Max Planck Institute for Mathematics
California Institute of Technology

# Volume Conjecture: Refined and Categorified

Sergei Gukov

based on: hep-th/0306165 (generalized volume conjecture)

with T.Dimofte, arXiv:1003.4808 (review/survey)

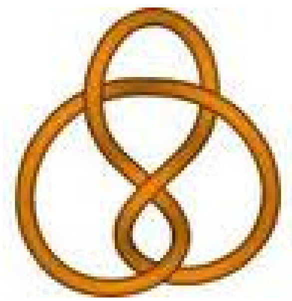
with H.Fuji and P.Sulkowski, arXiv:1203.2182 (new!)

Progress in Quantum Field Theory and String Theory
April 3-7, 2012
Media Center, Osaka City University, Japan

# Kashaev's observation

[R. Kashaev, 1996]

knot **K**



invariant $\langle K \rangle_n \in \mathbb{C}$
labeled by a positive integer **n**

- defined via R-matrix
- very hard to compute

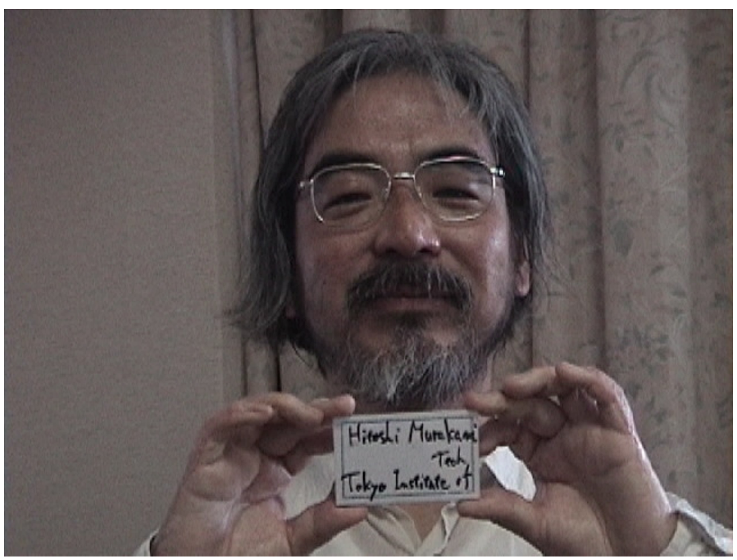
$$\lim_{n\to\infty} \frac{1}{n} \log \langle K \rangle_n = \mathrm{Vol}\,(S^3 \setminus K)$$

("volume conjecture")

# A first step to understanding the Volume Conjecture

$\langle K \rangle_n = J_n(q)$ colored Jones polynomial

with $q = \exp(2\pi i/n)$




Hitoshi Murakami

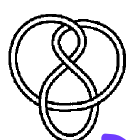
Jun Murakami (1999)

# Colored Jones polynomial

$J_2(q) = J(q) =$ Jones polynomial

- In Chern-Simons gauge theory [E.Witten]

Wilson loop operator

$\Big\langle$  $\Big\rangle_R =$ polynomial in $q$

$R$ ← 2-dimn'l representation of $SU(2)$

# Colored Jones polynomial

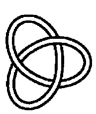
$$J_2(q) = J(q) = \text{Jones polynomial}$$

- Skein relations:

$$q^2 J(\;) - q^{-2} J(\;) = (q^{-1} - q) \cdot J(\;)$$

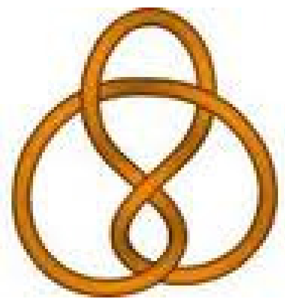
$$J(\text{unknot}) = q^{-1} + q$$

<u>Example:</u>

$$J\Big($$  $$\Big) = q + q^3 + q^5 - q^9$$

# Colored Jones polynomial

knot **K** $\longrightarrow$ **n**-colored Jones polynomial:



$J_n(K;q) \in \mathbb{Z}[q, q^{-1}]$

**R** = **n**-dimn'l representation of **SU(2)**

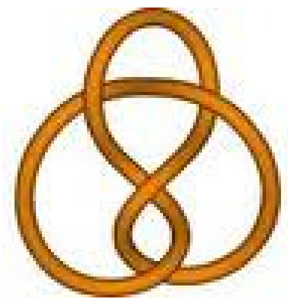
- "Cabling formula":

$$J_{\oplus_i R_i}(K;q) = \sum_i J_{R_i}(K;q)$$

$$J_R(K^n;q) = J_{R^{\otimes n}}(K;q)\,,$$

# Colored Jones polynomial

knot K



n-colored Jones polynomial:

$J_n(K;q) \in \mathbb{Z}[q, q^{-1}]$

R = n-dimn'l representation of SU(2)

$$J_1(K;q) = 1\,,$$
$$J_2(K;q) = J(K;q)\,,$$
$$\mathbf{2}^{\otimes 2} = \mathbf{1} \oplus \mathbf{3} \Rightarrow J_3(K;q) = J(K^2;q) - 1\,,$$
$$\mathbf{2}^{\otimes 3} = \mathbf{2} \oplus \mathbf{2} \oplus \mathbf{4} \Rightarrow J_4(K;q) = J(K^3;q) - 2J(K;q)$$
$$\ldots,$$

