Noncommutative Field Theory Representation of AdS/CFT Correspondence

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NC Fields As Large N Matrices

Consider a two-dimensional noncommutative (NC) space

$$[x, y] = i\theta \quad \Leftrightarrow \quad [a, a^\dagger] = 1.$$

Since $\mathcal{H} = \{\, |n\rangle\, ;\ n = 0,1,\cdots,\infty\}$ and $\sum_{n=0}^{\infty} |n\rangle\langle n| = 1_{\mathcal{H}}$, for $\Phi_1,\ \Phi_2 \in \mathcal{A}_\theta$,

$$\Phi_1(x,y) = \sum_{n,m=0}^{\infty} |n\rangle\langle n|\Phi_1(x,y)\,|m\rangle\langle m| \equiv M_{nm}|n\rangle\langle m|,$$

$$\Phi_2(x,y) = \sum_{n,m=0}^{\infty} |n\rangle\langle n|\Phi_2(x,y)\,|m\rangle\langle m| \equiv N_{nm}|n\rangle\langle m|,$$

$$(\Phi_1 \star \Phi_2)(x,y) = \sum_{n,l,m=0}^{\infty} |n\rangle\langle n|\Phi_1(x,y)|l\rangle\langle l|\Phi_2(x,y)\,|m\rangle\langle m| = M_{nl}N_{lm}|n\rangle\langle m|,$$

NC fields $\Phi_a(x,y)$ in $\mathcal{A}_\theta$ = adjoint operators acting on a separable Hilbert space $\mathcal{H}$ = $N \times N$ matrices in $End(\mathcal{H}) \equiv \mathcal{A}_N$ with $N \to \infty$.

Ordering in $\mathcal{A}_\theta$ = ordering in $\mathcal{A}_N$ and $Tr_N = Tr_{\mathcal{H}} = \int \frac{dxdy}{2\pi\theta}$.

Four-dimensional $\mathcal{N}=4\ U(N)$ Super Yang-Mills Theory

$\mathcal{N}=4$ vector multiplet: $(A_\mu, \lambda^i_\alpha, \Phi_a)$ where $i=0,\cdots,3$ and $a=1,\cdots,6$ are in the adjoint representation of $U(N)$.

The action is given by

$$S=\int d^4x\, Tr\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}D_\mu\Phi_a D^\mu\Phi^a+\frac{g^2}{4}[\Phi_a,\Phi_b]^2-\frac{i}{2}\bar{\lambda}_i\bar{\sigma}^\mu D_\mu\lambda^i$$
$$+\frac{g}{2}C^a_{ij}\lambda^i[\Phi_a,\lambda^j]+\frac{g}{2}C^{ij}_a\,\bar{\lambda}_i[\Phi^a,\bar{\lambda}_j]\}. \qquad (1)$$

$\mathcal{N}=4$ supersymmetry transformation is a dimensional reduction of ten-dimensional $\mathcal{N}=1$ super Yang-Mills theory to four dimensions:

$$\delta A_M=i\,\bar{\alpha}\Gamma_M\Psi,\ \delta\Psi=F_{MN}J^{MN}\alpha,\quad M,N=0,\cdots,9, \quad (2)$$
$$A_M=(A_\mu,\Phi_a),\ \ \Psi=\begin{pmatrix}P_+\lambda^i\\ P_-\tilde{\lambda}_i\end{pmatrix}\text{ with }\tilde{\lambda}_i=C\bar{\lambda}_i^{\,T},$$
$$\Gamma^M=(\gamma^\mu\otimes I_8,\gamma_5\otimes\Delta^a),$$
$$\Gamma_{11}=\gamma_5\otimes I_8,\ C_{10}=C\otimes\begin{pmatrix}0 & I_4\\ I_4 & 0\end{pmatrix}.$$

(Brink, Scherk & Schwarz, NPB 121 (1977) 77)

NC Field Representation of d=4 $\mathcal{N}=4\;U(N)$ SYM Theory

Consider a vacuum configuration of $\mathcal{N}$=4 super Yang-Mills theory

$$\langle\Phi_a\rangle_{vac} = B_{ab}\, y^b,\ \langle A_\mu\rangle_{vac} = 0,\ \langle\lambda^i\rangle_{vac} = 0. \qquad (3)$$

Assume that the vacuum expectation value $y^a \in \mathcal{A}_N$ $(N \to \infty)$ satisfies the Heisenberg-Moyal algebra

$$[y^a, y^b] = i\theta^{ab} I_{N\times N}$$

where $\theta^{ab} = \left(\frac{1}{B}\right)^{ab}$. It is obvious that the vacuum configuration (3) in the $N \to \infty$ limit is definitely a solution of the theory and preserves four-dimensional Lorentz symmetry.

