Progress in Quantum Field Theory and String Theory
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Counting BPS states in E-string theory

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String theory predicts the existence of nontrivial QFTs in 6D.

- The worldvolume theory of multiple M5 branes

  --- (2,0) SUSY (16 supercharges)

What about theories with (1,0) SUSY?

- Heterotic string theories on K3 (16 $\longrightarrow$ 8 supercharges)

What happens when instantons in K3 shrink to zero size?

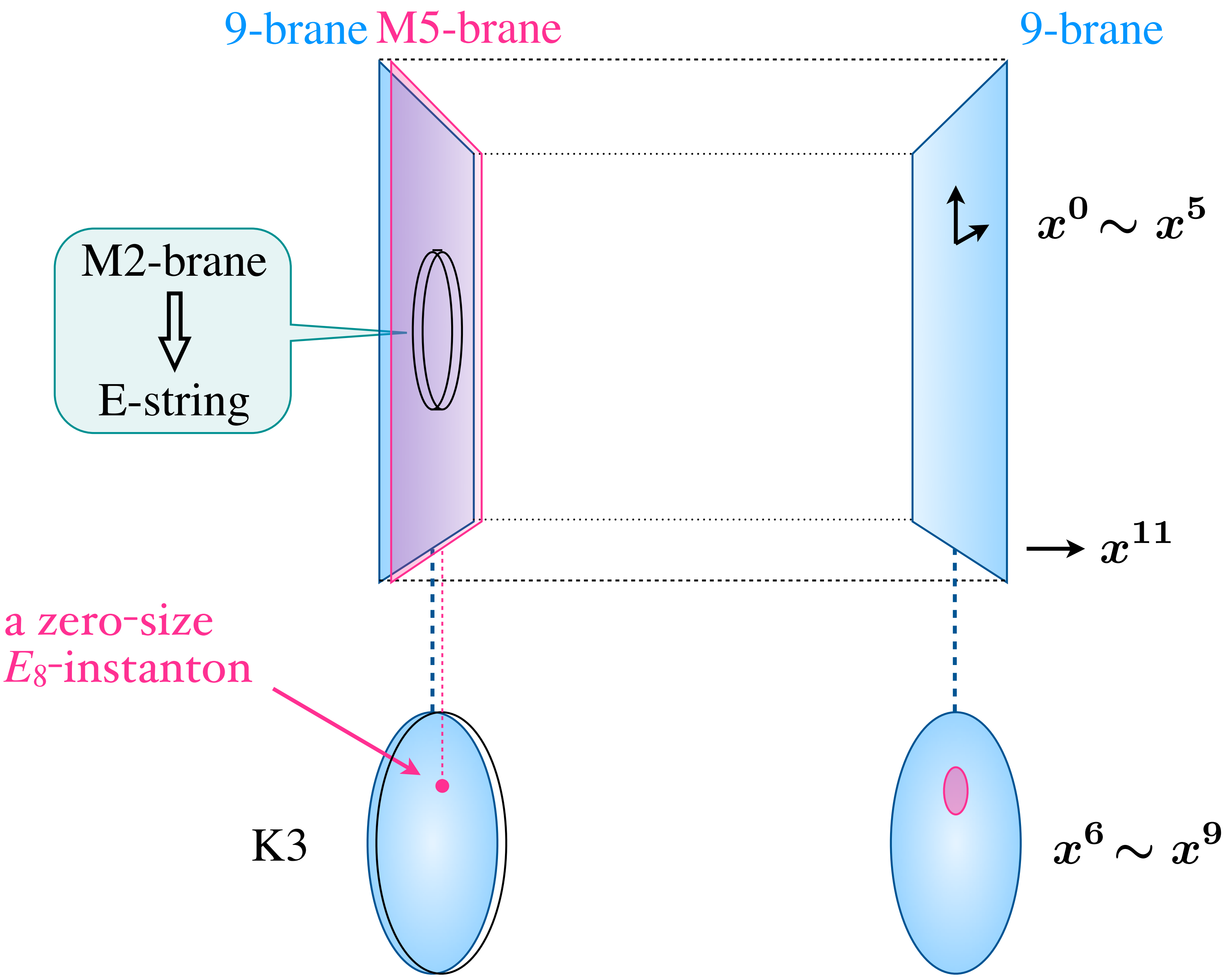
- Small SO(32) instantons

  $\Rightarrow$ extra Sp(n) gauge symmetry (Witten '95)

  $\Leftrightarrow$ worldvolume theory of n type I D5-branes

- What about small E_8 instantons?