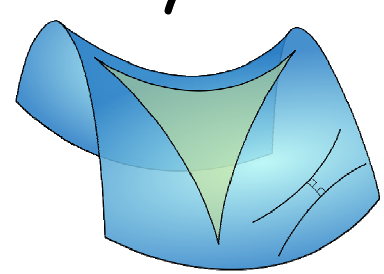
# Volume Conjecture

Murakami & Murakami:

$$\langle K \rangle_n = J_n(K; q = e^{2\pi i/n})$$

$$\lim_{n\to\infty} \frac{2\pi \log |J_n(K; q = e^{2\pi i/n})|}{n} = \mathrm{Vol}(M)$$

quantum group invariants (combinatorics, representation theory) $\longleftrightarrow$ classical hyperbolic geometry



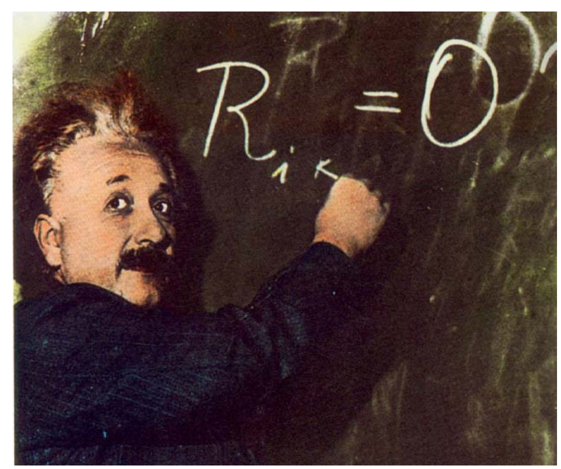
# Physical interpretation of the Volume Conjecture

[S.G., 2003]

- analytic continuation of **SU(2)** is **SL(2,ℂ)**

$$\lim_{n\to\infty} \frac{2\pi \log |J_n(K; q = e^{2\pi i/n})|}{n} = \mathrm{Vol}(M)$$

- classical **SL(2,ℂ)** Chern-Simons theory = classical 3d gravity (hyperbolic geometry)



# Physical interpretation of the Volume Conjecture

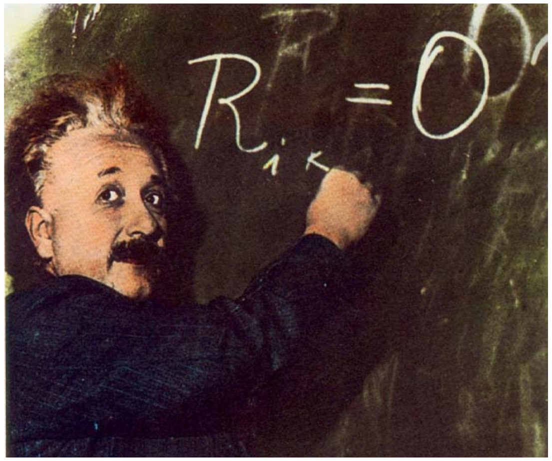
[S.G., 2003]

constant negative curvature metric on **M** $\longleftrightarrow$ flat $\mathbf{SL(2,\mathbb{C})}$ connection on $M = S^3 \setminus K$

$$\boxed{R_{ij} = -2g_{ij}}$$

$$\boxed{d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \ = \ 0}$$

- classical $\mathbf{SL(2,\mathbb{C})}$ Chern-Simons theory = classical 3d gravity (hyperbolic geometry)



# Physical interpretation of the Volume Conjecture

[S.G., 2003]

constant negative curvature metric on **M** $\longleftrightarrow$ flat **SL(2,$\mathbb{C}$)** connection on **M = S$^3$ \ K**

$$\boxed{R_{ij} = -2g_{ij}}$$

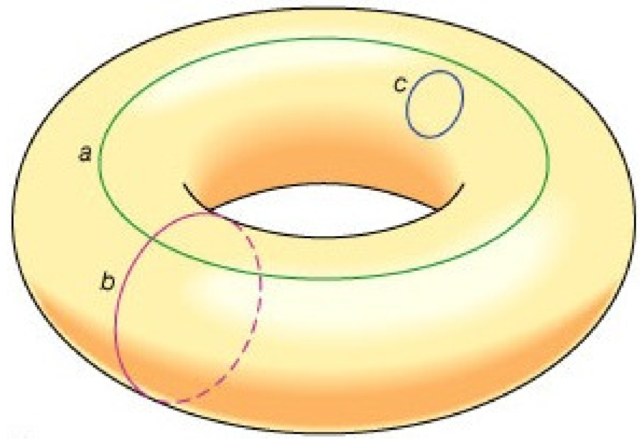
$$\boxed{d\mathcal{A} + \mathcal{A} \wedge \mathcal{A} \; = \; 0}$$

classical solution in 3D gravity with negative cosmological constant ( $\Lambda = -1$ )

classical solution in CS gauge theory

# Example: unknot = BTZ black hole

K = unknot ⇒ M = solid torus

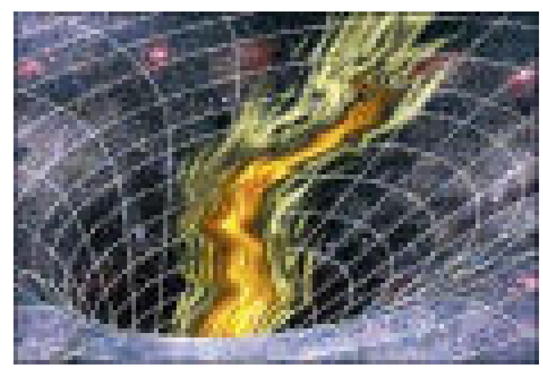



Euclidean BTZ black hole:

$$ds^2 = N^2 d\tau^2 + N^{-2} dr^2 + r^2 \left( d\phi^2 + N^\phi d\tau \right)^2$$

$$N = \sqrt{r^2 - M - \frac{J^2}{4r^2}} \quad , \quad N^\phi = -\frac{J}{2r^2}$$

$$r_\pm^2 = \frac{M}{2} \left[ 1 \pm \sqrt{1 + \left( \frac{J}{M} \right)^2} \right]$$



