Gluon scattering amplitudes from gauge/string duality and integrability

Yuji Satoh
(University of Tsukuba)

Based on

Y. Hatsuda (TIT), K. Ito (TIT), K. Sakai (YITP) and Y.S.
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Y. Hatsuda (TIT), K. Ito (TIT) and Y.S.
JHEP 1202(2012)003; also work in progress

1. Introduction

☆ Discovery of integrability in AdS/CFT

$\Rightarrow$ opened up new dimensions

- gauge/string duality beyond susy sectors
- quantitative analysis of
  gauge theory dynamics at strong coupling
- (insights into applications)

$\vdots$

In fact,

[cf. talks by Prof. Kazakov, Prof. Artyunov; poster by Dr. Suzuki]

- spectrum of planar AdS/CFT
  for arbitrary ’t Hooft coupling

Also,

- applications ?
  ⇒ gluon scatt. amplitudes/Wilson loops
  [cf. correlators, cusp anom. dim.]
  [cf. poster by Dr. Komatsu]

gluon scatt. amplitudes at strong coupling

$\Uparrow$ [$\Leftarrow$ AdS/CFT]

minimal surfaces in AdS5 x S5

$\Uparrow$ [$\Leftarrow$ integrability]

thermodynamic Bethe ansatz (TBA) equations

In this talk, I would like to

[cf. talk by Prof. Taylor]

- discuss maximally helicity violating (MHV) amplitudes/ Wilson loops of $\mathcal{N}=4$ SYM at strong coupling

  $\Uparrow$ underlying 2D integrable models and CFTs

- in particular, derive analytic expansions around certain kinematic points

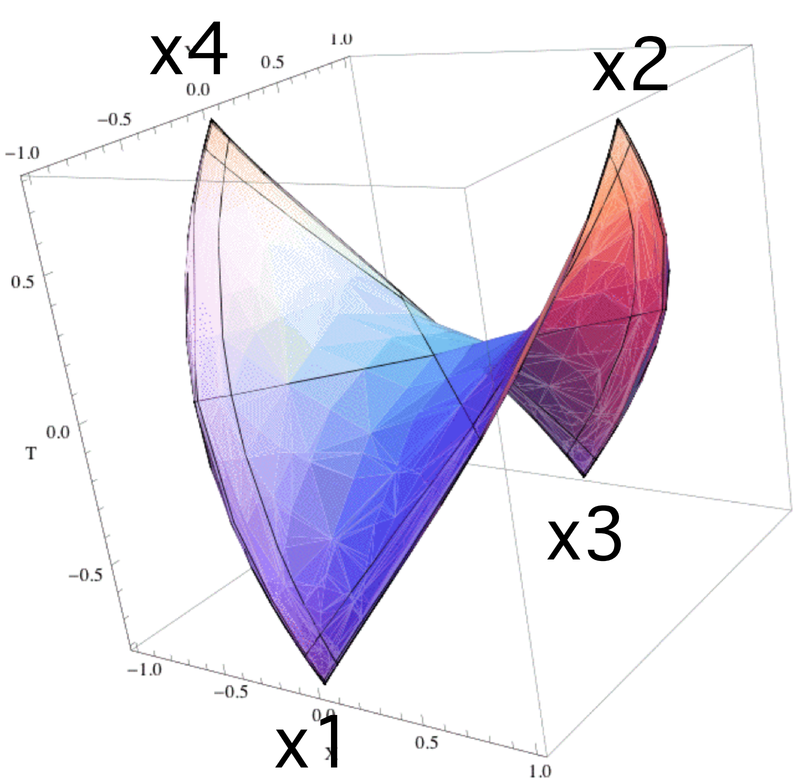
  [regular polygonal Wilson loops]

2. Gluon scattering amplitudes at strong coupling

[Alday-Maldacena '07]

amplitudes of $\mathcal{N}=4$ SYM at strong coupling = minimal surfaces in AdS

AdS/CFT



$$\mathcal{M} \sim e^{-\frac{\sqrt{\lambda}}{2\pi}(\text{Area})}$$

- $\mathcal{M}$: scalar part of MHV amplitudes
- λ : 't Hooft coupling
- null boundary at AdS boundary

$$x_{i+1}^{\mu} - x_i^{\mu} = 2\pi k_i^{\mu} \quad \text{(momentum of particle)}$$