M-theory description of the $E_8 \times E_8$ heterotic string theory on K3



E-string theory

(Ganor-Hanany '96)
(Seiberg-Witten '96)

- 6D (1,0)-supersymmetric local QFT
- decoupled from gravity
- no vector multiplets
- Coulomb branch --- a tensor multiplet
- Higgs branch $\cong$ the moduli space of an E_8 instanton
- fundamental excitations --- strings
- global E_8 symmetry

$$\boxed{\text{E-string} \quad = \quad \tfrac{1}{2}\ \text{heterotic}\ \boldsymbol{E_8 \times E_8}\ \text{string}}$$

Toroidal compactification down to 4D

a 4D $\mathcal{N} = 2$ theory $\Longrightarrow$ described by Seiberg-Witten theory

<table>
<tr><td>SU(n) gauge theories</td><td>E-string theory</td></tr>
<tr><td colspan="2">Seiberg-Witten curves are known</td></tr>
<tr><td colspan="2">realized by String Theory on certain CY_3</td></tr>
<tr><td>$\exists$ Lagrangian description</td><td>no Lagrangian description</td></tr>
<tr><td>toric CY_3</td><td>non-toric CY_3</td></tr>
<tr><td>Nekrasov partition functions
are known</td><td>Any analogue?
$\Longrightarrow$ Yes! (at least partly)</td></tr>
</table>

Heterotic -- F-theory duality (Morrison-Vafa '96)

Heterotic on K3 $\Rightarrow$ E-strings on $\mathbb{R}^6$

$\Updownarrow$

F-theory on $\boxed{\text{CY}_3}$ $\Rightarrow$ E-strings on $\mathbb{R}^6$

$\Updownarrow$

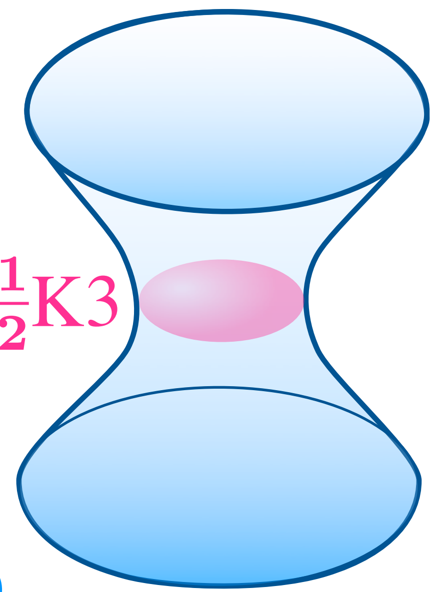
M-theory on $\boxed{\text{CY}_3}$ $\Rightarrow$ E-strings on $\mathbb{R}^5 \times \boldsymbol{S}^1$

$\|$

local $\frac{1}{2}$K3 surface

(the total space of
the canonical bundle
of a $\frac{1}{2}$K3 surface)

$\frac{1}{2}$K3



local $\frac{1}{2}$K3
(non-compact CY_3)

Partition function of BPS wound E-strings (= prepotential)

$$F_0(\varphi, \tau) = \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} N_{n,k} \sum_{m=1}^{\infty} \frac{p^{mn} q^{mk}}{m^3}$$

$$p := e^{2\pi i \varphi}, \qquad q := e^{2\pi i \tau}$$

n : winding number $\qquad$ k : momentum

$N_{n,k}$: BPS multiplicity (instanton number)

