary 3-string junction.

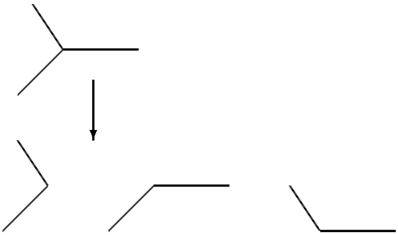
Although the string junction picture is useful in determining the region of existence of 1/4 BPS states, it is not very useful in finding the index B_6.

This can be done in special cases by representing them as bound states of multiple monopoles in gauge theories and solving the associated supersymmetric quantum mechanics problem. Bak, Lee, Yi; Gauntlett, Kim, Park, Yi; Stern, Yi



For this configuration $B_6 = (-1)^{s+1} s$

A three string junction exists in a certain region of the moduli space bounded by walls of marginal stability.



Along each wall the 3-string junction becomes marginally unstable against decay into a pair of half BPS states.

Beyond these walls the state stops existing.

$\Rightarrow \quad \mathbf{B_6 = \Delta B_6}$ **across the wall**

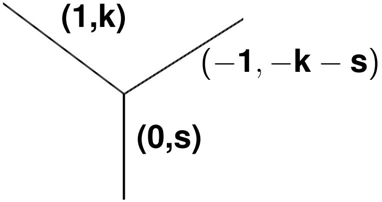
Relevant decays are to a pair of half BPS states.

A.S.; Mukherjee, Mukhi, Nigam; Dabholkar, Guica, Murthy, Nampuri

Across a wall along which (Q,P) decays into a pair of half BPS states with primitive charges (Q_1, P_1) and (Q_2, P_2) the jump in the index B_6 is

$$(-1)^{Q_1.P_2-P_1.Q_2+1}|Q_1.P_2 - Q_2.P_1|B_4(Q_1, P_1)B_4(Q_2, P_2)$$

Denef; Denef, Moore; A.S.; Dabholkar, Gaiotto, Nampuri; Cheng, Verinde



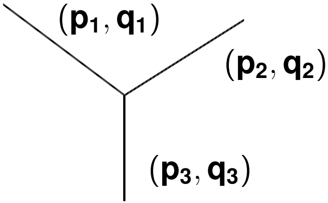
Consider the wall along which the (0,s) string shrinks to zero-size.

– decays to (1,k) string connecting a pair of D3-branes and a $(-1, -k - s)$ string connecting a pair of D3-branes, each with $B_4 = 1$.

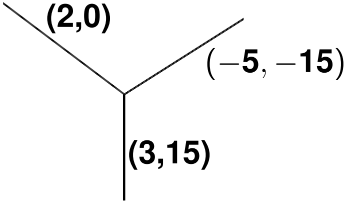
The jump in B_6 across this wall is given by $(-1)^{s+1}s$ in agreement with direct index computation.

Dabholkar, Nampuri, Narayan

Can we calculate the B_6 of a general 3-string junction?



In general the decays are not primitive, e.g. for



none of the decays are primitive.

We need to use general wall crossing formula for non-primitive decays in $\mathcal{N}$=4 supersymmetric string / field theories.

For the decay

$$(Q, P) \rightarrow (Q_1, P_1) + (Q_2, P_2)$$

$$\begin{aligned}\Delta B_6 &= (-1)^{Q_1\cdot P_2 - Q_2\cdot P_1 + 1}|Q_1 \cdot P_2 - Q_2 \cdot P_1| \\ &\times \sum_{r_1|Q_1,P_1} B_4(Q_1/r_1, P_1/r_1) \sum_{r_2|Q_2,P_2} B_4(Q_2/r_2, P_2/r_2)\end{aligned}$$

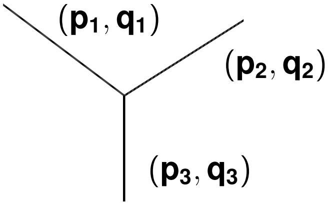
Banerjee, Srivastava, A.S; A.S.

For SYM theories,

$$\sum_{r|Q,P} B_4(Q/r, P/r) = 1$$

$\Rightarrow$ simple formula for B_6.

For



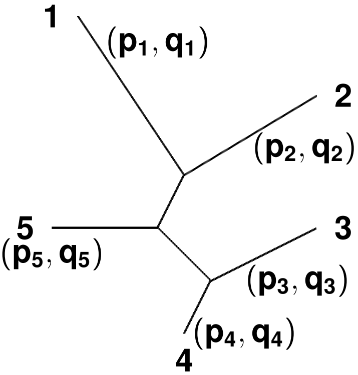
$$B_6 = (-1)^{p_1 q_2 - p_2 q_1 + 1} |p_1 q_2 - p_2 q_1|$$

– symmetric in the three strings.

This solves completely the problem of computing B_6 for $\mathcal{N}$=4 supersymmetric SU(N) theories.