$\Rightarrow$ n-pt. amplitude $\approx$ n-cusp min. surface

- 4pt. amplitude :
  precise agreement w/ BDS conjecture [Bern–Dixon–Smirnov '05]
  (all order in perturbation)

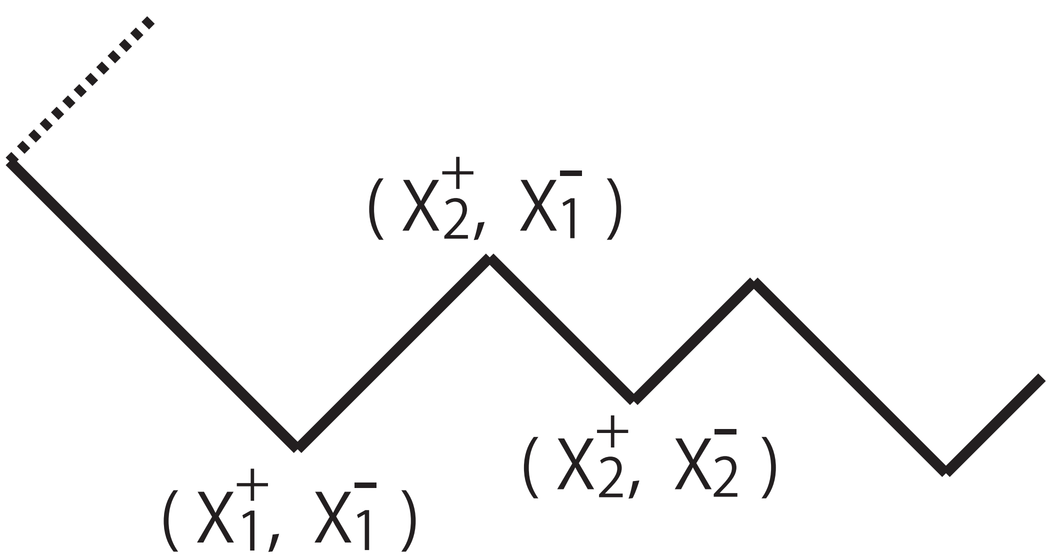
- remainder function : deviation from BDS formula
  = fn. of cross–ratios of x_a^μ $\Longleftarrow$ dual conformal sym.
  (cusp coordinate)
  [Drummond–Henn–Smirnov–Sokatchev '06
  Drummond–Henn–Korchemsky–Sokatchev '07]

3. Scattering amplitudes from TBA system

[Alday–Maldacena '09, Alday–Gaiotto–Maldacena '09,
Alday–Maldacena–Sever–Vieira '10, Hatsuda–Ito–Sakai–YS '10]

- difficult to construct min. surface in AdS w/ null boundary
- but, its area is obtained by using integrability w/o explicit solutions

In the following, we focus on the case of 2n–cusp min. surface in AdS3



- 4D external momenta in $R^{1,1}$
- # of cusps : even in AdS3 [mom. conservation]
- 2 light–cone coord. $x^{\pm}$ at ∂(AdS)