# Physical interpretation of the Volume Conjecture

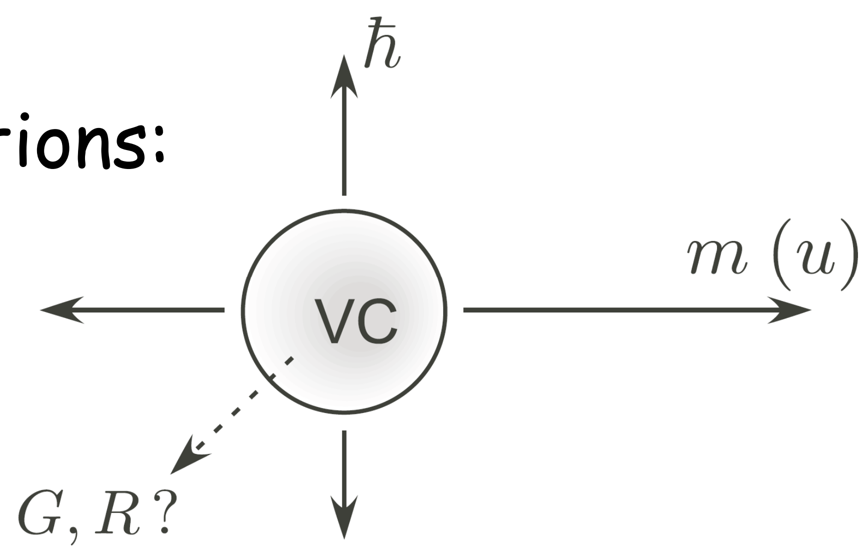
[S.G., 2003]

<u>Moral:</u>

$$\lim_{n \to \infty} \left( \text{SU(2) Chern-Simons} \right) \simeq \text{SL(2,C) Chern-Simons}$$

Classical limit $q = \exp(2\pi i/n) \to 1$

- leads to many generalizations:

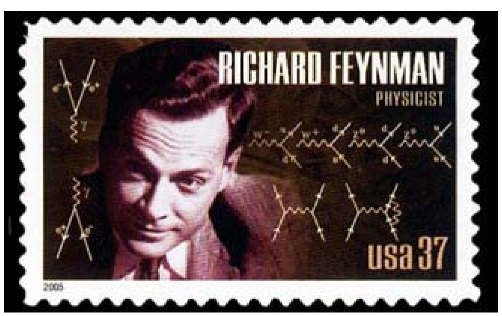


# Physical interpretation of the Volume Conjecture

Generalization 1: 't Hooft limit:

$$q = e^{\hbar} \to 1\,, \qquad n \to \infty\,, \qquad q^n = e^u \equiv x \quad \text{(fixed)}$$

$$\boxed{J_n(K; q = e^{\hbar}) \overset{n\to\infty}{\underset{\hbar\to 0}{\sim}} \exp\left(\frac{1}{\hbar} S_0(u) + \ldots\right)}$$



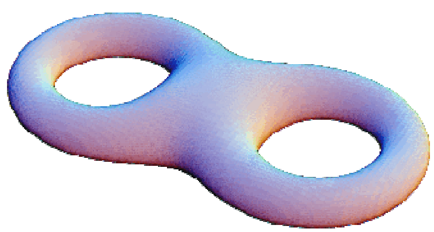

partition function of $SL(2,\mathbb{C})$ Chern-Simons theory

# Generalized Volume Conjecture

$S_0(u)$ = classical action of
**SL(2,ℂ)** Chern-Simons theory

$\mathcal{M}_{\text{flat}}(G_{\mathbb{C}}, M)$ : space of solutions

$$dA + A \wedge A = 0 \text{ on } S^3 \setminus K$$



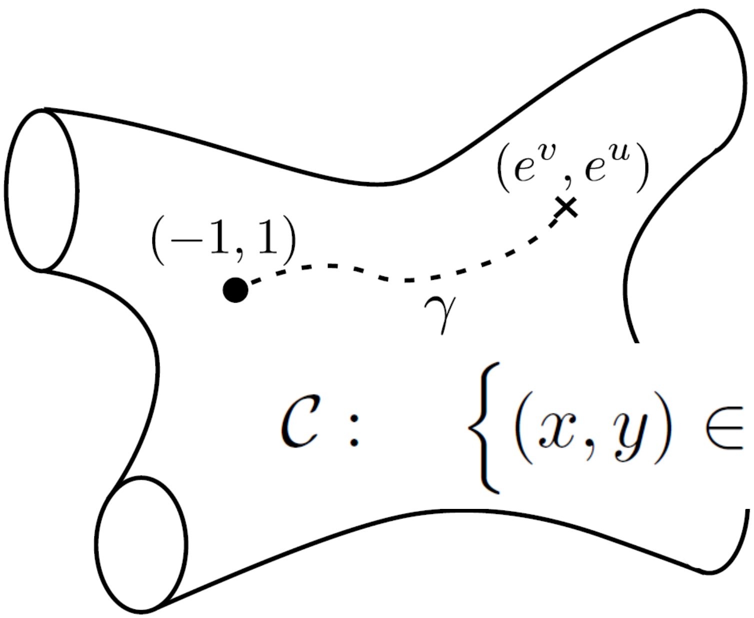
A-polynomial of a knot K

$$\mathcal{C}: \quad \left\{ (x,y) \in \mathbb{C}^* \times \mathbb{C}^* \,\middle|\, \underline{A(x,y) = 0} \right\}$$