For other gauge groups we can compute B_6 by identifying various SU(3) subgroups and computing B_6 for different chage vectors in each SU(3).

We shall now return to more general planar string network (possibly with internal loops).



As argued before, B_6 and any other index that is defined everywhere in the moduli space vanish for this configuration.

Strategy

Introduce an index that can be defined only in special subspace of the moduli space where all the D3-branes lie in a plane.

$\Leftrightarrow$ only two of the six adjoint Higgs fields take vev.

Make use of the SO(4)$\equiv$ SU$(2)_L \times$ SU$(2)_R$ rotational symmetry in the four directions transverse to the plane of the 3-branes to introduce a twisted index.

$SU(2)_L \times SU(2)_R \times SU(2)_{rotation}$ transformation laws of various supersymmetries:

State	unbroken susy	broken susy
Half BPS	(1,2,2) + (2,1,2)	(1,2,2) + (2,1,2)
Class A quarter BPS	(1,2,2)	(1,2,2) + 2 (2,1,2)
Class B quarter BPS	(2,1,2)	2(1,2,2) + (2,1,2)

Class A states and half BPS states have four broken and four unbroken $SU(2)_L$ invariant SUSY

Of these two of each are invariant under $I_{3R} + J_3$

(I_{3L}, I_{3R}, J_3): Cartan gen. of $SU(2)_L \times SU(2)_R \times SU(2)_{rotation}$

Following earlier logic we can now introduce protected index

$$\mathbf{B_2(z) = -\frac{1}{2!}Tr\{(-1)^{2J_3} z^{2I_{3L}} (2J_3)^2\}}$$

$$\mathbf{B_1(y,z) = -\frac{1}{y - y^{-1}} Tr \left\{e^{2i\pi J_3} z^{2I_{3L}}\, y^{2I_{3R}+2J_3}\,(2J_3)\right\}}$$

Similar index for N=2 SUSY theories are known.

Gaiotto, Moore, Neitzke

One can show that for class A states

$$\mathbf{B_2(z) = lim_{y\to 1} B_1(y,z)}$$

$$\mathbf{B_6 = lim_{z\to 1}(z + z^{-1} - 2)^{-2} B_2(z)}$$

Thus $\mathbf{B_1(y,z)}$ is the most general index.

Can we compute $B_1(y,z)$ for a general string network?

1. For half BPS states we have all the information and $B_1(y,z)$ is straightforward to compute.

$$B_1(y,z) = (z + z^{-1} - y - y^{-1})(y - y^{-1})$$

for each primitive half BPS state.

2. For collinear configuration of D3-branes only half BPS states contribute to $B_1(y,z)$.

Strategy: Start from this collinear configuration and apply wall crossing formula to find the result elsewhere in the moduli space.

Caution: The structure of marginal stability walls is more complicated than before.

$B_1(y, z)$ follows a wall crossing formula similar to the motivic KS formula.

Kontsevich, Soibelman

There are many physical ‘derivations’ of this formula by now.

Cecotti, Vafa; Gaiotto, Moore, Neitzke; Dimofte, Gukov; Andriyash, Denef, Jafferis, Moore; Lee, Yi; Kim, Park, Wang, Yi; Manschot, Pioline, A.S $\cdots$

We can derive the wall crossing formula for $B_1(y, z)$ using any of these approaches.

1. Given $\alpha = (\mathbf{Q}, \mathbf{P})$ and $\alpha' = (\mathbf{Q}', \mathbf{P}')$, define

$$\langle \alpha, \alpha' \rangle = \mathbf{Q}.\mathbf{P}' - \mathbf{Q}'.\mathbf{P}$$

2. Define

$$\bar{\mathbf{B}}_1(\alpha; \mathbf{y}, \mathbf{z}) \equiv \sum_{\mathbf{m}|\alpha} \mathbf{m}^{-1} \frac{\mathbf{y} - \mathbf{y}^{-1}}{\mathbf{y}^{\mathbf{m}} - \mathbf{y}^{-\mathbf{m}}} \mathbf{B}_1(\alpha/\mathbf{m}; \mathbf{y}^{\mathbf{m}}, \mathbf{z}^{\mathbf{m}})$$

3. Introduce the algebra:

$$[\mathbf{e}_\alpha, \mathbf{e}_{\alpha'}] = \frac{(-\mathbf{y})^{\langle \alpha, \alpha' \rangle} - (-\mathbf{y})^{-\langle \alpha, \alpha' \rangle}}{\mathbf{y} - \mathbf{y}^{-1}} \mathbf{e}_{\alpha+\alpha'}$$

Then wall crossing formula tells us that:

$$\mathbf{P} \left(\prod_{\alpha} \exp \left[\bar{\mathbf{B}}_1(\alpha; \mathbf{y}, \mathbf{z}) \mathbf{e}_\alpha \right] \right)$$

is unchanged across a wall.

P: ordering according to Arg(central charge).

