Notes on F-Theory Compactifications

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§ Motivation

To understand the flavor structure of the Standard Model is a big challenge.

Besides the phenomenological approach, we would like to invoke the top-down approach — string theory.

In an $\mathcal{N}=1$ supersymmetric compactification, the holomorphy facilitates the computation of the superpotential including "Yukawa couplings."

However, "physical" Yukawa couplings can be obtained, only after the kinetic terms take the canonical form. Thus, one needs to know the kinetic terms, but they are included in the D-term, which is not protected by the supersymmetry.

In this talk, I will report the results of our first attempt to compute the kinetic terms of charged matters in F-theory compactifications.

In an F-theory compactification, the charged matters of a grand unified model are located at "intersections" of 7-branes. Although their kinetic terms are apparently given by the integral over the whole 7-branes supporting the GUT gauge field, We will see below that they actually localize just on the "intersections".

In our paper, although we can see the localization in more realistic compactifications, I will explain it with simple examples in this talk.

§ F-theory compactifications

An F-theory compactification to 4 dimensions

$\parallel$

type IIB theory compactified

on a 6-dim. Kähler mfd B

w/ the R-R 0-form field C_0 & the dilaton ϕ

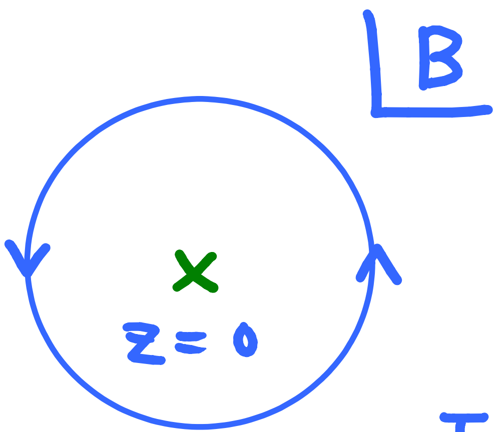
varying over B.

The S-duality in type IIB theory transforms

$$\tau = C_0 + i e^{-\phi} \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}$$

w/ $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(\mathbb{Z}, \mathbb{Z})$

On the Kähler mfd $\mathcal{B}$, τ may have a monodromy;



$$z \longrightarrow z e^{2\pi i},$$

$$\tau \rightarrow \tau' = \frac{a\tau + b}{c\tau + d}.$$

It suggests that a composite of 7-branes is located at $z = 0$.

Thus, the 7-branes are of codimension 1 in 6-dim. mfd B, and namely, they are wrapped on a 4-dim. Kähler submfd $S \subset B$ and extend in the 4-dim. Minkowski space $\mathbb{R}^{3,1}$.

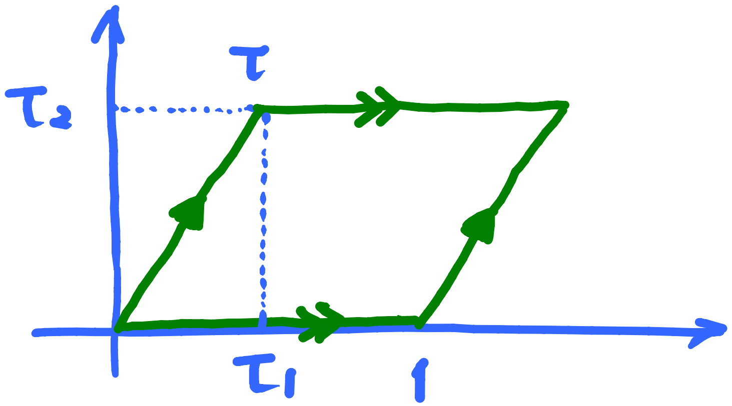
$$\underbrace{0\ 1\ 2\ 3}_{\underset{(3+1)\,\text{dim.}}{\underbrace{\mathbb{R}^{3,1}}}}\quad \underbrace{4\ 5\ 6\ 7\ 8\ 9}_{\substack{B \\ \cup \\ \underbrace{S}_{4\,\text{dim}}}}\ ;\quad z = 0$$