Thermodynamic Bethe ansatz (TBA) eq. for AdS3 min. surface

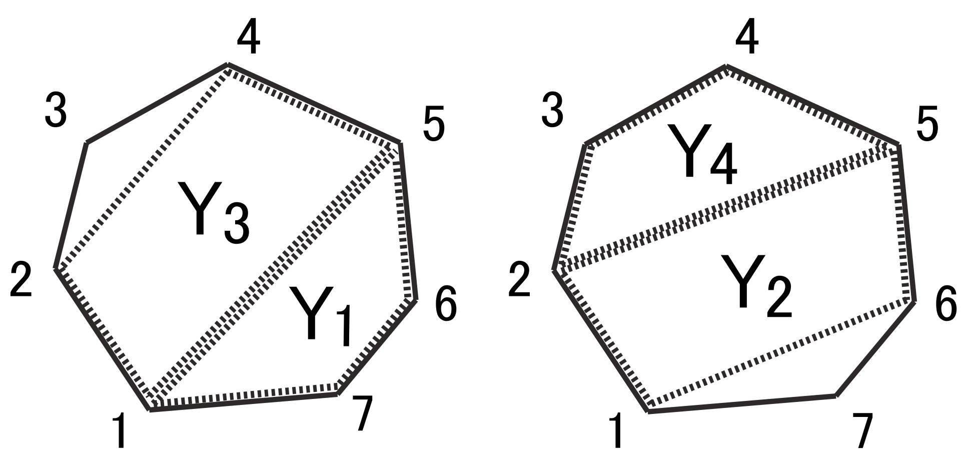
[Alday–Maldacena–Sever–Vieira '10, Hatsuda–Ito–Sakai–YS '10]

- to compute amplitudes (2n–pt.), first need to solve

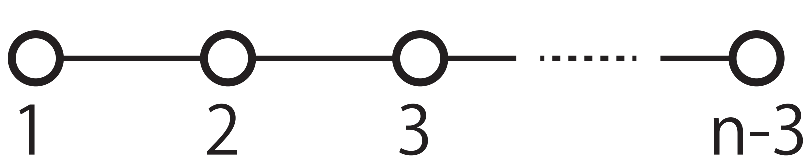
$$\log Y_s(\theta) = -m_s \cosh\theta + \sum_r K_{sr} * \log(1+Y_r)$$

$$Y_0 = Y_{n-2} = 0 \qquad (s = 1, ..., n-3)$$

⇒ TBA eq. of hom. sine–Gordon model



n = 7



- θ : spectral parameter
- Ys : (extended) cross–ratios of $x_a^{\pm}$

  e.g.) $Y_1(-\frac{\pi i}{2}) = \frac{x_{15}^+ x_{67}^+}{x_{56}^+ x_{17}^+}, \quad Y_1(0) = \frac{x_{15}^- x_{67}^-}{x_{56}^- x_{17}^-}$

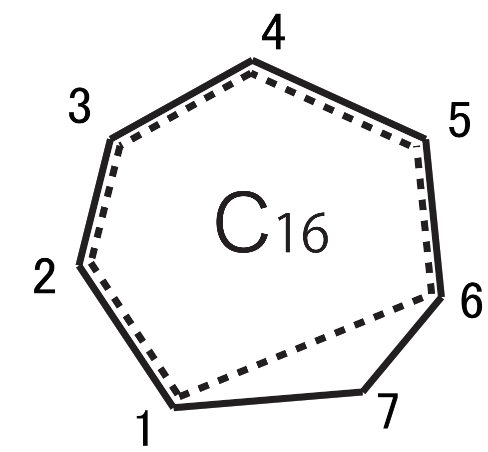
- m_s : complex (mass) param.

  ≈ shape of surface ⇔ momenta

- $K_{sr} = I_{sr}/\cosh\theta$
- I_{sr} : incidence matrix for A_{n-3}

Remainder function [overall coupling dependence : omitted]

- Once Y-fn. are obtained



$$R_{2n} := A \text{ [amp.]} - A_{BDS} \text{ [BDS formula]}$$

$$= \frac{7\pi}{12}(n-2) + A_{\text{periods}} + \Delta A_{BDS} + A_{\text{free}}$$

$$c_{16} = \frac{x_{23}\, x_{45}\, x_{16}}{x_{12}\, x_{34}\, x_{56}}$$

$$A_{\text{periods}} = -\frac{1}{4} m_r\, I^{-1}_{rs}\, \overline{m}_s$$

$$\Delta A_{BDS} = \frac{1}{4} \sum_{i,j=1}^{n} \log \frac{c^+_{i,j}}{c^+_{i,j+1}} \log \frac{c^-_{i-1,j}}{c^-_{i,j}}\ , \quad c^{\pm}_{i,j} = \frac{x^{\pm}_{i+2,i+1} x^{\pm}_{i+4,i+3} \cdots x^{\pm}_{j,i}}{x^{\pm}_{i+1,i} x^{\pm}_{i+3,i+2} \cdots x^{\pm}_{j,j-1}}$$

$$A_{\text{free}} = \sum_{s=1}^{n-3} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m_s \cosh\theta \log(1 + Y_s(\theta))$$

free energy of TBA system

$\Rightarrow$ non-trivial part : A_{free} , ΔA_{BDS}

- TBA system : solved numerically
- exact solutions $\begin{cases} m_s \to 0 & : \text{ CFT limit} \\ m_s \to \infty \end{cases}$ regular polygon