Consider fluctuations of large N matrices around the vacuum (3)

$$D_\mu(x,y) = \partial_\mu - i\,A_\mu(x,y),\ \ D_a(x,y) \equiv -i\,\Phi_a(x,y) = -iB_{ab}y^b - iA_a(x,y),$$

$$\Psi(\mathrm{x},\mathrm{y}) = \begin{pmatrix} P_+\lambda^i \\ P_-\widetilde{\lambda}_i \end{pmatrix}(x,y).$$

According to the map between NC $\ast$-algebra $\mathcal{A}_\theta$ and $\mathcal{A}_N = End(\mathcal{H})$, large N matrices ($\mathcal{N}$=4 vector multiplet) on $\mathbb{R}^{3,1}$ are mapped to NC fields ($\mathcal{N}$=1 vector multiplet) in $S(C^\infty(\mathbb{R}^{3,1}) \otimes \mathcal{A}_\theta)$.

Therefore let us introduce 10-dimensional coordinates $X^M = (x^\mu, y^a)$ and 10-dimensional connections defined by

$$D_M(X) = \partial_M - iA_M(x,y) = \left(D_\mu, D_a\right)(x,y)$$

whose field strength is given by

$$F_{MN}(X) = \partial_M A_N - \partial_N A_M \; - i[A_M, A_N]_\star.$$

In the end, the 4-dimenisonal $U(N)$ super Yang-Mills theory (1) has been transformed into 10-dimensional $U(1)$ super Yang-Mills theory

$$S_{10} = \int d^{10}X \left\{ -\frac{1}{4g_{YM}^2}(F_{MN} - B_{MN})^2 + \frac{i}{2}\,\overline{\Psi}\Gamma^M D_M \Psi \right\} \qquad (4)$$

where $B_{MN} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix}$. The action (4) is invariant under $\mathcal{N}$=1 supersymmetry transformation given by

$$\delta A_M = i\,\bar{\alpha}\Gamma_M \Psi, \qquad \delta\Psi = (F_{MN} - B_{MN})\, J^{MN}\alpha.$$

We want to emphasize that the relationship between the 4-dimenisonal $U(N)$ super Yang-Mills theory (1) and 10-dimensional $U(1)$ super Yang-Mills theory (4) is not a dimensional reduction but they are simply equivalent to each other.

$U(N)$ *vs.* NC *U(1)* Wilson lines

- Because the orderings in $U(N)$ and NC $U(1)$ gauge theories are compatible each other, any quantity in $U(N)$ gauge theory can be transformed into an object in NC $U(1)$ gauge theory.

For example, a Wilson loop in $U(N)$ gauge theory

$$W_N = \frac{1}{N} Tr\, P \exp\left(i \oint (A_\mu \dot{x}^\mu + \Phi_a \dot{y}^a) ds\right)$$

can be translated into a corresponding NC $U(1)$ Wilson loop defined by

$$\widehat{W} = \frac{1}{V_6} \int d^6 y\, P \exp(i \oint (B_{ab}\, \dot{y}^a y^b + A_M \dot{x}^M)\, ds\,)$$

where V_6 is a volume of extra six-dimensional space. P denotes a path ordering which is taken only for loop variables $x^M(s)$ satisfying $\dot{x}^2 - \dot{y}^2 = 0$ to preserve supersymmetry (a minimal surface on the boundary of AdS_5) and then the phase factor $B_{ab}\, \dot{y}^a y^b$ vanishes because of $\dot{y}^a = \frac{y^a}{\rho^2}\, \dot{x}^2$ with $\rho^2 = \sum_{a=1}^{6} y^a y^a$.

(Drukker, Gross & Ooguri, hep-th/9904191)

Emergent Gravity from NC Gauge Fields

A great advantage of NC field representation for large N gauge theory is that it provides an efficient way to identify a higher-dimensional gravitational metric which is dual to the $\mathcal{N}=4$ vector multiplet.

An underlying idea is simple. The adjoint map of NC $*$-algebra $\mathcal{A}_\theta$ defined by

$$ad_f^\star : g \mapsto -i[f, g]_\star, \qquad f, g \in \mathcal{A}_\theta \qquad (5)$$

is a derivation, i.e., satisfies the Leibniz rule. So it generates generalized vector fields, so-called polydifferential operators on $\mathcal{A}_\theta$, given by

$$ad_f^\star \equiv X_f^\star = X_f + \sum_{n=2}^{\infty} \xi_f^{A_1 \cdots A_n} \partial_{A_1} \cdots \partial_{A_n}.$$

The polydifferential operator $X_f^\star$ recovers the usual vector field $X_f \in \Gamma(TM)$ in the commutative $(\theta \to 0)$ limit.