# Algebraic curves and knots

$$S_0(u) \;=\; \int v\,du \;=\; \int \log y \frac{dx}{x}$$

$x = e^u \qquad y = e^v$



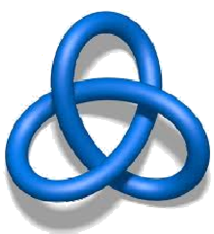

$$\mathcal{C}: \quad \left\{ (x,y) \in \mathbb{C}^* \times \mathbb{C}^* \,\middle|\, \underline{A(x,y) \;=\; 0} \right\}$$

A-polynomial
of a knot K

# Physical interpretation of the Volume Conjecture

[S.G., 2003]

Generalization 2:



knot **K**

**n**-colored Jones polynomial:

$$J_n(K;q)$$

recursion relation:

$$\alpha(q^n, q)\, J_{n-1} + \beta(q^n, q)\, J_n + \gamma(q^n, q)\, J_{n+1} \;=\; 0$$

rational functions

# Quantum Volume Conjecture

using $x \equiv e^u = q^n = \text{fixed}$ [S.G., 2003]

we can write this recursion relation as:

$$\boxed{\widehat{A}\, J_*(K;q) \;\simeq\; 0}$$

where $\widehat{A}(\widehat{x},\widehat{y};q) = \alpha\widehat{y}^{-1} + \beta + \gamma\widehat{y}$

$$\widehat{x} J_n = q^n J_n$$

$$\widehat{y} J_n = J_{n+1}$$

so that

$$\widehat{y}\widehat{x} = q\widehat{x}\widehat{y}$$

# Quantum Volume Conjecture

[S.G., 2003]

- In the classical limit $q \to 1$ the operator $\widehat{A}(\widehat{x}, \widehat{y}; q)$ becomes $A(x,y)$ and the way it comes about is that

$$x \,,\, y \quad \rightsquigarrow \quad \widehat{x} \,,\, \widehat{y}$$

$$A(x,y) \;=\; 0 \quad \rightsquigarrow \quad \widehat{A}(\widehat{x}, \widehat{y})\, Z_{\mathrm{CS}}(M) \;=\; 0$$

- in the mathematical literature was independently proposed around the same time, and is know as the AJ-conjecture

[S.Garoufalidis, 2003]

# Quantization and B-model

$$\log T(u) \;=\; \lim_{u_1 \to u_2 = u} \int \left( \frac{du_1\, du_2}{(u_1 - u_2)^2} - B(u_1, u_2) \right)$$

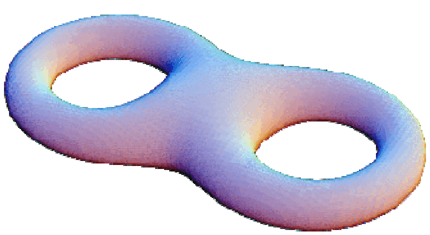
Bergman kernel

- simple formula that turns classical curves $A(x,y) = 0$ into quantum operators

[S.G., P.Sulkowski]

$$x\;,\;y \;\rightsquigarrow\; \hat{x}\;,\;\hat{y}$$

$$A(x,y) \;=\; 0 \;\rightsquigarrow\; \hat{A}(\hat{x},\hat{y})\, Z_{\mathrm{CS}}(M) \;=\; 0$$



$$\hat{A}(\hat{x},\hat{y};q) \;=\; \sum_{m,n} a_{m,n}\, q^{c_{m,n}}\, \hat{x}^m\, \hat{y}^n$$

# Quantization and B-model

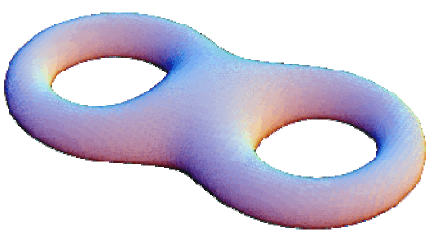
$$\log T(u) \;=\; \lim_{u_1 \to u_2 = u} \int \left( \frac{du_1\, du_2}{(u_1 - u_2)^2} - B(u_1, u_2) \right)$$

$$\boxed{\sum_{m,n} a_{m,n}\, c_{m,n}\, x^m y^n \;=\; \frac{1}{2} \left( \frac{\partial_u A}{\partial_v A} \partial_v^2 + \frac{\partial_u T}{T} \partial_v \right) \; A}$$

[S.G., P.Sulkowski]

$$x \,,\, y \;\rightsquigarrow\; \widehat{x} \,,\, \widehat{y}$$

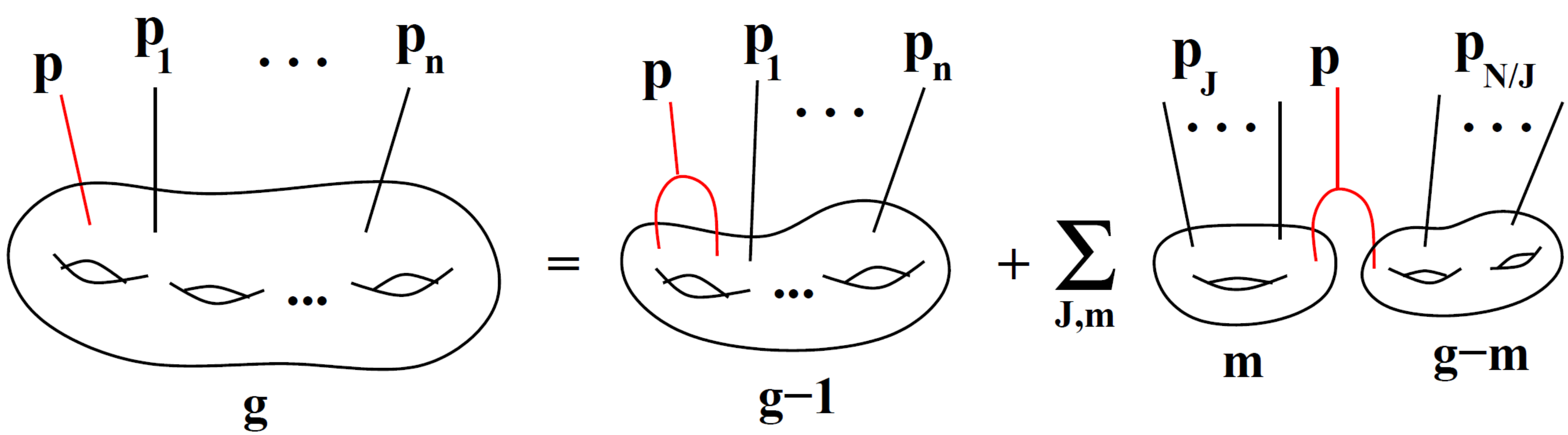
$$A(x,y) \;=\; 0 \;\rightsquigarrow\; \widehat{A}(\widehat{x}, \widehat{y})\, Z_{\mathrm{CS}}(M) \;=\; 0$$