(Klemm-Mayr-Vafa '96)

$n \backslash k$	0	1	2	3	4	5	$\cdots$
1	1	252	5130	54760	419895	2587788	
2	0	0	-9252	-673760	-20534040	-389320128	
3	0	0	0	848628	115243155	6499779552	
4	0	0	0	0	-114265008	-23064530112	
5	0	0	0	0	0	18958064400	
$\vdots$							$\ddots$

Winding number expansion

$$F_0(\varphi, \tau) = \sum_{n=1}^{\infty} Q^n Z_n(\tau)$$

$$Q := q^{1/2} p, \qquad p := e^{2\pi i \varphi}, \qquad q := e^{2\pi i \tau}$$

Z_n : partition function of $\mathcal{N} = 4$ U(n) topological SYM on $\frac{1}{2}$K3

(Minahan-Nemeschansky-Vafa-Warner '98)

$$Z_1 = \frac{E_4}{\eta^{12}}, \qquad Z_2 = \frac{E_2 E_4^2 + 2E_4 E_6}{24\eta^{24}}, \qquad \cdots$$

$$Z_n = \frac{P_{6n-2}(E_2, E_4, E_6)}{\eta^{12n}}$$

$E_{2n}(\tau)$: Eisenstein series $\qquad$ $\eta(\tau)$: Dedekind eta function

- The expansion coefficients Z_n at low orders can be computed either from (Minahan-Nemeschansky-Warner '97)

the Seiberg-Witten curve

$$y^2 = 4x^3 - \frac{1}{12}E_4(\tau)u^4x - \frac{1}{216}E_6(\tau)u^6 + 4u^5$$

or from

the modular anomaly equation

$$\partial_{E_2} Z_n = \frac{1}{24}\sum_{k=1}^{n-1} k(n-k)Z_k Z_{n-k}$$

and the gap condition

$$q^{n/2} Z_n = \frac{1}{n^3} + \mathcal{O}(q^n)$$

Nekrasov-type expression for the prepotential

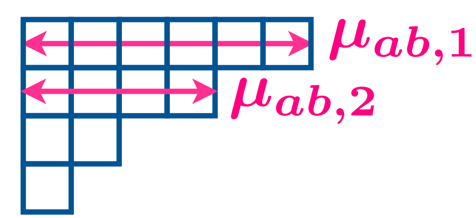
(K.S. '12)

$$F_0 = \left(2\hbar^2 \ln \mathcal{Z}\right)\Big|_{\hbar=0}$$

$$\mathcal{Z} = \sum_{R} Q^{|R|} \prod_{a,b,c,d} \prod_{(i,j)\in R_{ab}} \frac{\vartheta_{ab}\left(\frac{1}{2\pi}(j-i)\hbar, \tau\right)^2}{\vartheta_{1-|a-c|,1-|b-d|}\left(\frac{1}{2\pi}h_{ab,cd}(i,j)\hbar, \tau\right)^2}$$

$R = (R_{11}, R_{10}, R_{00}, R_{01})$

R_{ab} : partition

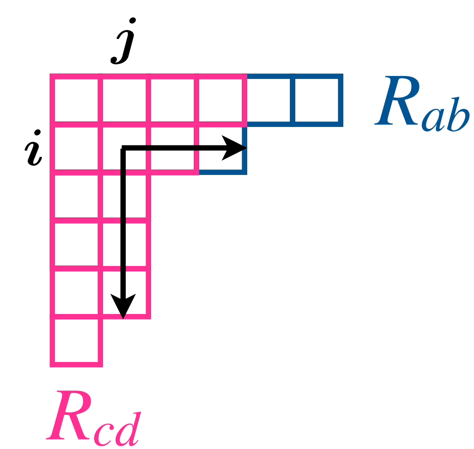


$a, b, c, d = 0, 1$

$\vartheta_{ab}(z,\tau)$: Jacobi theta functions

$h_{ab,cd}(i,j) := \mu_{ab,i} + \mu^{\vee}_{cd,j} - i - j + 1$

(relative hook-length)



