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Strange Metals and Holographic Entanglement Entropy

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① Introduction

AdS/CFT is a very powerful method to understand strongly coupled condensed matter systems.

Especially, the calculations become most tractable in the strong coupling and large N limit of gauge theories.

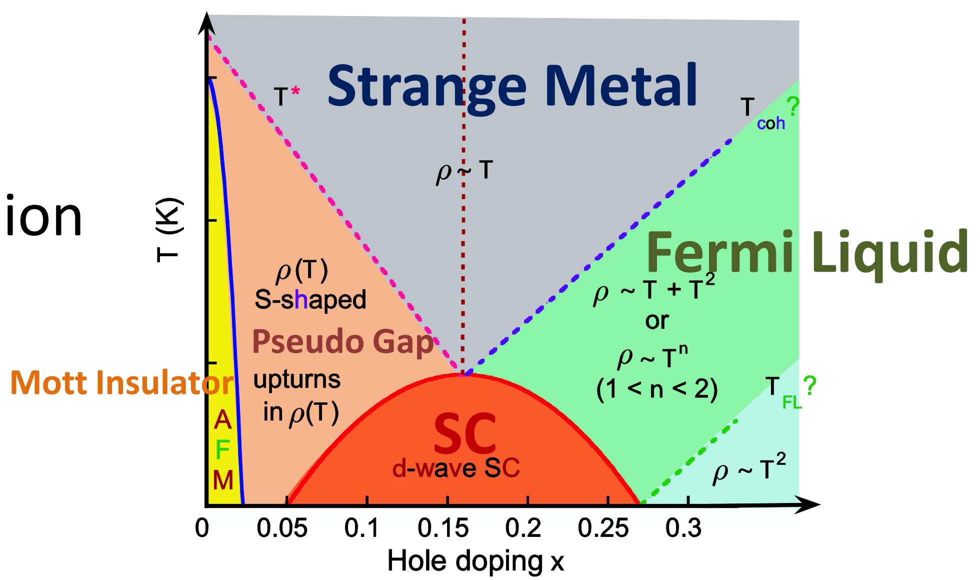
In this limit, the AdS side is given by a classical gravity and we can naturally expect universal behaviors such as the no hair theorem in GR, $\eta/s=1/4\pi$, etc.

So, we concentrate on this limit in this talk.

In this talk, we would like to consider what is a universal properties for metallic condensed matter systems via AdS/CFT.

The metals are usually described by the Landau's Fermi liquids. It is well-known that Fermi liquid states are stable against perturbations by the Coulomb forces.

However, in strongly correlated electron systems such as the strange metal phase of high Tc superconductors or heavy fermion systems etc., we encounter so called non-Fermi liquids.



Figs taken from Sachdev 0907.0008

So, the main purpose of this talk is to answer the question:

Can we obtain Landau's Fermi liquids in the classical gravity limit ?

⇒ **We will show that the answer is NO !**

Note: Several interesting setups of (non-)Fermi liquids have already been found.

(i) Probe fermions in Charged AdS BH (Emergent AdS2 in the IR)
[Faulkner-Liu-McGreevy-Vegh 2009]

(ii) Electron stars (Lifshitz metric in the IR) [Hartnoll-Tavanfar 2010]
[Confined version: Soliton Star, Bhattacharya-Ogawa-Takayanagi-Ugajin 2012]

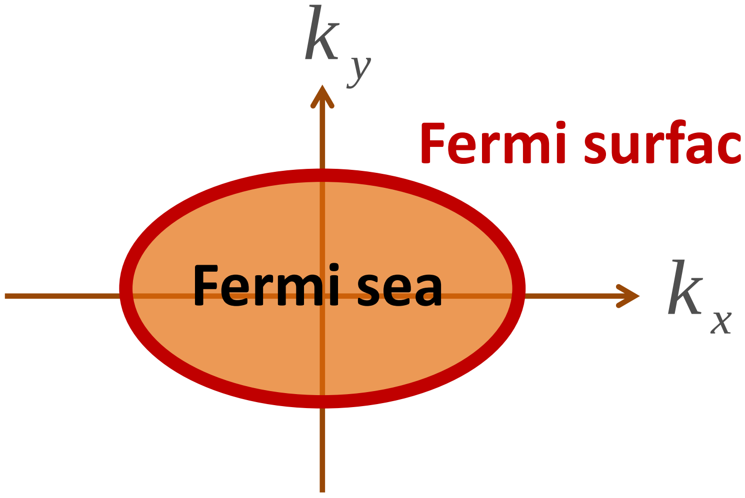
⇒ Both of them do not have any Fermi surfaces in the leading order $O(N^2)$ of large N limit !