Furthermore, the Jacobi identity of NC $*$-algebra $\mathcal{A}_\theta$ guarantees the (deformed) Lie algebra homomorphism

$$[X_f^\star, X_g^\star] = X_{[f,g]_\star}^\star.$$

Consider 10-dimensional $\mathcal{N}=1$ supersymmetric $U(1)$ gauge theory (4) and apply the adjoint operation (5) to the 10-dimensional NC gauge fields

$$V_A^\star \equiv \mathrm{ad}^\star_{\mathrm{D_A}} = \mathrm{V_A^M(x,y)}\partial_M + \cdots$$
$$= V_A + \mathcal{O}(\theta^3)\,.$$

In the commutative limit, the vector fields $V_A = V_A^M(x,y)\partial_M \in \Gamma(TM)$ are related to the vielbeins $E_A \in \Gamma(TM)$ by

$$V_A = \lambda E_A$$

where $\lambda^2 = \nu(V_0, \cdots, V_9)$ with a volume form $\nu = d^4x \wedge \nu_6$.
Thus the 10-dimensional geometry dual to the gauge theory (1) or (4) can easily be determined by

$$ds^2 = \lambda^2 V^A \otimes V^A$$
$$= \lambda^2(\eta_{\mu\nu}dx^\mu dx^\nu + V_b^a V_c^a(dy^b - \boldsymbol{A}^b)(dy^c - \boldsymbol{A}^c))$$

where $\boldsymbol{A}^a = A_\mu^a(x,y)dx^\mu$.

First consider a vacuum geometry with $A_M = (A_\mu, A_a) = 0,\ \Psi = 0$ which is given by

$$ds^2 = \lambda^2(\eta_{\mu\nu}dx^\mu dx^\nu + dy^a dy^a).$$

$$\nu_6 = \mathrm{d}^6\mathrm{y} \quad \Rightarrow \quad M_{10} = \mathbb{R}^{9,1},$$

$$\nu_6 = \frac{d^6 y}{\rho^2} \quad \Rightarrow \quad M_{10} = AdS_5 \times \mathbb{S}^5$$ where $\rho^2 = \sum_{a=1}^{6} y^a y^a$.

What makes this difference ? In order to pose this question, we might address an issue about the origin of four-dimensional spacetime.
So let us start with 0-dimensional IKKT matrix model

$$S = -\frac{1}{4g^2} Tr[X_A, X_B]^2 + Fermions, \qquad A = 0,1,\cdots,9.$$

One can consider two kinds of vacuum defined by

$$(I): \quad \langle X_\mu \rangle_{vac} = B_{\mu\nu}\, x^\nu, \qquad \langle X_a \rangle_{vac} = B_{ab}\, y^b,$$

$$(II): \quad \langle X_\mu \rangle_{vac} = B_{\mu\nu}\, x^\nu + A_\mu(x), \qquad \langle X_a \rangle_{vac} = B_{ab}\, y^b,$$

where $A_\mu(x)$ describes a uniform condensate of NC $U(1)$ instantons over $\mathbb{R}^4$. Then the IKKT matrix model on the vacuum (I) and (II) becomes 4-dimensional (supersymmetric) NC $U(N \to \infty)$ gauge theory whose commutative limit is precisely $\mathcal{N}=4$ super Yang-Mills theory.
Then the vacuum (I) gives rise to $M_{10} = \mathbb{R}^{10}$ and the vacuum (II), I speculate, corresponds to $M_{10} = AdS_5 \times \mathbb{S}^5$. This speculation seems to be consistent with the instanton calculus (Bianchi, Green, et al, hep-th/9807033 & Dorey, Hollowood, et al, hep-th/9901128) if we can identify NC $U(1)$ instantons with D-instantons in type IIB string theory.

Calabi-Yau Manifolds from NC U(1) Instantons

Return to the $\mathcal{N}$=4 super Yang-Mills theory. Remember that the adjoint scalar fields $\Phi_a(x) \in U(N)$ are mapped to NC $U(1)$ gauge fields in extra dimensions and obey the relation

$$-i[\Phi_a, \Phi_b] = -B_{ab} + F_{ab},$$

where $F_{ab}(x, y) = \partial_a A_b - \partial_b A_a - i[A_a, A_b]_\star$. Therefore, topological solutions made out of $\Phi_a(x) \in U(N)$ will be given by NC $U(1)$ instantons in four or six dimensions.

First turn off $A_\mu = 0, \ \Psi = 0$ for simplicity and consider 4-dimensional NC $U(1)$ instantons satisfying the self-duality equation

$$F_{ab} = \pm\frac{1}{2}\varepsilon_{ab}{}^{cd}F_{cd}$$

where $a, b, c, d = 1, \cdots, 4$ and two remaining gauge fields are chosen to describe a Riemann surface Σ_g.

Then the resulting ten-dimensional spacetime becomes

$$M_{10} = M_4 \times CY_2 \times \Sigma_g .$$

(Salizzoni & HSY , PRL 96 (2006) 201602; HSY, EPL 88 (2009) 31002)

If we consider 6-dimensional NC Hermitian $U(1)$ instantons defined by

$$F_{ab} = \pm \frac{1}{2} \varepsilon_{ab}{}^{cdef} F_{cd} J_{ef} ,$$

$$J^{ab} F_{ab} = 0,$$

where $B_{ab} = \kappa J_{ab}$. Then the ten-dimensional spacetime as an emergent geometry from NC Hermitian $U(1)$ instantons is given by

$$M_{10} = M_4 \times CY_3 .$$

(S. Yun & HSY, to appear).