$$\widehat{A}(\widehat{x}, \widehat{y}; q) \;=\; \sum_{m,n} a_{m,n}\, q^{c_{m,n}}\, \widehat{x}^m\, \widehat{y}^n$$

# Quantization and B-model

[B.Eynard, N.Orantin]
[V.Bouchard, A.Klemm, M.Marino, S.Pasquetti]
[A.S.Alexandrov, A.Mironov, A.Morozov]
[R.Dijkgraaf, H.Fuji, M.Manabe]
:



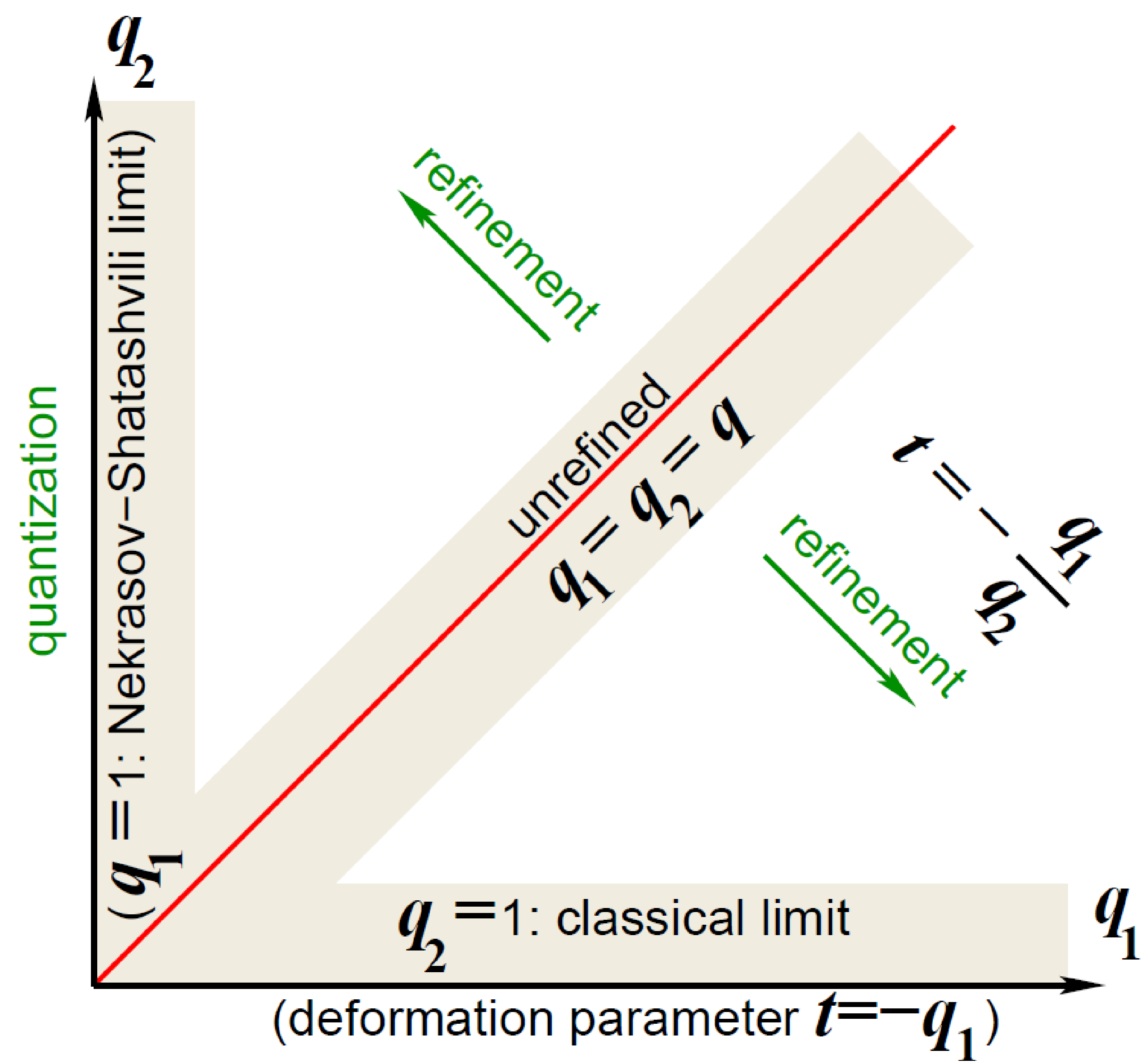

$x(p)$ and $y(p)$ $\rightsquigarrow$ $W_n^g$ $\rightsquigarrow$ $Z$ $\boxed{\hat{A}(\hat{x},\hat{y})}$