Systems with Fermi surfaces

⇔ Fermi liquids or non-Fermi liquids

So, we concentrate on systems with Fermi surfaces.

To make the presentation simpler, we will work for 2+1 dim. systems with Fermi surfaces. But our analysis can be generalized to higher dimensions, straightforwardly.



How to characterize the Fermi surfaces ?

Metals $\Rightarrow$ Conductivity ?

But it seems difficult to find universal results for conductivity in the gravity dual. This is because it is related to the propagation of U(1) gauge fields in AdS, whose behavior largely depends on the precise Lagrangian of gauge fields e.g. $f(\phi)F^2$.

So we want to find a quantity whose gravity dual is closely related to the metric (i.e. gravity field).

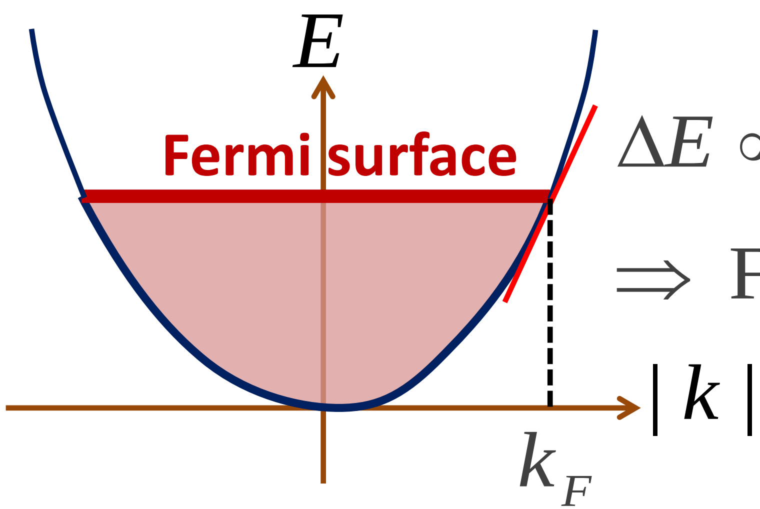
$\Rightarrow$ We should look at a thermodynamical quantity !

One traditional candidate is the specific heat C.

For (Landau's) fermi liquids, we always have the behavior

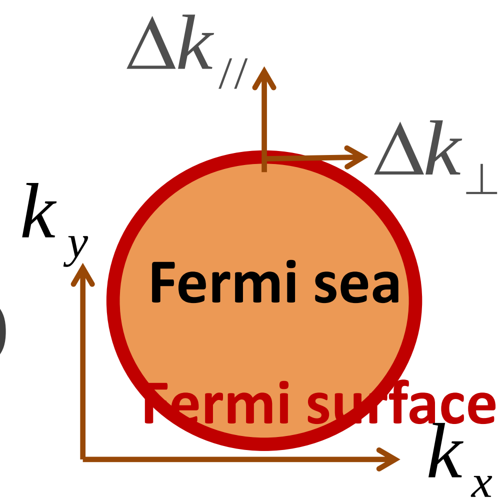
$$C \propto S \propto T \cdot V.$$

This linear specific heat can be understood if we note that we can approximate the excitations of Fermi liquids by an infinite copies of 2 dim. CFTs.



$$\Delta E \propto \Delta k_{//} + O(\Delta k_{\perp}^{2})$$

$$\Rightarrow \ \mathrm{FL} \approx \prod_{\vec{k}F} \mathrm{CFT}_2(\vec{k}_F)$$



In 2d CFT, we know

$$C \propto S \propto T \cdot L.$$

In this way, we can estimate the specific heat of the Fermi liquids

$$C \propto S \propto T \cdot L \cdot (Lk_F) = k_F \cdot T \cdot V.$$

However, the linear specific heat is not true for non-Fermi liquids.

This is because they have anomalous dynamical exponents **z**.