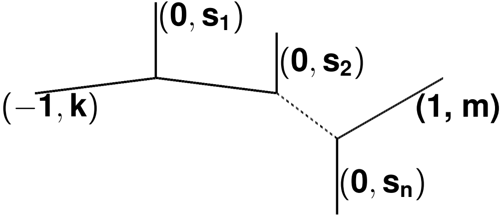
Using the wall crossing formula and the result for half BPS states we can compute $B_1(y,z)$ for any class A quarter BPS states in any chamber in the moduli space.

The corresponding index for class B states can be defined and computed in a similar way by exchanging

$$SU(2)_L \Leftrightarrow SU(2)_R$$

Example 1: Consider the network

Stern, Yi



We can shrink the $(0, s_i)$ strings one by one and apply the primitive wall crossing formula.

Final result for $B_1(y, z)$:

$$(-1)^{\sum |s_i| + n} \{z + z^{-1} - y - y^{-1}\}^{n+1} \prod_i \frac{y^{|s_i|} - y^{-|s_i|}}{y - y^{-1}}$$

We get the same result by shrinking the $(-1, k)$ string and applying semi-primitive wall crossing formula.

$$B_1(y,z) = (-1)^{\sum |s_i|+n} \left\{z + z^{-1} - y - y^{-1}\right\}^{n+1} \prod_i \frac{y^{|s_i|} - y^{-|s_i|}}{y - y^{-1}}$$

Consistency check:

1. For $y = 1$, $z = -1$ we get $-4 \prod_i (-1)^{s_i}(4s_i)$

– agrees with an appropriate index computed in supersymmetric quantum mechanics of multiple monopoles.

Stern, Yi

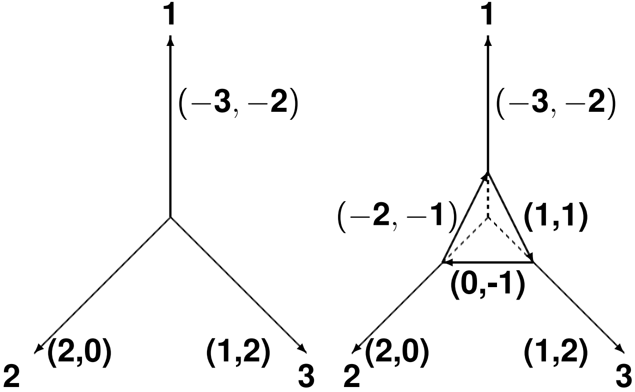
This has also been derived by applying primitive wall crossing formula.

Dabholkar, Nampuri, Narayan

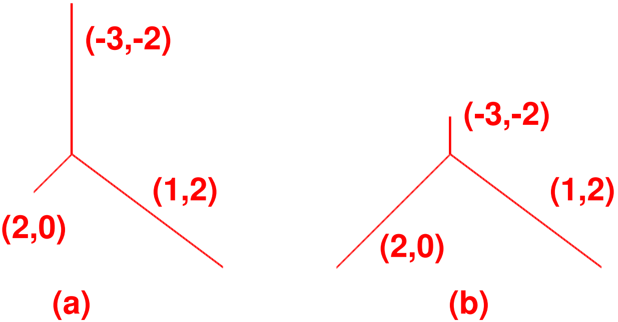
2. y-dependence agrees with quiver quantum mechanics analysis.

Denef

Example 2:



We can try to compute the index by shrinking any of the external strings to zero size.

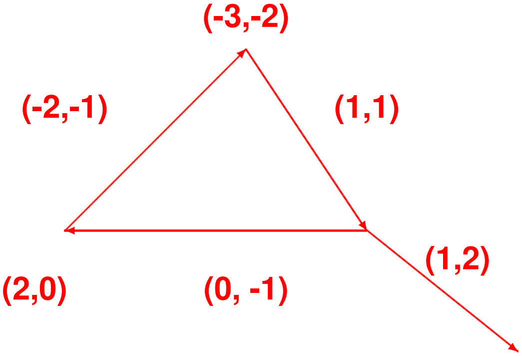


Result for $B_1(y,z)$ for cases (a) and (b):

$$\left\{z+z^{-1}-y-y^{-1}\right\}^2\left\{-y^3-\frac{1}{y^3}-y-\frac{1}{y}\right\}$$

$$\left\{z+z^{-1}-y-y^{-1}\right\}^2\left\{-y^3-\frac{1}{y^3}-2y-\frac{2}{y}+z+\frac{1}{z}\right\}$$

$\Rightarrow$ there must be marginal stability wall separating (a) and (b).



– **can break apart into the (-2,-1) string and the rest.**

The jump in $B_1(y,z)$ across this wall precisely accounts for the previous difference.

Conclusion

The protected information about the spectrum of BPS states in $\mathcal{N} = 4$ supersymmetric Yang-Mills theories is encoded in the index $\mathbf{B_1(y, z)}$.

We now have a complete algorithm for computing this index for any charge vector at any point in the moduli space.