○ solutions to TBA system : not fully investigated

○ momentum dependence : ms ↔ momentum

○ progress in analytic results at 2 loops

Any analytic results at strong coupling except
$m_s \to 0 \;,\; \infty$??

⇒ we discuss expansions around CFT lim.

$$l = ML \to 0 \quad (m_s = M_s L = \tilde{M}_s M L)$$

[M: mass scale, L: length scale]

4. Analytic expansion of amplitudes

[Hatsuda–Ito–Sakai–YS ' 11, Hatsuda–Ito–YS ' 11]

- basis of the expansion : [Hatsuda–Ito–Sakai–YS ' 10]

TBA for AdS3 2n–pt. amplitude	$\Leftarrow$	HSG from $\frac{\widehat{\mathrm{su}}(n-2)_2}{[\widehat{\mathrm{u}}(1)]^{n-3}}$

[w/ imaginary resonance param.]

- Homogeneous sine–Gordon (HSG) model for AdS3: integrable deformation of the coset CFT

[Fernandez Pousa–Gallas–Hollowood–Miramontes ' 96]

$$S_{HSG} = S_{\mathrm{gWZNW}} + \beta \int d^2x \, \Phi$$

Φ : comb. of weight 0 adjoint op.

$$\Delta = \bar{\Delta} = \frac{n-2}{n}, \qquad \beta = -\kappa_n M^{2(1-\Delta)}$$

A_{free} : free-energy part

$[\, R_{2n} \sim A_{\text{free}} + \Delta A_{BDS} \,]$

- free energy around CFT limit $\Leftarrow$ CFT perturbation

ΔA_{BDS} $(\leftarrow\ c_{ij}^{\pm})$

- can show cross-ratios $c_{ij}^{\pm}$: nothing but T-fn.
  [≈ transfer mat.]

$$c_{ij}^{+} = T_{|i-j|-1}^{[i+j]}(0), \qquad c_{ij}^{-} = T_{|i-j|-1}^{[i+j+1]}(0)$$

where $\quad Y_s = T_{s+1} T_{s-1}\ , \quad 1 + Y_s = T_s^{[+1]} T_s^{[-1]}$

$$T_s^{[k]}(\theta) := T_s\big(\theta + \frac{\pi i}{2} k\big)$$

$$\Longrightarrow \quad \Delta A_{BDS} = \Delta A_{BDS}(T_s)$$

[everything fits into language of HSG model]

T–function

- periodicity

$$\Rightarrow \quad T_s(\theta) = \sum_{p,q=0}^{\infty} t_s^{(p,q)} l^{(1-\Delta)(p+q)} \cosh\left(\frac{2p\theta}{n}\right) \qquad \text{[real } m_s \text{]}$$

- T–function

  $\approx$ g–function (boundary entropy)

  [Bazhanov–Lukyanov–Zamolodchikov '94; Dorey–Runkel–Tateo–Watts '99; Dorey–Lishman–Rim–Tateo '05]

  $\Uparrow$

- reflection factors for HSG model w/ boundary

  satisfying boundary YB eq., unitarity, crossing sym.

