Instanton calculus for quiver gauge theories

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Outline

- 4d N=2 quiver theories & classification
- Instanton partition function [LMNS, Nekrasov]
- Limit shape problem [Nekrasov-Okounkov]
- Cut-crossing and quiver Weyl group
- The spectral curve
- Integrable systems: G-bundles, Nahm transform and Hitchin systems

4d N=2 quiver theories & classification

Assumptions

- gauge group $G_{\rm g} = \times_{i\in {\rm Vert}} U(N_i)$
- fundamental & bifundamental hypers

$$c_{ii} = 2$$

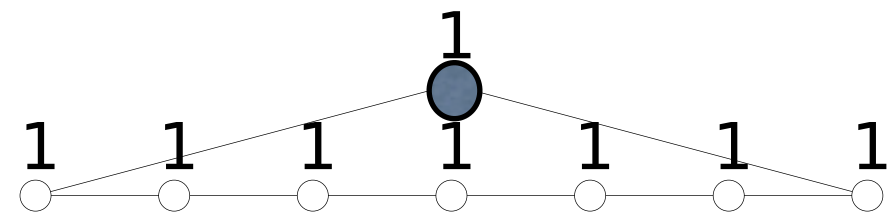
$$c_{ij} = c_{ij} = -\#_{\text{bifund hypers of } i \text{ and } j}$$

$$N_i^{\rm f} = \#_{\text{fund hypers of } i}$$

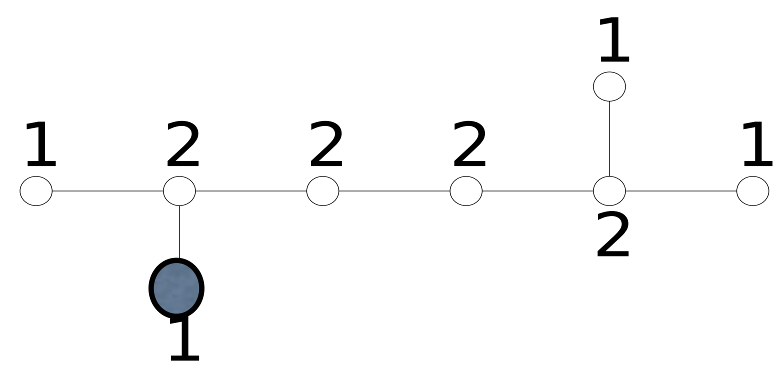
$$\beta_i \leq 0 \quad \longrightarrow \quad c_{ij} N_j \geq N_i^{\rm f} \geq 0$$