The expression coincides with a special case of the elliptic generalization of the Nekrasov partition function for SU(4) SYM with $N_{\mathrm{f}} = 8$ massless fundamental hypermultiplets (K.S. '12)

(Nekrasov '02) (Hollowood-Iqbal-Vafa '03)

$$\mathcal{Z} = \sum_{R} (-p)^{|R|} \prod_{k=1}^{4} \prod_{(i,j)\in R_k} \frac{\vartheta_1\left(a_k + \frac{1}{2\pi}(j-i)\hbar, \tau\right)^8}{\prod_{l=1}^{4} \vartheta_1\left(a_k - a_l + \frac{1}{2\pi} h_{kl}(i,j)\hbar, \tau\right)^2}$$

$$a_1 = 0, \qquad a_2 = \frac{1}{2}, \qquad a_3 = -\frac{1+\tau}{2}, \qquad a_4 = \frac{\tau}{2}$$

$$R = (R_1, R_2, R_3, R_4) = (R_{11}, R_{10}, R_{00}, R_{01})$$

The prepotential represents $g=0$ topological string amplitude

$$F_0 = F_0^{\frac{1}{2}\mathrm{K3}}$$

However,

$$\mathcal{Z} \neq \mathcal{Z}^{\frac{1}{2}\mathrm{K3}}$$

$$:= \exp\left(\sum_{g=0}^{\infty} \hbar^{2g-2} F_g^{\frac{1}{2}\mathrm{K3}}\right)$$

Difference of modular anomalies

$$\partial_{E_2}\mathcal{Z} = \frac{1}{12}\hbar^2\partial_\phi^2\mathcal{Z}$$

$$\partial_{E_2}\mathcal{Z}^{\frac{1}{2}\mathrm{K3}} = \frac{1}{24}\hbar^2\partial_\phi(\partial_\phi+1)\mathcal{Z}^{\frac{1}{2}\mathrm{K3}}$$

(Hosono-Saito-Takahashi '99)

$$\Downarrow \qquad \Downarrow$$

$$\partial_{E_2}F_0 = \frac{1}{24}\left(\partial_\phi F_0\right)^2$$

$$\partial_\phi = \frac{1}{2\pi i}\partial_\varphi = Q\partial_Q$$

Part II

Topological string amplitudes for the most general local $\frac{1}{2}$K3

Topological string amplitudes at higher genus

- Holomorphic anomaly equation (Bershadsky-Cecotti-Ooguri-Vafa '93)

$$\bar{\partial}_{\bar{\imath}} F_g = \frac{1}{2} \bar{C}_{\bar{\imath}\bar{\jmath}\bar{k}} e^{2K} G^{j\bar{\jmath}} G^{k\bar{k}} \left(D_j D_k F_{g-1} + \sum_{h=1}^{g-1} D_j F_h D_k F_{g-h} \right)$$

Recent developments

- F_g are polynomials in a finite number of generators

(Yamaguchi-Yau '04) (Alim-Länge '07)

- F_g are quasi-automorphic forms (Aganagic-Bouchard-Klemm '06)

For SU(2) gauge theories:

F_g are polynomials in $\tilde{E}_2 := E_2(\tilde{\tau})$

- Direct integration method (Grimm-Klemm-Mariño-Weiss '06)

HAE: $\partial_{\tilde{E}_2} F_g = \cdots$

Topological string amplitudes for the most general local $\frac{1}{2}$K3

$$\mathcal{Z}^{\frac{1}{2}\mathrm{K3}}(\varphi,\tau,\vec{\mu};\hbar)=\exp\left[\sum_{g=0}^{\infty}\hbar^{2g-2}\sum_{n=1}^{\infty}e^{2\pi i n\varphi}Z_{g,n}(\tau,\vec{\mu})\right]$$

For low g and n, $Z_{g,n}$ can be determined by

- symmetry

$$Z_{g,n}(\tau,\vec{\mu})=\frac{e^{\pi i n\tau}}{\eta(\tau)^{12n}}T_{g,n}(\tau,\vec{\mu})$$