(~infinite copies of 2d Lifshitz theory: $(t, x) \sim (\lambda^z t, \lambda x)$)

$$C \propto S \propto T^{1/z} \cdot V.$$

To characterize the existence of Fermi surfaces, we need to look at a *property which is common to both the FL and non-FL*.

⇒ The entanglement entropy is a suitable quantity.

After we concentrate on the systems with Fermi surfaces, we can distinguish between FL and non-FL by calculating the specific heat.

This is our strategy in the present talk.

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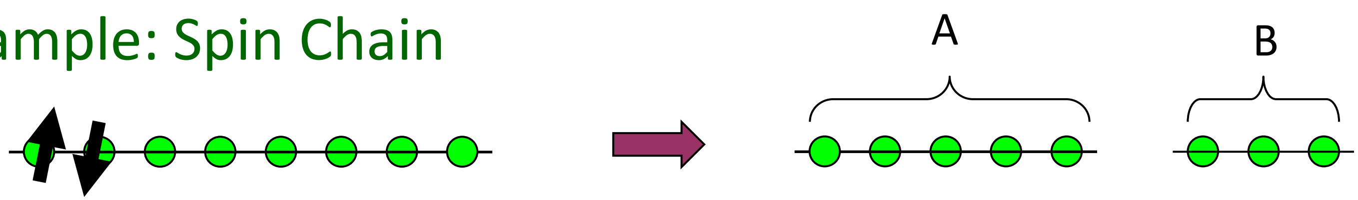
② Entanglement Entropy and Fermi surfaces

(2-1) Definition and Properties of Entanglement Entropy

Divide a quantum system
into two subsystems A and B: $H_{tot} = H_A \otimes H_B$.

Example: Spin Chain



We define the reduced density matrix ρ_A for A by

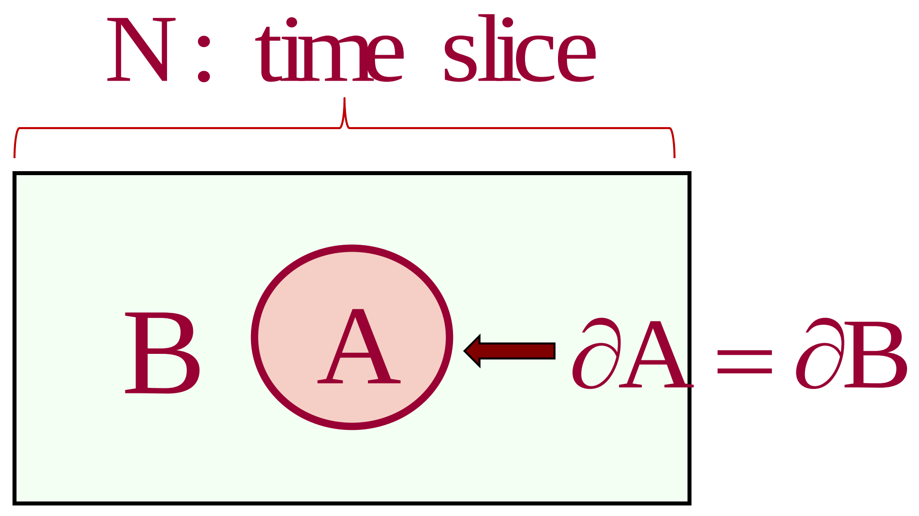
$$\rho_A = \mathrm{Tr}_B \rho_{tot} ,$$

taking trace over the Hilbert space of B .

Now the entanglement entropy S_A is defined by the von-Neumann entropy

$$S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \quad .$$

In QFTs, it is defined geometrically:



(2-2) Area law [Bombelli-Koul-Lee-Sorkin 86, Srednicki 93]

EE in QFTs includes UV divergences.