$$(7+1)\ \text{dim} \longrightarrow 7\text{-branes}$$

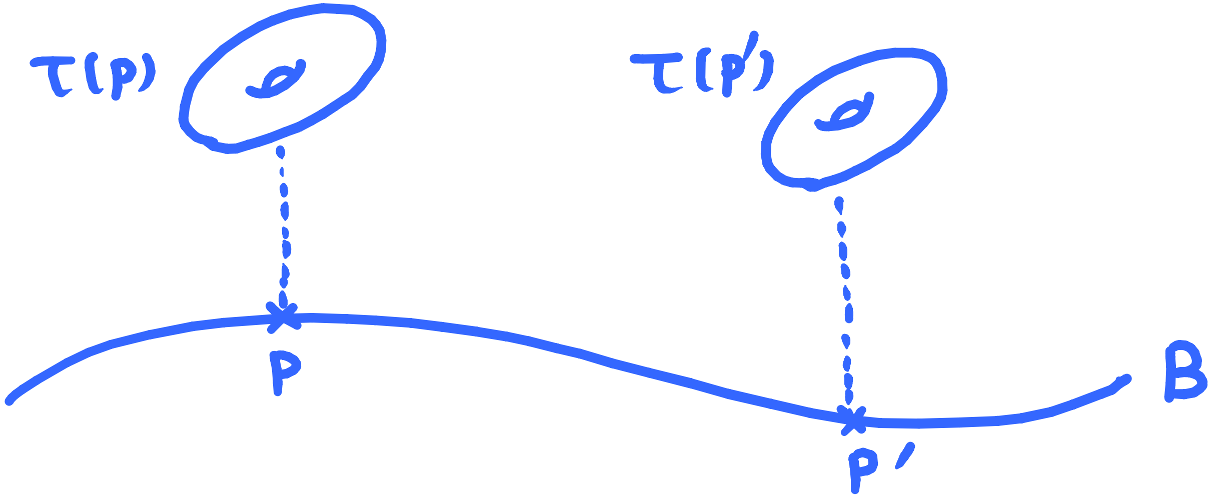
The S-duality $\longleftrightarrow$ the modular transformation of a torus

identify

$\Downarrow$



One can think about an elliptic fibration X over B



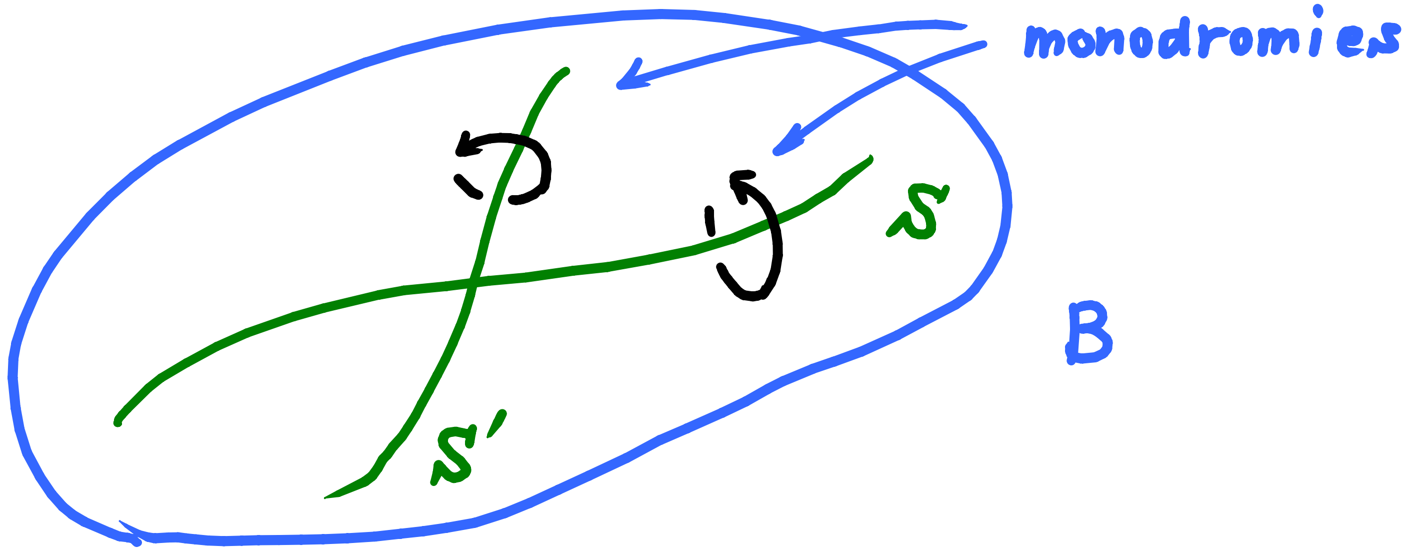
X (8-dim.)