- boundary CFT perturbation

$$\Rightarrow \quad \frac{t_s^{(2,0)}}{t_s^{(0,0)}} = -\frac{\kappa_n G(\tilde{M}_j) B(1-2\Delta,\Delta)}{2(2\pi)^{1-2\Delta}} \left(\frac{\sin(\frac{3(s+1)\pi}{n})}{\sin(\frac{(s+1)\pi}{n})} \sqrt{\frac{\sin(\frac{\pi}{n})}{\sin(\frac{3\pi}{n})}} - \sqrt{\frac{\sin(\frac{3\pi}{n})}{\sin(\frac{\pi}{n})}} \right)$$

$$t_s^{(0,0)} = \frac{\sin\left(\frac{(s+1)\pi}{n}\right)}{\sin(\frac{\pi}{n})} \quad , \quad \langle \Phi(z)\Phi(0) \rangle = \frac{G^2(\tilde{M}_s)}{|z|^{4\Delta}}$$

[given by modular S-matrix]

- $t_s^{(2,0)},\ t_s^{(0,4)} \quad \Leftarrow \quad t_1^{(2,0)} \propto \kappa_n G(\tilde{M}_s)$

T-system [rel. among T-fn.]

Combining all, 2n–pt. remainder fn. is expanded as

$$R_{2n} = R_{2n}^{(0)} + l^{\frac{8}{n}} R_{2n}^{(4)} + \mathcal{O}(l^{\frac{12}{n}})$$

$$R_{2n}^{(0)} = \frac{\pi}{4n}(n-2)(3n-2) - \frac{n}{2}\sum_{s=1}^{(n-3)/2} \log^2\left(\frac{\sin(\frac{(s+1)\pi}{n})}{\sin(\frac{s\pi}{n})}\right)$$

$$R_{2n}^{(4)} = \frac{\pi}{6} C_n^{(2)} \kappa_n^2 G^2(\tilde{M}_j) - \frac{n}{4}\left[\sum_{s=1}^{(n-3)/2} A_{n,s} - 2\left(\frac{t_{(n-3)/2}^{(2,0)}}{t_{(n-3)/2}^{(0,0)}}\right)^2 \sin^2\left(\frac{\pi}{n}\right)\right]$$

$$A_{n,s} := \left[\left(\frac{t_{s-1}^{(2,0)}}{t_{s-1}^{(0,0)}}\right)^2 + \left(\frac{t_{s}^{(2,0)}}{t_{s}^{(0,0)}}\right)^2\right]\cos\left(\frac{2\pi}{n}\right) - \frac{2t_{s-1}^{(2,0)} t_s^{(2,0)}}{t_{s-1}^{(0,0)} t_s^{(0,0)}}$$
$$+ \left[\left(\frac{t_{s-1}^{(2,0)}}{t_{s-1}^{(0,0)}}\right)^2 - \left(\frac{t_{s}^{(2,0)}}{t_{s}^{(0,0)}}\right)^2 - 4\left(\frac{t_{s-1}^{(0,4)}}{t_{s-1}^{(0,0)}} - \frac{t_{s}^{(0,4)}}{t_{s}^{(0,0)}}\right)\right]\log\left(\frac{t_s^{(0,0)}}{t_{s-1}^{(0,0)}}\right)$$

$$C_n^{(2)} = 3(2\pi)^{\frac{2(n-4)}{n}} \gamma^2\left(\frac{n-2}{n}\right)\gamma\left(\frac{4-n}{n}\right), \quad \gamma(x) = \Gamma(x)/\Gamma(1-x)$$

- fn. of $t_1^{(2,0)} \propto \kappa_n G(\tilde{M}_s)$

To further express amplitudes by momenta

- need
  - $\kappa_n G \;\Leftrightarrow\; m_s$ [mass-coupling relation]
  - recover phases of m_s
  - invert expansion of Y-fn. (cross-ratio) by m_s

$$\Rightarrow \; m_s = m_s(k_a) \qquad [k_a \; : \; \text{momenta}]$$

- this can be done for single mass cases (e.g. $M_s = \delta_{s,r} M$)
  - HSG model $\Rightarrow$ simpler models (RSOS, SU(2) coset)

[Zamolodchikov '95; Fateev '94]

$\Rightarrow$ amplitudes along certain trajectories in mom. space

- For 10-pt. case,

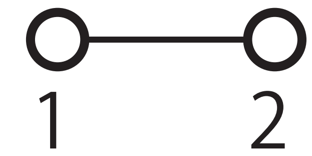
  these are enough to completely fix $\kappa_n G \;\Leftrightarrow\; m_s$