c_{ij} is Cartan matrix of Fin or Aff ADE Dynkin graph

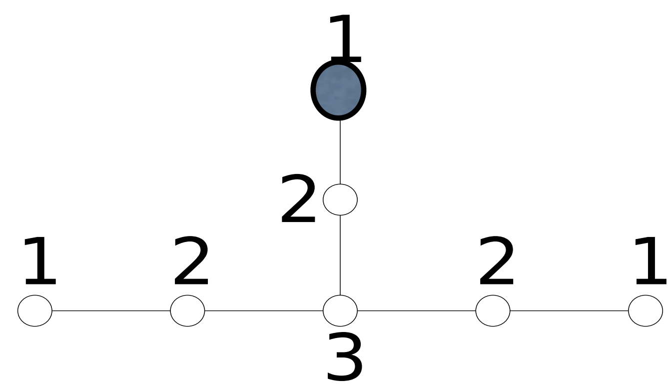
Aff A_r



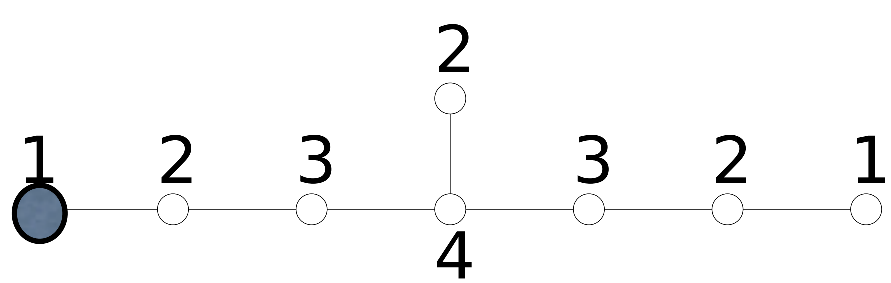
Aff D_r



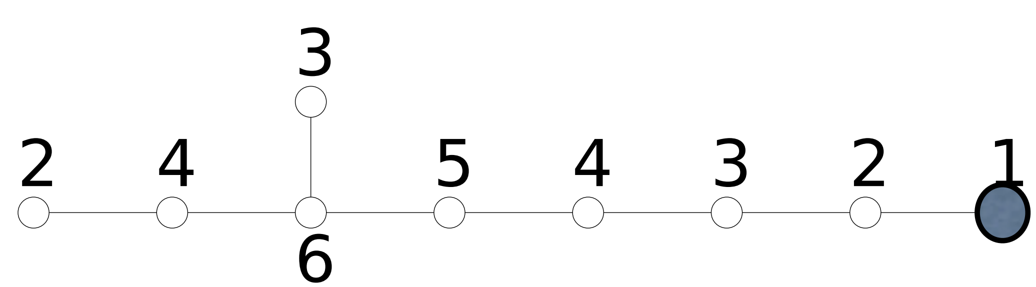
Aff E_6



Aff E_7



Aff E_8



Instanton partition function [LMNS'97,98; Nekrasov'02]

$$Z_{\mathbb{T}}^{\text{inst}}(\mathbf{q}) = \sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^{|\text{Vert}|}} \mathbf{q}^{\mathbf{k}} \int_{\mathcal{M}_{\mathbf{k}}} \text{eu}_{\mathbb{T}}(\ker \not{D}_R)$$

instanton charges $\mathbf{k} = \{k_i | i \in \text{Vert}\}$

coupling constants $\mathbf{q} = \{e^{2\pi\imath\tau_i} | i \in \text{Vert}\}$

framed instanton moduli $\mathcal{M}_{\mathbf{k}} = \times_{i \in \text{Vert}} \mathcal{M}_{k_i}$

matter bundle $R = \oplus_i \mathbf{N}_i^{\oplus N_i^{\text{f}}} \oplus_{i<j} (\mathbf{N}_i, \overline{\mathbf{N}}_j)^{\oplus c_{ij}}$

equivariant Euler class $\text{eu}_{\mathbb{T}}$

torus $\mathbb{T} = T_G \times T_F \times T_L$

$(\mathbf{a}, \mathbf{m}, \boldsymbol{\epsilon})$ equivariant parameters in $\text{Lie}(\mathbb{T})$

Complete partition function

$$Z_{\mathbb{T}}(\mathbf{q}) = Z_{\mathbb{T}}^{\mathrm{tree}} Z_{\mathbb{T}}^{1-\mathrm{loop}} Z_{\mathbb{T}}^{\mathrm{inst}}(\mathbf{q})$$

$$Z_{\mathbb{T}}(\mathbf{q}) = \sum_{\mathbf{k}} \mathbf{q}^k \int_{\mathcal{A}/\mathcal{G}_{\mathrm{gauge}}} \mathrm{eu}_{\mathbb{T}}(\Omega^{2+} \otimes \mathrm{ad}\, G_{\mathrm{g}}) \mathrm{eu}_{\mathbb{T}}((S^- \ominus S^+) \otimes R)$$