$N=1$ SUSY in 4 dim. $\Rightarrow$ X should be a Calabi-Yau 4-fold.

An F-theory GUT model

$\parallel$

The worldvolume theory on 7-branes wrapped on divisors (4-dim. submfds such as S) in the base B.



§ 7-brane worldvolume theory

- the gauge group G
- the field contents

  8-dim. gauge field $A_{\mu=0,\cdots,3}$, $A_{m=1,2}$, $\bar{A}_{\bar{m}=1,2}$

  ($z^{m=1,2}$; complex local coordinates of S)

  a (2,0)-form field in the adj. rep.

  $\varphi_{mn} = -\varphi_{nm}$ (+ $\bar{\varphi}_{\bar{m}\bar{n}}$)

  + their fermionic superpartners

- The covariant derivatives

$$D_\mu \bar{\Phi} = \partial_\mu \bar{\Phi} + [A_\mu, \bar{\Phi}],$$

$$\bar{\partial}_A \bar{\Phi} = \bar{\partial} \bar{\Phi} + [\bar{A}, \bar{\Phi}],$$

where

$$\bar{\Phi} = \frac{1}{2} \Phi_{mn} dz^m \wedge dz^n = \Phi_{12} dz^1 \wedge dz^2,$$

$$\bar{A} = A_{\bar{m}} d\bar{z}^{\bar{m}}, \quad \bar{\partial} = d\bar{z}^{\bar{m}} \partial_{\bar{m}}.$$

- The field strength

$$F^{(0,1)}_{A\,\mu} = \partial_\mu \bar{A} - \bar{\partial} A_\mu + [A_\mu, \bar{A}],$$

$$F^{(0,2)}_A = \bar{\partial} \bar{A} + \bar{A} \wedge \bar{A},$$

$$F^{(1,1)}_A = \partial \bar{A} + \bar{\partial} A + A \wedge \bar{A} + \bar{A} \wedge A.$$

- The bosonic action

$$
\begin{aligned}
I = -\int d^4x \int_S \mathrm{tr}\Big[& \underbrace{\tfrac{1}{2}\,\omega\wedge\omega}_{\mathrm{dvol}(S)} \left(-\tfrac{1}{4}F_{\mu\nu}^2 + \tfrac{1}{2}D^2\right) \\
& + D_\mu\Phi^\dagger D^\mu\Phi - i\,\omega\wedge F^{(1,0)}_{A\mu}\,F^{(0,1)\mu}_{A} && \leftarrow \text{Kinetic terms of } A,\ \Phi \\
& - D\left([\Phi,\Phi^\dagger] + i\,\omega\wedge F^{(1,1)}_A\right) && \leftarrow \text{D-term potential} \\
& - z\,F^{(2,0)}_A\wedge F^{(0,2)}_A + z\,i\,\omega\wedge\bar{G}\wedge G && \leftarrow \text{F-term} \\
& + z\,\bar{G}\wedge\partial_A\Phi^\dagger + z\,\bar{\partial}_A\Phi\wedge G \Big],
\end{aligned}
$$

where

$\omega = i\,g_{m\bar{n}}\,dz^m\wedge d\bar{z}^{\bar{n}}$; the Kähler form of S,

$G = G_{\bar{m}}\,d\bar{z}^{\bar{m}}$, D ; auxiliary fields in the adj. rep. of G

- The superpotential

$$W = \int_S \mathrm{tr}\left[F_A^{(0,2)} \wedge \Phi \right].$$

- The BPS condition

  - F-term condition

$$F_A^{(0,2)} = 0, \qquad \bar{\partial}_A \Phi = 0.$$

  - D-term condition

$$\omega \wedge F_A^{(1,1)} = i\,[\Phi, \Phi^\dagger].$$

§ BPS backgrounds (= vacuum solutions)

Let us take the gauge group $G = SU(N)$ for simplicity.