10 pt. remainder function

$$[\ l = ML,\ \ m_s = \tilde{M}_s l\ ]$$

$$R_{10} = R_{10}^{(0)} + R_{10}^{(4)} \cdot l^{8/5} + \mathcal{O}(l^{12/5})$$

A2



$$R_{10}^{(0)} = \frac{39}{20}\pi - \frac{5}{2}\log^2\left(2\cos\frac{\pi}{5}\right)$$

$$R_{10}^{(4)} = \left(-\frac{1}{5}\tan\frac{\pi}{5} + C_1\right) \cdot \left|t_1^{(2,0)}\right|^2$$

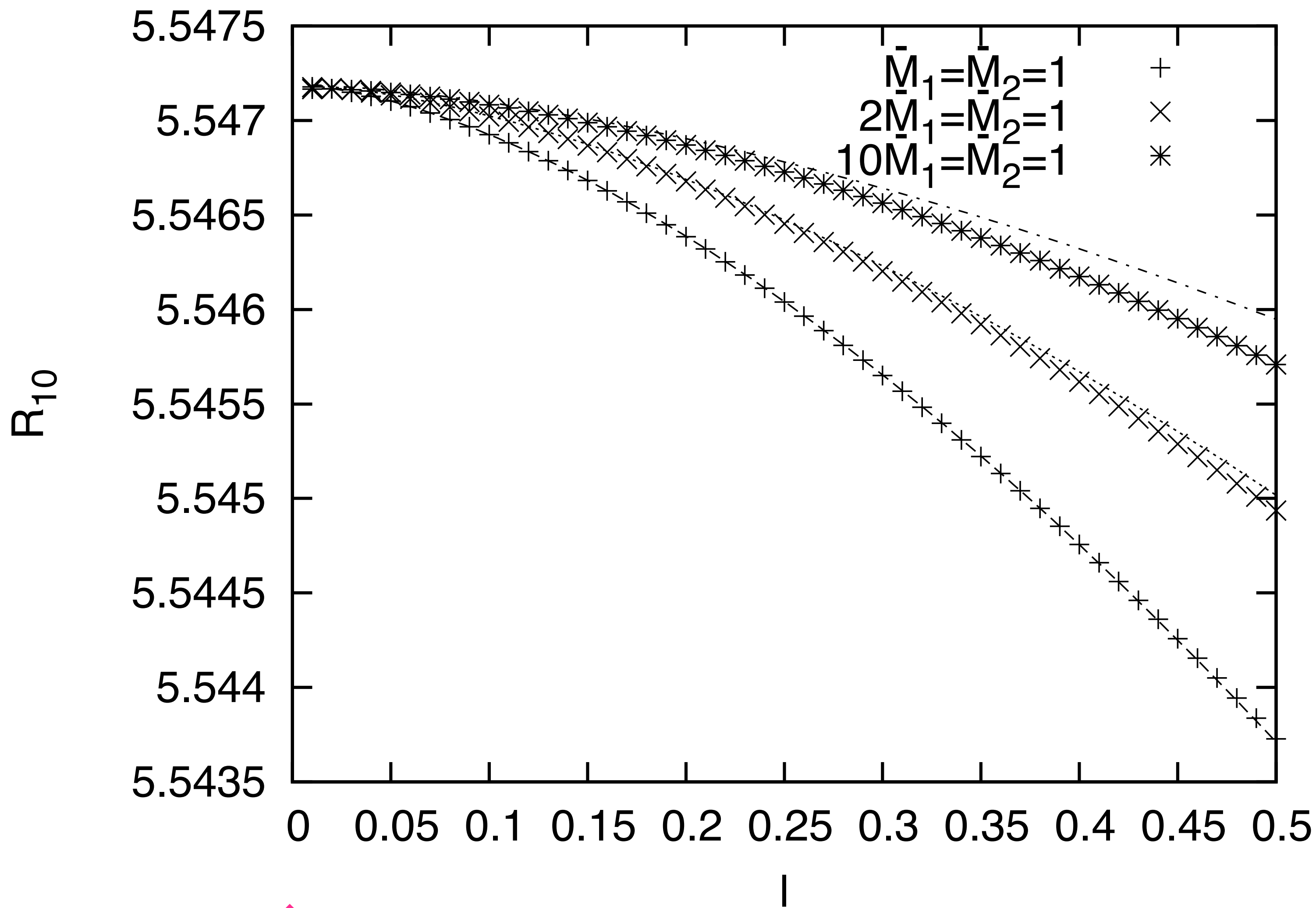
$$t_1^{(2,0)} = C_2\left(\tilde{M}_1^{4/5} + \tilde{M}_2^{4/5} - C_3 \tilde{M}_1^{2/5}\tilde{M}_2^{2/5}\right)$$