$\mathcal{A}$ = {G_{g} connections framed at infinity on R^4}

Seiberg-Witten limit

$$Z_{\mathbb{T}}(\mathbf{a},\mathbf{m},\boldsymbol{\epsilon};\mathbf{q}) = \exp(-\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}_{\mathbb{T}}(\mathbf{a},\mathbf{m},\boldsymbol{\epsilon};\mathbf{q}))$$

$$\lim_{\epsilon_1,\epsilon_2\to 0}\mathcal{F}_{\mathbb{T}}(\mathbf{a},\mathbf{m},\boldsymbol{\epsilon};\mathbf{q}) = \mathcal{F}_{SW}(\mathbf{a},\mathbf{m};\mathbf{q})$$ [Nekrasov '02]

Proofs

- [Nekrasov-Okounkov'03] for $U(N)$ with adj/fund matter
  (limit shape of partition profiles)
- [Nakajima-Yoshioka'03] for $U(N)$ without matter
  (blow-up recursion formula)
- [Braverman-Etingof'04] for generic G without matter
  (surface operators for parabolic subgroups P of G)

Limit shape method [Nekrasov-Okounkov'03]

- relies on ADHM
- originally described the shape of the dominant partitions labeling $\mathbb{T}$ -fixed points on $\mathcal{M}_{\mathrm{inst}}$
- applicable to Lagrangian theories with classical gauge groups and tensor matter
- examples:
  - [Nekrasov-Shadchin'04] $SO(N)$ and $Sp(N)$ with adj/fund
  - [Shadchin'05] $U(N_1)$x$U(N_2)$ bifund and $U(N_1)$ sym/antisym

Review of limit shape method

$G = SU(N)$

$$Z_k = \int_{\mathcal{M}_{N,k}} \mathrm{eu}_{\mathbb{T}}(\ker \not{D}_R^{\vee})$$

Nakajima's lectures
Losev-Marshakov-Nekrasov'03
Nekrasov-Okounkov'03
Shadchin'05

ADHM complex

$$\mathcal{C}_{\mathrm{ADHM}}: \quad V \otimes K^{\frac{1}{2}} \xrightarrow{(B_2, B_1, J)} V \otimes L \otimes K^{\frac{1}{2}} \oplus W \xrightarrow{(-B_1, B_2, I)} V \otimes \Lambda^2 L \otimes K^{\frac{1}{2}}$$

$$[B_1 B_2] + IJ = 0$$

where

$$V = \mathbb{C}^k, W = \mathbb{C}^N, L \simeq \mathbb{C}^2, K = \Lambda^2 L^{\vee}$$

then $$\mathcal{E}_z = H^1(\mathcal{C}_{\mathrm{ADHM}, B-z})$$

$$\begin{aligned}\mathrm{ch}_{\mathbb{T}} \mathcal{E}_{z=0} \equiv \mathcal{E}_{\mathbb{T}} &= W_{\mathbb{T}} - K_{\mathbb{T}}^{\frac{1}{2}} (1 - L_{\mathbb{T}} + \Lambda^2 L_{\mathbb{T}}) V_{\mathbb{T}} \\ &= W_{\mathbb{T}} - e^{-\imath\epsilon_+}(1 - e^{\imath\epsilon_1})(1 - e^{\imath\epsilon_2}) V_{\mathbb{T}} = \\ &= W_{\mathbb{T}} - 2\mathrm{sh}\frac{\imath\epsilon_1}{2} 2\mathrm{sh}\frac{\imath\epsilon_2}{2} V_{\mathbb{T}}\end{aligned}$$

$$W_{\mathbb{T}} = \sum_{\alpha=1}^{N} e^{\imath a_\alpha}$$

$$V_{\mathbb{T}} = \sum_{i=1}^{k} e^{\imath\phi_i}$$

$$L_{\mathbb{T}} = e^{\imath\epsilon_1} + e^{\imath\epsilon_2}$$

$$\epsilon_+ = \frac{1}{2}(\epsilon_1 + \epsilon_2)$$

$$\mathrm{ch}_{\mathbb{T}} \ker \not{D}_R = \int_L \hat{A}_{\mathbb{T}}(L)\mathrm{ch}_{\mathbb{T}}(\mathcal{E}\otimes R) = \frac{\mathrm{ch}_{\mathbb{T}}(\mathcal{E}_{z=0}\otimes R)}{2\mathrm{sh}\frac{\imath\epsilon_1}{2}\,2\mathrm{sh}\frac{\imath\epsilon_2}{2}}$$