( recall: $\hat{A}(\hat{x},\hat{y})\, Z(\mathbf{M}) = 0$ )

# Refinement / Categorification

| | Algebraic curve | Refinement |
|---|---|---|
| Knots / SL(2,C) CS | A-polynomial | Homological invariants $P_n(q,t)$ |
| Matrix models | spectral curve | $\beta$-deformation |
| 4d gauge theories<br>3d superconformal indices | Seiberg-Witten curve<br>A-polynomial | refinement $t = -\frac{q_1}{q_2}$ |
| Topological strings (B-model) | mirror Calabi-Yau geometry $A(x,y) = zw$ | |

# Deformation vs Quantization



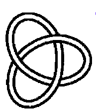
# Knot Homologies

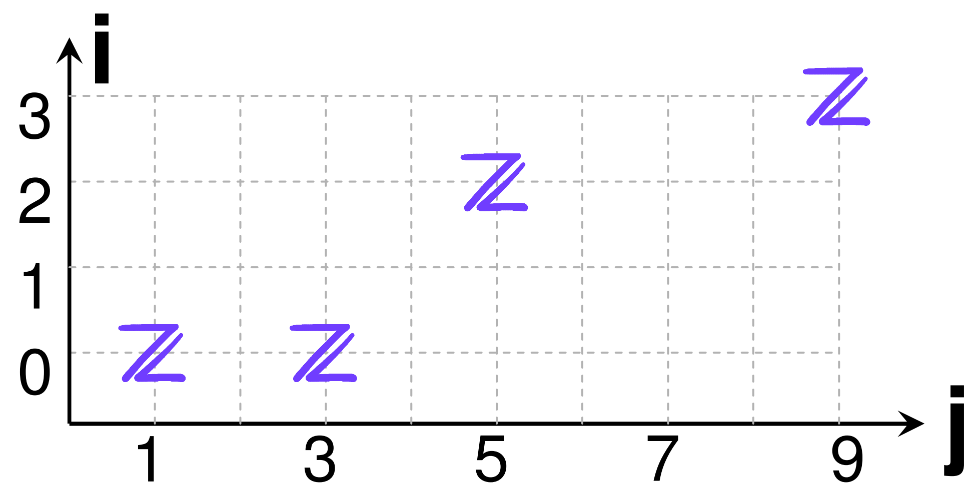
- Khovanov homology: $H^{Kh}_{i,j}(K)$

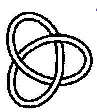
[M.Khovanov]

$$J(q) = \sum_{i,j} (-1)^i \, q^j \, \dim H^{Kh}_{i,j}(K)$$

<u>Example:</u>

$H^{Kh}_{i,j}($  $)$




$J($  $) = q + q^3 + q^5 - q^9$

# (Large) Color Behavior of Knot Homologies

- similarly, $\mathcal{H}^{sl(2),V_n}(K)$ is the **n**-colored sl(2) knot homology:

$$P_n(K;q,t) \; = \; \sum_{i,j} q^i t^j \dim \mathcal{H}^{sl(2),V_n}_{i,j}(K)$$

- categorify **n**-colored Jones polynomials:

$$J_n(K;q) \; = \; P_n(K;q,t=-1)$$

# (Large) Color Behavior of Knot Homologies

- similarly, $\mathcal{H}^{sl(2),V_n}(K)$ is the n-colored sl(2) knot homology:

$$P_n(K;q,t) \; = \; \sum_{i,j} q^i t^j \dim \mathcal{H}^{sl(2),V_n}_{i,j}(K)$$

- satisfy recursion relations
- exhibit beautiful large-n asymptotic behavior

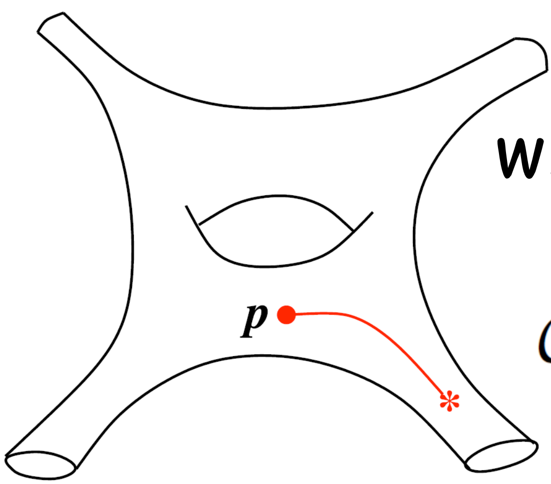
Both controlled by a "refined" algebraic curve

# Refinement / Categorification

<u>Generalized Volume Conjecture:</u> [S.G., H.Fuji, P.Sulkowski]

$$q = e^{\hbar} \to 1 \,, \qquad t = \text{fixed} \,, \qquad x \equiv e^{u} = q^{n} = \text{fixed}$$

$$\boxed{P_n \simeq \exp\left( \frac{1}{\hbar} S_0(u,t) + \sum_{n=0}^{\infty} S_{n+1}(u,t)\,\hbar^n \right)}$$

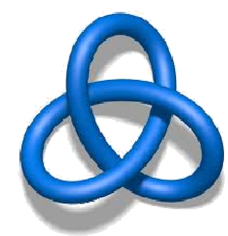



where $S_0(u,t) = \int v du = \int \log y \frac{dx}{x}$

$$\mathcal{C}^{\text{ref}} : \quad \left\{ (x,y) \in \mathbb{C}^* \times \mathbb{C}^* \Big| A^{\text{ref}}(x,y,t) = 0 \right\}$$