$$C_1 = 20\cos^4\left(\frac{2\pi}{5}\right)\left(1 - 5^{-1/2}\log\left(2\cos\frac{\pi}{5}\right)\right)$$

$$C_2 = \frac{1}{4 \cdot 6^{1/5}}\Gamma(-1/5)\left[10\cos\frac{\pi}{5}\gamma(3/5)\gamma(4/5)\right]^{1/2}$$

$$C_3 = 2 - \left(\frac{3}{\pi^2}\right)^{1/5}\gamma(1/4)^{4/5}$$

$$\gamma(x) = \Gamma(x)/\Gamma(1-x)$$



$(\ \varphi_1 = \pi/20, \ \varphi_2 = \pi/5 \ )$

⇑ CFT lim.

l- dependence of 10–pt. remainder function $[\, m_s = \tilde{M}_s l \, e^{i\varphi_s} \,]$

- dashed lines : $R_{10}^{(0)} + R_{10}^{(4)} l^{8/5}$
- good agreement w/ numerics

5. Comparison with 2-loop results

2-loop results

[Heslop-Khoze' 10 ; Gaiotto-Maldacena-Sever-Vieira ' 11]

- analytic results at 2 loops are known for AdS3 amplitudes
- for the same cross-ratios, have similar expansion

$$R_{2n}^{\text{2-loop}} = R_{2n}^{\text{2-loop}\,(0)} + l^{\frac{8}{n}} R_{2n}^{\text{2-loop}\,(4)} + \mathcal{O}(l^{\frac{12}{n}})$$

rescaled remainder fn.

[Brandhuber-Heslop-Khoze-Travaglini ' 09]

- for comparison, useful to define

$$\overline{R}_{2n} := \frac{R_{2n} - R_{2n,\text{UV}}}{R_{2n,\text{UV}} - (n-2)R_6}$$

  - normalized s.t. $\overline{R}_{2n} \;\; \begin{aligned} &\to \;\; 0 \quad (l \to 0) \\ &\to -1 \quad (l \to \infty) \end{aligned}$

- dependence on m_s (mometum) :

  encoded in $t_1^{(2,0)} \propto \kappa_n G(\tilde{M}_s)$ at leading order

$\Rightarrow$ ratio of $\overline{R}_{2n}$: number

$$\frac{\bar{R}_8^{\text{strong}}}{\bar{R}_8^{\text{2-loop}}} \approx 1.0257\,, \qquad \frac{\bar{R}_{10}^{\text{strong}}}{\bar{R}_{10}^{\text{2-loop}}} \approx 0.9841$$

$$\frac{\bar{R}_{12}^{\text{strong}}}{\bar{R}_{12}^{\text{2-loop}}} \approx 0.9609\,, \qquad \frac{\bar{R}_{14}^{\text{strong}}}{\bar{R}_{14}^{\text{2-loop}}} \approx 0.9463$$

$$\frac{\bar{R}_{16}^{\text{strong}}}{\bar{R}_{16}^{\text{2-loop}}} \approx 0.9366\,, \qquad \frac{\bar{R}_{18}^{\text{strong}}}{\bar{R}_{18}^{\text{2-loop}}} \approx 0.9297$$

$$\vdots$$

$$\frac{\bar{R}_{2n\gg 1}^{\text{strong}}}{\bar{R}_{2n\gg 1}^{\text{2-loop}}} \approx 0.905 - \frac{0.118}{n}$$

close to 1

[not 1000, 0.0001 ...]