The BPS conditions $F_A^{(0,2)} = 0$, $\bar{\partial}_A \Phi = 0$ can be solved locally in terms of an $N \times N$ matrix V and a holomorphic 2-form ϕ by

$$\bar{A} = V \bar{\partial} V^{-1}, \qquad \Phi = V \phi V^{-1}.$$

Under a complex gauge transformation,

$$V \to V \cdot \Lambda, \quad \phi \to \Lambda^{-1} \phi \Lambda,$$

with Λ a holomorphic matrix in $GL(N, \mathbb{C})$.

The remaining BPS condition is the D-term condition $\omega \wedge F_A^{(1,1)} = i[\Phi, \Phi^\dagger]$, and it may be rewritten as

$$\omega \wedge \bar{\partial}(H^{-1}\partial H) = i[\phi, H^{-1}\phi^\dagger H].$$

with the Hermitian metric $H = V^\dagger \cdot V$.

A solution (H, ϕ) to this equation yields a BPS background, around which fluctuations give rise to the 4-dim. spectrum of the theory.

* examples of BPS backgrounds

1. the intersecting brane background

For $G = SU(2)$, for simplicity,

$$H = 1, \quad \Phi = \frac{1}{2}\begin{pmatrix} x & \\ & -x \end{pmatrix} \quad \text{w/ } \begin{array}{l} x = z_1, \\ y = z_2 \end{array}$$

since $[\Phi, \Phi^\dagger] = 0$, it solves the D-term condition.

② the $\mathbb{Z}_2$ monodromy background

$$H = \begin{pmatrix} e^{\Sigma(r)} & & \\ & e^{-\Sigma(r)} & \\ & & 1 \end{pmatrix}, \quad \phi = \begin{pmatrix} x & 1 & \\ & & 0 \\ & & \end{pmatrix},$$

w/ $r \equiv \frac{4}{3\ell} |x|^{\frac{3}{2}}$ & $\omega = \ell^2 (dz^1 \wedge d\bar{z}^{\bar{1}} + dz^2 \wedge d\bar{z}^{\bar{2}})$.

In terms of them, the D-term condition gives the sinh-Gordon equation

$$\left(\frac{d}{dr^2} + \frac{1}{r}\frac{d}{dr}\right)\Sigma(r) = \frac{1}{2}\sinh(2\Sigma(r)),$$

which has been analytically solved. (McCoy, Tracy, Wu, ('77) Zamolodchikov ('94))

§ Localized Matters

Around a BPS background (V, ϕ), let us expand the fields $(\bar{A}, \Phi)$ with fluctuations $(\bar{\alpha}, \varphi)$;

$$\bar{A} = \langle \bar{A} \rangle + \bar{a} = V \bar{\partial} V^{-1} + V \bar{\alpha} V^{-1},$$

$$\Phi = \langle \Phi \rangle + V \varphi V^{-1} = V \phi V^{-1} + V \varphi V^{-1}.$$

The fluctuations $(\bar{\alpha}, \varphi)$ satifying the BPS condition (the zeromodes) gives rise to 4-dim. massless particles.

In fact, the zero modes are locally given by

$$\bar{\alpha} = \bar{\partial} v , \qquad \varphi = h + [\Phi , v]$$

in terms of an $N \times N$ matrix v and a holomorphic $N \times N$ matrix 2-forms h.

Under an infinitesimal complex gauge transformation,

$$v \to v + \epsilon , \qquad h \to h - [\Phi , \epsilon]$$

with an $N \times N$ holomorphic matrix ϵ.