$$\mathrm{ch}_{\mathbb{T}} \ker \not{D}_{\mathcal{E}_i\otimes\mathcal{E}_j^{\vee}} = \frac{(\mathcal{E}_i)_{\mathbb{T}}(\mathcal{E}_j^{\vee})_{\mathbb{T}}}{2\mathrm{sh}\frac{\imath\epsilon_1}{2}\,2\mathrm{sh}\frac{\imath\epsilon_2}{2}}$$

then transform

$$\mathrm{ch}_{\mathbb{T}}(M) \rightsquigarrow \mathrm{eu}_{\mathbb{T}}(M)$$

using

$$-\ \frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^{\infty} dt\, t^{s-1}e^{\imath w t}\Big|_{s=0} = \log(\imath w)$$

obtain

$$\exp\left(-\frac{d}{ds}\frac{1}{\Gamma(s)}\int_0^{\infty} dt\, t^{s-1}\mathrm{ch}_{\mathbb{T}^t}M\right)_{s=0} = \mathrm{eu}_{\mathbb{T}}M$$

where M is any $\mathbb{T}$-module

Let $\rho_{\mathbb{T};M}(x)$ be Fourier transform of $\mathrm{ch}_{\mathbb{T}^t} M$

$$\mathrm{ch}_{\mathbb{T}^t} M = \sum e^{\imath w t}$$

$$\mathrm{ch}_{\mathbb{T}^t} M = \int dx\, \rho_{\mathbb{T};M}(x) e^{\imath x t} \qquad \rho_{M;\mathbb{T}}(x) = \sum \delta(x-w)$$

$$\rho_i(x) \equiv \rho_{\mathbb{T};\mathcal{E}_i}(x)$$

$$\mathrm{eu}_{\mathbb{T}} \ker \not{D}_{\mathcal{E}_i \otimes \mathcal{E}_j^\vee} = \exp\left(-\frac{d}{ds} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \frac{\int dx \rho_i(x) e^{\imath x t} \int dx \rho_j(x') e^{-\imath x' t}}{2\mathrm{sh}\frac{\imath \epsilon_1 t}{2} 2\mathrm{sh}\frac{\imath \epsilon_2 t}{2}} e^{\imath m_{ij} t} \right) =$$

$$= \exp\left(- \int dx dx' \rho_i(x) \gamma_{\epsilon_1,\epsilon_2}(x - x' + m_{ij}) \rho_j(x') \right)$$

Barnes double gamma-function

$$\gamma_{\epsilon_1,\epsilon_2}(x) = \frac{d}{ds} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \frac{e^{ixt}}{2\mathrm{sh}\frac{\imath \epsilon_1 t}{2} 2\mathrm{sh}\frac{\imath \epsilon_2 t}{2}}$$

Assume $\sum_{\text{cycle}} m_{ij} = 0$

(always true for all Fin and Aff DE quivers b/c there are no cycles)

Absorb bi-fund masses to *U(1)* Coloumb moduli

$$Z_{\mathbb{T}} = \sum_{\mathbf{k}} \int [D\rho]_{\mathbf{k}} e^{-\frac{1}{\epsilon_1 \epsilon_2}\mathcal{F}}$$