6. Summary

- Gluon scatt. amplitudes of $\mathcal{N}=4$ SYM at strong coupling

  $\Uparrow$ minimal surfaces in AdS [$\Leftarrow$ AdS/CFT]

  $\Uparrow$ TBA eq. of HSG model [$\Leftarrow$ integrability]

- Underlying 2D integrable (HSG) model

  $\Rightarrow$ analytic expansions of amplitudes/Wilson loops around CFT lim. [regular polygon]

  - A_{free} $\Leftarrow$ bulk CFT perturbation
  - ΔA_{BDS} $\Leftarrow$ T–function
  - T–/Y–fn. $\Leftarrow$ g–fn. $\Leftarrow$ boundary CFT perturbation

- Rescaled remainder fn. at strong coupling, 2 loops: close

- Why minimal surface ⇔ TBA ?

  T–function ⇔ g–function (boundary entropy) ?

- Why strong coupling ⇔ 2 loops : so close ?

  [full structure of amplitudes]

- General case : AdS4 ⇐ HSG model for $\frac{\widehat{\mathrm{su}}(m-4)_4}{[\widehat{\mathrm{u}}(1)]^{m-5}}$ [m–pt.]

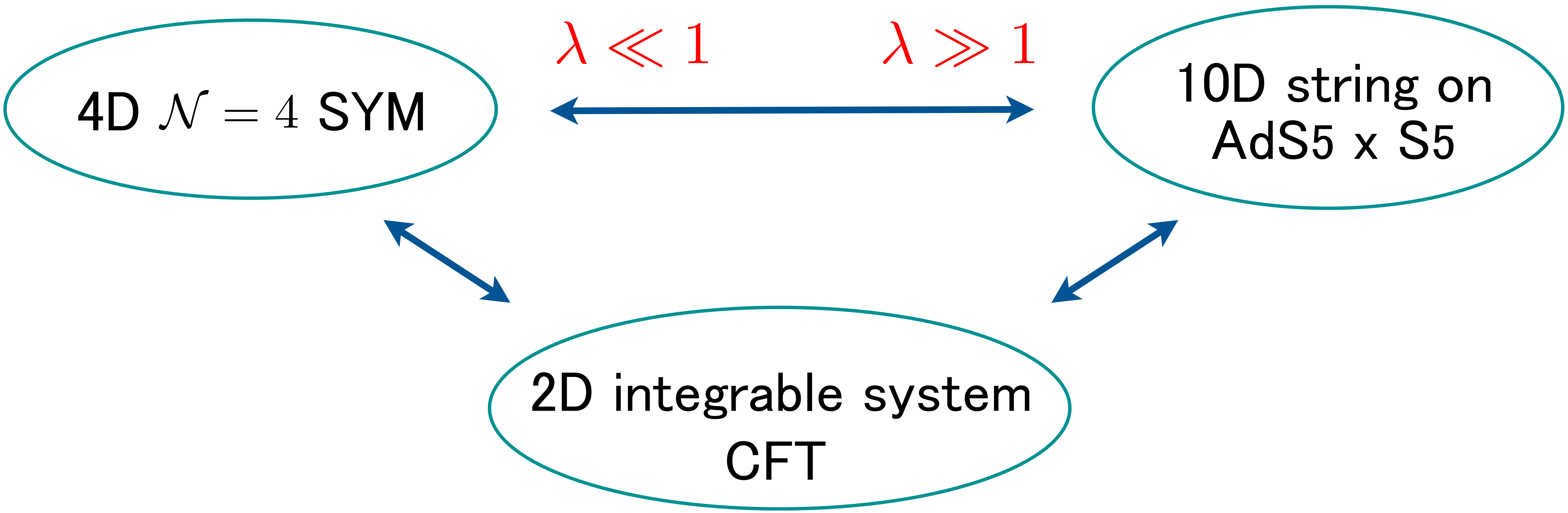
  AdS5 ⇐ ??

  - exact form of $\kappa_n G \;\Leftrightarrow\; m_s$ [mass–coupling relation]

- Strong coupling corrections ?

  [cf. Alday–Gaiotto–Maldacena–Sever–Vieira ’10]

⋮



[λ : 't Hooft coupling]