- examples of the zero-mode solutions

① the intersecting brane background

$$\phi = \frac{1}{2}\begin{pmatrix} x & \\ & -x \end{pmatrix}, \quad \text{w/ } (z_1, z_2) = (x, y).$$

One can find the zero modes (v, h)

$$h = \begin{pmatrix} h_0(x,y) & h_+(y) \\ h_-(y) & -h_0(x,y) \end{pmatrix} \quad \text{in a gauge,}$$

where the bi-fundamentals $h_\pm(y)$ are localized on the curve Σ ; $x = 0$, and may be written as

$$\begin{pmatrix} & h_+ \\ h_- & \end{pmatrix} = \left[\phi, \frac{1}{x}\eta\right], \text{ w/ } \eta = \begin{pmatrix} & +h_+ \\ -h_- & \end{pmatrix}.$$

② the $\mathbb{Z}_2$ monodromy background

$$H = \begin{pmatrix} e^{\Sigma} & & \\ & e^{-\Sigma} & \\ & & 1 \end{pmatrix}, \quad \Phi = \begin{pmatrix} & 1 & \\ x & & \\ & & 0 \end{pmatrix},$$

where we took $G = SU(3)$, instead of $SU(2)$.

The localized zero modes are given by

$$h = \begin{pmatrix} 0 & & h_+(y) \\ & 0 & \\ h_-(y) & & 0 \end{pmatrix} = \left[\Phi, \frac{1}{x} \eta \right],$$

with

$$\eta = \begin{pmatrix} 0 & & +h_+(y) \\ & 0 & \\ 0 & -h_-(y) & 0 \end{pmatrix}.$$

More gererally, localized modes on a curve $f = 0$ can be given in the form

$$h = \left[\phi, \frac{1}{f} \eta \right]$$

(Cecotti, Cordova, Heckman, Vafa ('11))

The charged matters of a grand unified model in an F-theory compactification are given by the localized zero modes.

Therefore, let us focus on them and discuss their kinetic terms.

The remaining massless (D-term) condition

$$\omega \wedge \partial (H \cdot \bar{\partial} v \cdot H^{-1}) + i [\phi^{+}, H \varphi H^{-1}] = 0$$

determines v in terms of h.

In the example 1, in fact, one finds that

$$v_{\pm}(x,y) = h_{\pm}(y) \frac{1}{x} \left(e^{-\frac{|x|^2}{\ell}} - 1 \right)$$

with the Kähler form

$$\omega = \frac{i}{2} \ell^2 (dx \wedge d\bar{x} + dy \wedge d\bar{y}) .$$

In the example 2, one alse observes a similar localization with decays at infinity $|x| \sim \infty$.

§ The Kähler potential of localized modes

Substituting the localized modes

$$\bar{A} = \underbrace{V \bar{\partial} V^{-1}}_{\text{a BPS sol.}} + \sum_i \underbrace{\chi_i(x^\mu)}_{\text{4-dim. fields}} \cdot V \cdot \underbrace{\bar{\partial} v_i}_{\text{the localized solution}} \cdot V^{-1},$$

$$\Phi = \underbrace{V \phi V^{-1}}_{} + \sum_i \underbrace{\chi_i(x^\mu)}_{} \cdot V \cdot \underbrace{\varphi_i}_{h_i + [\phi, v_i]} \cdot V^{-1}$$

into the kinetic term

$$\int d^4x^\mu \int_S \mathrm{tr}\left[- D_\mu \Phi^\dagger D^\mu \Phi + i\omega \wedge F_{A\,\mu}^{(1,0)} \wedge F_A^{(0,1)\mu} \right],$$

one obtains

$$- \sum_{i,j} \int d^4x^\mu \, N_{ij} \, \partial_\mu \chi_i^\dagger \partial^\mu \chi_j \, ,$$

where the normalization N_{ij} is given by

$$N_{ij} = \int_S \mathrm{tr}\left[h_i^\dagger \wedge H \left(h_j + [\Phi, v_j] \right) H^{-1} \right].$$

For the localized mode

$$h_i = \left[\Phi, \frac{1}{f_i} \eta_i \right] \quad \text{on } f_i = 0,$$

it yields

$$N_{ij} = \int_S \mathrm{tr}\left[\eta_i^\dagger \, \underbrace{\partial\left(\frac{1}{f_i}\right)}_{= -i\pi\delta^2(f_i)\, df_i} \wedge \left(-i\omega \wedge H \bar{\partial} v_j \cdot H^{-1} \right) \right]$$

Thus, the normalization $\mathcal{N}_{ij}$ is given by the integration over the matter curve $\Sigma_i : f_i = 0$, but not over the whole surface S.