(no fundamentals)

$$\mathcal{F} = \sum_i \pi \imath \tau_i \int dx\, x^2\, \rho(x) - \sum_{i,j} \int dx\, dx'\, \rho_i(x) k(x-x') \mathrm{c}_{ij} \rho_j(x')$$

$$k(x) \equiv -\epsilon_1 \epsilon_2 \gamma_{\epsilon_1 \epsilon_2}(x)$$

$$k(x) = \frac{x^2}{2}(\log x - \frac{3}{2}) \quad \text{as} \quad \epsilon_i \to 0$$

$$\int_{I_{i\alpha}} dx \rho_i(x) = 1$$

$$\int_{I_{i\alpha}} dx\, x \rho_i(x) = a_{i\alpha}$$

$$\int_{I_i} dx\, x^2 \rho_i(x) = \mathbf{a_i}^2 - 2\epsilon_1 \epsilon_2 k$$

Critical density (with constraints):

$$\frac{d^2}{dx^2} \frac{\delta \mathcal{F}}{\delta \rho} = 0$$

critical density solves

(no fundamentals)

$$\psi_j(x+i0)+\psi_j(x-i0)=2\pi\imath\tau_i-\sum_{j\neq i}\mathrm{c}_{ij}\psi_j(x),\quad x\in I_i$$

where potentials

$$\psi_j(x)=\int k_{xx}(x-x')\rho(x')dx' \qquad k_{xx}(x)=\log x,\quad \epsilon_i\to 0$$

in term of exp-potentials $y_i(x)=\exp\psi_i(x)$

critical equations with fundamentals

$$y_i(x+i0)y_i(x-i0)=Q(x)\prod_{j\neq i}y_j^{-\mathrm{c}_{ij}}(x)$$

involve matter polynomials

$$Q_i(x)=q_i\prod_{f=1}^{N_i^{\mathrm{F}}}(x+m_{if})$$

Cut-crossing and quiver Weyl group

cut-crossing jump $\psi_i(x+i0) \to \psi_i(x-i0)$

from the critical equations: $T_i(x) = \log Q_i(x)$

$$s_i: \quad \psi_j(x) \mapsto \begin{cases} \psi_j(x) - \sum_k \psi_k(x) \mathrm{c}_{kj} + T_j(x), & j = i \\ \psi_j(x), & j \neq i \end{cases}$$

The cut-crossing transformations generate the quiver Weyl group W_{q} !

quiver Cartan $\mathfrak{t}_{\mathrm{q}}$ ⟷ $\{\psi_i(x)\}$

$$\mathbf{t}(x) = \mathring{\mathbf{t}}(x) + \sum_i \alpha_i^\vee \psi_i(x) \qquad \mathring{\mathbf{t}}(x) = -\sum_i \Lambda_i^\vee T_i(x).$$

$$s_i: \quad \mathbf{t} \mapsto \mathbf{t} - \alpha_i(\mathbf{t})\alpha_i^\vee.$$

s_i - simple root reflections

W_{q} invariants

Define canonical W_{q} invariant functions

$$\chi_i(\mathbf{t}) = e^{-\Lambda_i(\mathring{\mathbf{t}})} \sum_{w \in W/W_i} e^{\Lambda_i^w(\mathbf{t})}$$

(also we can use characters)

equivalently

$$y_i = x^{N_i} + O(1), \quad x \to \infty$$

$$\chi_i = \sum_{w \in W/W_i} w(y_i) = y_i + \dots$$

is Weyl orbit of i-th fundamental weight

All terms $\prod_j y_j(x)^\bullet Q_j(x)^\bullet$ contain integer powers of $y_i(x)$ and positive integer powers of $Q_i(x)$ in combinations such that each term is majorated by $O(x^{N_i})$ as $x \to \infty$

$$\chi_i = c_i(\mathbf{q}) x^{N_i} + O(1)$$

Solution to the quiver limit shape problem is given by the system of analytic equations on the exp-potentials $y_i(x)$

$$\{\chi_i(\mathbf{y};\mathbf{Q}) = P_i(x), \quad \deg P_i = N_i\}$$

This defines the curve

$$\Sigma \subset \mathbb{C}_{\langle x \rangle} \times (\mathbb{C}^{\times}_{\langle y_i \rangle})^n, \quad n = \#\text{gauge groups}$$