$T_{g,n}(\tau,\vec{\mu}):W(E_8)$-invariant quasi-Jacobi form of weight $2g-2+6n$ and index n

- holomorphic anomaly equation

$$\partial_{E_2}\mathcal{Z}^{\frac{1}{2}\mathrm{K3}}=\frac{1}{24}\hbar^2\partial_\phi(\partial_\phi+1)\mathcal{Z}^{\frac{1}{2}\mathrm{K3}}$$

- gap condition

No BPS states for $k<n$ (except for $n=1$)

(Minahan-Nemeschansky-Vafa-Warner '98) (Hosono-Saito-Takahashi '99)

Seiberg-Witten curve for the E-string theory (Eguchi-K.S. '02)

$$y^2 = 4x^3 - fx - g$$

$$f = \sum_{n=0}^{4} a_n u^{4-n}, \qquad g = \sum_{n=0}^{6} b_n u^{6-n}.$$

$a_n(\tau, \vec{\mu}),\ b_n(\tau, \vec{\mu})$ are expressed in terms of $W(E_8)$-invariant Jacobi forms $A_n(\tau, \vec{\mu}),\ B_n(\tau, \vec{\mu})$

$$a_0 = \frac{1}{12}E_4, \qquad a_1 = 0, \qquad a_2 = \frac{6}{E_4 \Delta}\left(-E_4 A_2 + A_1^2\right), \qquad \cdots$$

$$b_0 = \frac{1}{216}E_6, \qquad b_1 = -\frac{4}{E_4}A_1, \qquad b_2 = \frac{5}{6E_4^2 \Delta}\left(E_4^2 B_2 - E_6 A_1^2\right), \qquad \cdots$$

$W(E_8)$-invariant Jacobi forms

(K.S. '11)

$$\Theta(\tau,\vec{\mu}) = \frac{1}{2}\sum_{k=1}^{4}\prod_{j=1}^{8}\vartheta_k(\mu_j,\tau)$$

Theta function associated with the E_8 root lattice

$$A_2(\tau,\vec{\mu}) = \tfrac{8}{9}\mathscr{H}\{\Theta(2\tau,2\vec{\mu})\}\,, \qquad A_1(\tau,\vec{\mu}) = \Theta(\tau,\vec{\mu}),$$

$$A_3(\tau,\vec{\mu}) = \tfrac{27}{28}\mathscr{H}\{\Theta(3\tau,3\vec{\mu})\}\,, \qquad A_4(\tau,\vec{\mu}) = \Theta(\tau,2\vec{\mu}),$$

$$A_5(\tau,\vec{\mu}) = \tfrac{125}{126}\mathscr{H}\{\Theta(5\tau,5\vec{\mu})\}\,,$$

$$B_2(\tau,\vec{\mu}) = \tfrac{32}{5}\mathscr{H}\{e_1(\tau)\Theta(2\tau,2\vec{\mu})\}\,, \qquad B_3(\tau,\vec{\mu}) = \tfrac{81}{80}\mathscr{H}\left\{h(\tau)^2\Theta(3\tau,3\vec{\mu})\right\},$$

$$B_4(\tau,\vec{\mu}) = \tfrac{16}{15}\mathscr{H}\left\{\vartheta_4(2\tau)^4\Theta(4\tau,4\vec{\mu})\right\}, \qquad B_6(\tau,\vec{\mu}) = \tfrac{9}{10}\mathscr{H}\left\{h(\tau)^2\Theta(6\tau,6\vec{\mu})\right\}.$$