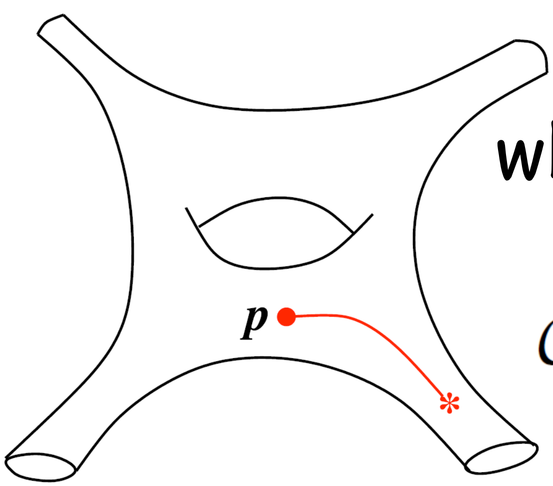
# Refined Algebraic Curves

<u>Example:</u> $$A(x,y) \;=\; (y-1)(y+x^3)$$



refinement:

$$A^{\text{ref}}(x,y;t) \;=\; y^2 - \frac{1-xt^2+x^3t^5+x^4t^6+2x^2t^2(t+1)}{1+xt^3}\,y + \frac{(x-1)x^3t^4}{1+xt^3}$$




where $$S_0(u,t) = \int v\,du \;=\; \int \log y \frac{dx}{x}$$

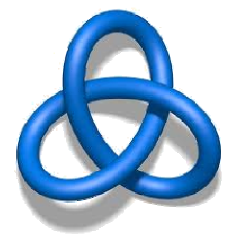
$$\mathcal{C}^{\text{ref}}: \quad \left\{ (x,y) \in \mathbb{C}^* \times \mathbb{C}^* \,\middle|\, A^{\text{ref}}(x,y;t) = 0 \right\}$$

# Refinement / Categorification

<u>Quantum Volume Conjecture:</u> [S.G., H.Fuji, P.Sulkowski]

$$\boxed{\widehat{A}^{\mathrm{ref}}(\widehat{x},\widehat{y};q,t)\; P_*(K;q,t) \;\simeq\; 0}$$

<u>Example:</u> $\alpha P_{n-1} + \beta P_n + \gamma P_{n+1} \;=\; 0$



$$\alpha = \frac{x^3(x-q)t^4}{q(q+x^2t^3)(1+xqt^3)}\,,$$

$$\beta = -\frac{t^2x^2}{1+x^2qt^3} - \frac{q-xqt^2+x^4t^6+x^2t^2(1+t+qt)}{(q+x^2t^3)(1+xqt^3)}$$

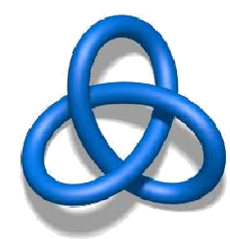
$$\gamma = \frac{1}{q+x^2q^2t^3}\,.$$

# Refinement / Categorification

Quantum Volume Conjecture: [S.G., H.Fuji, P.Sulkowski]

$$\boxed{\widehat{A}^{\mathrm{ref}}(\widehat{x},\widehat{y};q,t)\; P_*(K;q,t) \;\simeq\; 0}$$

Example: $\widehat{A}^{\mathrm{ref}}(\widehat{x},\widehat{y};q,t) \;=\; \alpha\widehat{y}^{-1}+\beta+\gamma\widehat{y}$



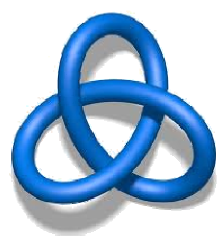
$$\alpha = \frac{x^3(x-q)t^4}{q(q+x^2t^3)(1+xqt^3)}\,,$$

$$\beta = -\frac{t^2x^2}{1+x^2qt^3} - \frac{q-xqt^2+x^4t^6+x^2t^2(1+t+qt)}{(q+x^2t^3)(1+xqt^3)}$$

$$\gamma = \frac{1}{q+x^2q^2t^3}\,.$$

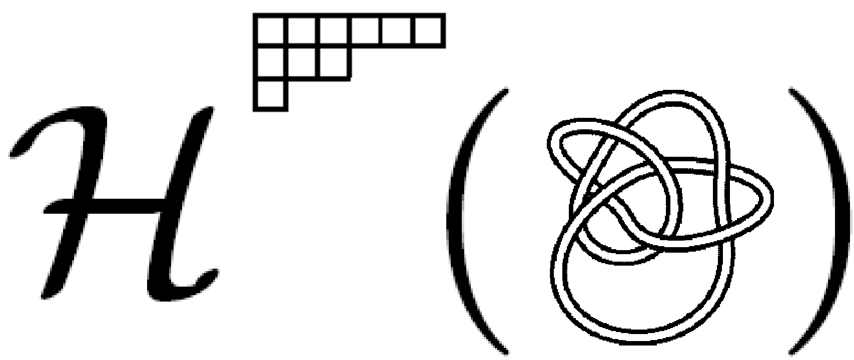
# Deformation vs Quantization

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\hat{A}(x,y;q) \arrow[r, shift left, "\text{deformation (``categorification'')}"] \arrow[d, dashed, shift left, "q=1"] & \hat{A}^{\mathrm{ref}}(x,y;q,t) \arrow[l, dashed, shift left, "t=-1"] \arrow[d, dashed, shift left, "q=1"] \\
A(x,y) \arrow[r, shift left, "\text{deformation (``categorification'')}"] \arrow[u, shift left, "\text{quantization}"] & A^{\mathrm{ref}}(x,y;t) \arrow[l, dashed, shift left, "t=-1"] \arrow[u, shift left, "\text{quantization}"]
\end{tikzcd}
$$