$$(\chi_1 =) \quad y_1 + Q_1 \frac{y_2}{y_1} + Q_1 Q_2 \frac{1}{y_2} = P_1$$

$$(\chi_2 =) \quad y_2 + Q_2 \frac{y_1}{y_2} + Q_1 Q_2 \frac{1}{y_2} = P_2$$

Example: A_2 quiver

canonical Weyl invariant system

$$(\chi_1 =) \quad y_1 + Q_1 \frac{y_2}{y_1} + Q_1 Q_2 \frac{1}{y_2} = P_1$$

$$(\chi_2 =) \quad y_2 + Q_2 \frac{y_1}{y_2} + Q_1 Q_2 \frac{1}{y_2} = P_2$$

implies Seiberg-Witten curve equation

$$y_1^3 - P_1(x) y_1^2 + P_2(x) Q_1(x) y_1 - Q_1(x)^2 Q_2(x) = 0$$

Spectral curve in terms of characteristic polynomial for Fin quivers

To each irrep R of $G_{\rm q}$ we can associate the spectral polynomial and the algebraic curve:

$$\det{}_R(y - e^{\mathbf{t}}) = 0$$

The coefficients of powers of y in the expansion

$$\det{}_R(y - e^{\mathbf{t}}) = \sum_{k=0}^{\dim R} y^{\dim R - k} (-1)^k \mathrm{tr}_{\Lambda^k R} e^{\mathbf{t}}$$

are expressed polynomially in terms of invariants χ_i and, hence, $P_i(x)$ and $Q_i(x)$

Aff quivers, theta-functions and G_{q} bundles on elliptic curve

$E_{\langle t \rangle} = \mathbb{C}^{\times}/q^{\mathbb{Z}}$ elliptic curve $q = \prod_{i=0}^{r} q_i^{a_i^{\vee}}$

$a_i^{\vee}$ Dynkin marks

$\hat{G}_q$ orbits on $\hat{\mathfrak{t}}_{\mathrm{q}}$

G_{q}-flat connections on E

ADE theta-functions

$G_{\mathrm{q}}^{\mathbb{C}}$ polystable degree zero holomorphic bundles on E

$\mathbb{WP}_{a_0^{\vee},\ldots,a_r^{\vee}}$

Algebraic integrable system associated to affine ADE quiver

$$\mathcal{M} = \mathrm{Bun}^{0,\mathrm{ss}}_{G_{\mathrm{q}},N}(\overline{\mathbf{C}}_{\langle x \rangle} \times E_{\langle t \rangle}, \infty \times E)$$

$$\downarrow$$

$$\mathcal{B} = \mathrm{Maps}_N(\overline{\mathbf{C}}_{\langle x \rangle}, \mathrm{Bun}^{0,\mathrm{ss}}_{G_{\mathrm{q}}}(E))|_{\infty \to P_{\check{t}}}$$

$$\dim_{\mathbb{C}} \mathcal{M} = 2Nh^{\vee}$$

$$\dim_{\mathbb{C}} \mathcal{B} = Nh^{\vee}$$

The fibers are Lagrangian tori for holomorphic symplectic structure on $\mathcal{M}$

Affine AD quivers, Nahm transform and Hitchin systems

holomorphic $SU(r+1)$ / $SO(2r)$ / $Sp(2r)$ -bundles of instanton charge N on $\mathbf{C} \times E$

↕ Nahm transform

ADHM Higgs bundles on $E^{\vee}$ with (ADHM dual) $SU(N)$ / $Sp(2N)$ / $SO(N)$ structure group

↕

$SL(N)$ / $Sp(2N)$ / $SO(N)$ Hitchin system on elliptic curve $E^{\vee}$ (with $r+1$ / $2r$ / $2r$ punctures)

THANK YOU!