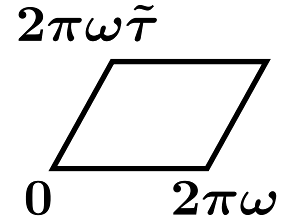
$\mathscr{H}\{\cdot\}$: sum of all possible $\mathbf{SL(2,\mathbb{Z})}$ transforms

$$e_1(\tau) = \tfrac{1}{12}\left(\vartheta_3(\tau)^4+\vartheta_4(\tau)^4\right), \qquad A_n(\tau,\vec{0}) = E_4,$$

$$h(\tau) = \vartheta_3(2\tau)\vartheta_3(6\tau)+\vartheta_2(2\tau)\vartheta_2(6\tau). \qquad B_n(\tau,\vec{0}) = E_6.$$

Amplitudes for the local $\frac{1}{2}$K3

(K.S. '11)



$$t := 2\pi i(\tilde{\tau} - \tau)$$
$$\tilde{E}_{2n} := E_{2n}(\tilde{\tau})$$
$$\tilde{\Delta} := \eta(\tilde{\tau})^{24}$$

$$F_0 = -\partial_\phi^{-2} t,$$

$$F_1 = \frac{1}{2}\ln\omega - \frac{1}{12}\ln\tilde{\Delta} + \frac{1}{12}\ln\Delta - \frac{1}{2}\phi,$$

$$F_2 = -\frac{1}{96}E_2 + \frac{1}{48}\tilde{E}_2\,\partial_\phi^2\ln\omega + \frac{1}{96}\tilde{E}_2(\partial_\phi\ln\omega)^2 - \frac{1}{576}(\tilde{E}_2^2 - \tilde{E}_4)\partial_\phi t\,\partial_\phi\ln\omega$$
$$-\frac{1}{1920}(5\tilde{E}_2^2 + 3\tilde{E}_4)\partial_\phi^2 t - \frac{1}{207360}(35\tilde{E}_2^3 + 51\tilde{E}_4\tilde{E}_2 - 86\tilde{E}_6)(\partial_\phi t)^2,$$

$$F_3 = \cdots$$

Structure of F_g $(g \geq 2)$

- polynomial in $\tilde{E}_{2k}$, $\partial_\phi^m\ln\omega$, $\partial_\phi^n t$
- each term contains $2g-2$ ∂_ϕ's
- quasi-modular form of weight $2g-2$

$$\Big([\tilde{E}_{2k}] = 2k, \quad [\partial_\phi^m\ln\omega] = 0, \quad [\partial_\phi^n t] = -2\Big)$$

Summary

- We have found a Nekrasov-type expression
  for the Seiberg-Witten prepotential for the E-string theory.
- We have constructed topological string amplitudes
  for the most general local $\frac{1}{2}$K3 (up to genus three).

“SU(2) Seiberg-Witten theories” with various flavor symmetries

- Seiberg-Witten curves --- constructed for all theories

(Seiberg-Witten ’94) (Minahan-Nemeschansky ’96) (Ganor-Morrison-Seiberg ’96) (Minahan-Nemeschansky-Warner ’97) (Eguchi-K.S. ’02)

- “All-genus” partition functions:

6D The $\hat{E}_8$-string theory needs further investigation!

5D / 4D

$$\begin{tikzcd}
 & & & & & & & E_1 \arrow[r] & E_0 \\
E_8 \arrow[r] \arrow[d] & E_7 \arrow[r] \arrow[d] & E_6 \arrow[r] \arrow[d] & E_5 \arrow[r] \arrow[d] & E_4 \arrow[r] \arrow[d] & E_3 \arrow[r] \arrow[d] & E_2 \arrow[ur] \arrow[r] \arrow[d] & \tilde{E}_1 \arrow[d] & \\
E_8 \arrow[r] & E_7 \arrow[r] & E_6 \arrow[r] & D_4 \arrow[r] & D_3 \arrow[r] \arrow[d] & D_2 \arrow[r] \arrow[d] & D_1 \arrow[r] \arrow[d] & D_0 & \\
 & & & & A_2 \arrow[r] & A_1 \arrow[r] & A_0 & &
\end{tikzcd}$$

Partition functions were constructed by “the ruled vertex formalism”
(Diaconescu-Florea ’05)

Nekrasov partition functions are available for these theories
(Nekrasov ’02)