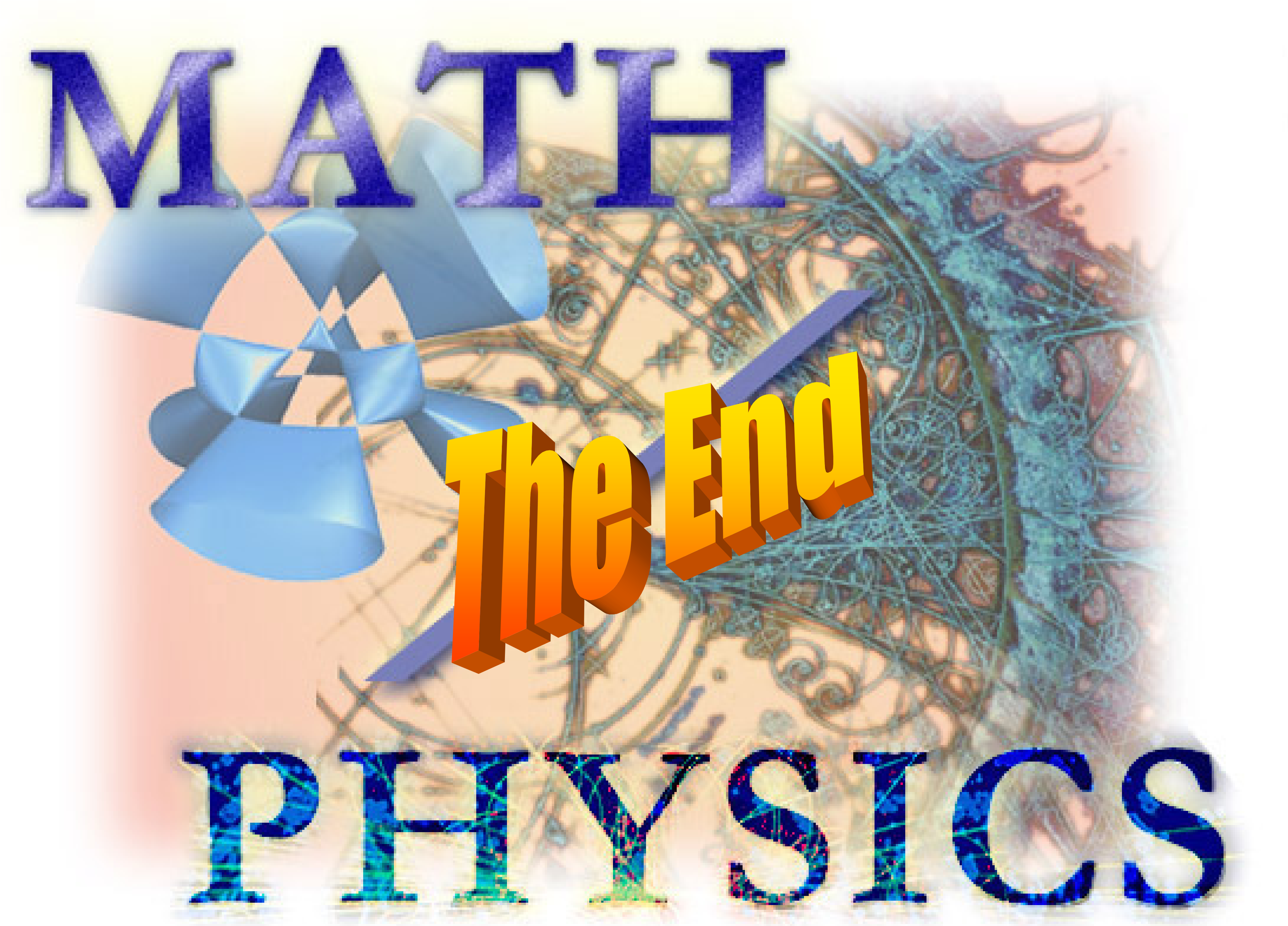


# Powerful Calculations

- (3,4) torus knot
- Colored HOMFLY homology



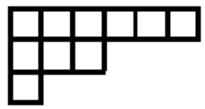
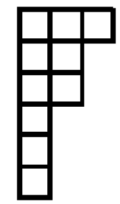
has dimension 8,170,288,749,315

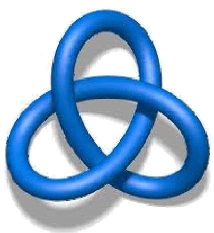

MATH
The End
PHYSICS

# A-polynomial

| Knot | A-polynomial | Volume |
|---|---|---|
| $3_1$ | $lm^6 + 1$ | non-hyperbolic |
| $4_1$ | $-2 + m^4 + m^{-4} - m^2 - m^{-2} - l - l^{-1}$ | $2.0298832\ldots$ |
| $5_1$ | $lm^{10} + 1$ | non-hyperbolic |
| $5_2$ | $1 + l(-1 + 2m^2 + 2m^4 - m^8 + m^{10}) + l^2(m^4 - m^6 +$ $+2m^{10} + 2m^{12} - m^{14}) + l^3m^{14}$ | $2.8281220\ldots$ |
| $7_1$ | $m^{14} + l$ | non-hyperbolic |
| $(-2, 3, 7)$-pretzel | $-m^{110} + lm^{90}(m^2 - 1)^2 + l^2(2m^{74} + m^{72})-$ $-l^4(m^{38} + 2m^{36}) - l^5m^{16}(m^2 - 1)^2 + l^6$ | $2.8281221\ldots$ |

# Mirror Symmetry for Knots

$\lambda =$  $\longleftrightarrow$ $\lambda^t =$ 



$$\mathcal{H}^{\lambda}(K) \cong \mathcal{H}^{\lambda^t}(K)$$