For the example ①, ②, one can more explicitly obtain the kinetic term

$$- \int d^4x^\mu \left(\mathcal{N}_{++}\, \partial_\mu \chi_+^\dagger\, \partial^\mu \chi_+ + \mathcal{N}_{--}\, \partial_\mu \chi_-^\dagger\, \partial^\mu \chi_- \right)$$

where

$$\mathcal{N}_{\pm\pm} = \pi \sqrt{\frac{1}{2}} \int \ell \,(h_+^\dagger h_+)\, dy \wedge d\bar{y} \, .$$

§ Discussions

Since a localized mode is localized on a curve $\Sigma : f = 0$, its effective action may be obtained by the K.-K. reduction of 6-dim. $\mathcal{N}=1$ hypermults on the curve Σ with the metric $ds^2 = 2 g_{z\bar{z}}\, dz\, d\bar{z}$. to 4 dimensions.

However, the reduction on the generic Kähler mfd Σ break supersymmetry completely. Therefore, one needs to "twist" the bosonic fields $(\tilde{\zeta}, \zeta)$ of the hypermultiplets. (Beasley, Heckman & Vafa, '08)

After "twisting", the two complex one-time scalars $(\tilde{\zeta}, \zeta)$ take values in $(K_\Sigma)^{1/2}$, where K_Σ is the canonical bundle on Σ.

Their Lagrangian is given by

$$- \int dz\, d\bar{z} \sqrt{g_{z\bar{z}}} \Big[(D_\mu \tilde{\zeta})^\dagger (D^\mu \tilde{\zeta}) + (D_\mu \zeta)^\dagger (D^\mu \zeta) + 2 g^{z\bar{z}} \Big((D_z \tilde{\zeta})^\dagger (D_{\bar{z}} \tilde{\zeta}) + (D_z \zeta)^\dagger (D_{\bar{z}} \zeta) \Big) \Big].$$

For 4-dim. massless particles given by

$$D_{\bar{z}}\,\zeta = 0\,, \qquad D_{\bar{z}}\,\tilde{\zeta} = 0$$

one can expand $(\zeta, \tilde{\zeta})$ in terms of holomorphic sections $h_i(z)$ of $(K_\Sigma)^{1/2} \otimes$ the vector bundle as

$$\begin{cases} \zeta = \sum_i \chi_i(x^\mu)\, h_i(z)\,, \\ \tilde{\zeta} = \sum_i \tilde{\chi}_i(x^\mu)\, h_i(z)\,, \end{cases}$$

and substitute them into the Lagrangian to obtain

$$-\int d^4x^\mu \sum_{i,j} N_{ij} \left(\partial_\mu \chi_i^\dagger\, \partial^\mu \chi_j + \partial_\mu \tilde{\chi}_i^\dagger\, \partial^\mu \tilde{\chi}_j \right),$$

where

$$N_{ij} = \int_{\Sigma} dz\, d\bar{z}\, \sqrt{g_{z\bar{z}}}\; h_i^{\dagger} \cdot h_j \, .$$

The normalization N_{ij} is given by the integral over the curve, and is consistent with our previous results derived from the 7-brane worldvolume theory.

Summary

$$I = -\int d^4x \int_S \mathrm{tr}\left[D_\mu \Phi^\dagger D^\mu \Phi - i\,\omega \wedge F^{(1,0)}_{A\mu} F^{(0,1)\mu}_{A} \right]$$

$$\to -\int d^4x \; N_{ij}\, D_\mu \chi_i^\dagger D^\mu \chi_j \quad \text{for charged matters,}$$

where

$$N_{ij} = -\pi \int_{f_i = 0} \omega \, \mathrm{tr}\left[\eta_i^\dagger \cdot H \cdot \frac{\partial}{\partial \bar{f}_i} v_j \cdot H^{-1} \right] .$$

the integral over the curve ; $f_i = 0$.