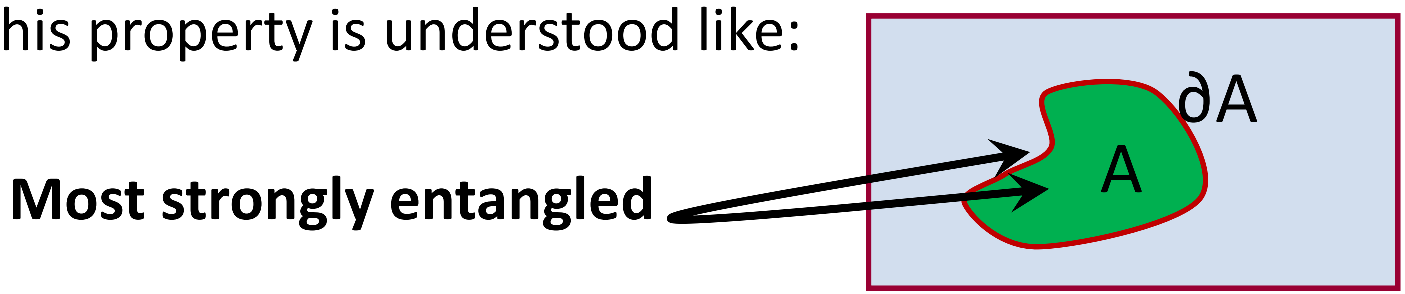
Area Law

In a (d+1) dim. QFT with a UV fixed point, the leading term of EE is proportional to the area of the (d-1) dim. boundary ∂A :

$$S_A \sim \frac{\mathrm{Area}(\partial A)}{a^{d-1}} + (\text{subleading terms}),$$

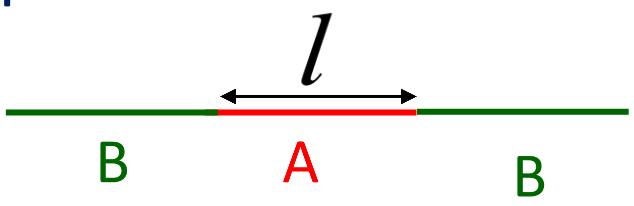
where a is a UV cutoff (i.e. lattice spacing).

Intuitively, this property is understood like:



However, there are two known exceptions:

(a) 1+1 dim. CFT $S_A = \frac{c}{3}\log\frac{l}{a}.$

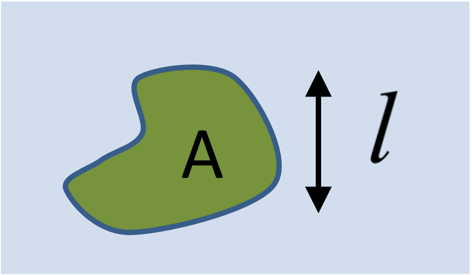


[Holzhey-Larsen-Wilczek 94, Calabrese-Cardy 04]

(b) ∃ Fermi surfaces ($k_F \sim a^{-1}$)

$$S_A \sim \left(\frac{l}{a}\right)^{d-1} \cdot \log\frac{l}{a} + \ldots$$

[Wolf 05, Gioev-Klich 05]

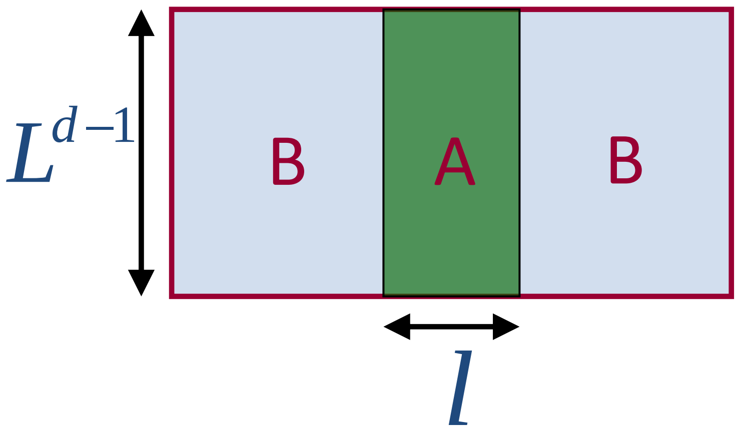


(2-3) Fermi Surfaces and Entanglement Entropy

Why do Fermi Liquids violate the area law ?

This can be understood if we remember that the Fermi liquids can be though of as infinite copies of 2d CFTs:

$$S_A \sim \underbrace{\left(\frac{L}{a}\right)^{d-1}}_{\#\text{ of 2d CFTs}} \cdot \underbrace{\log\frac{l}{a}}_{\text{Each of 2d CFT}}.$$



We will mainly assume this choice of subsystem A below.

Recently, there have been evidences that this logarithmic behavior is true also for non-Fermi liquids (e.g.spin liquids).

[Swingle 09,10, Zhang-Grover-Vishwanath 11 etc.]

Intuitively, we can naturally expect this because the logarithmic behavior does not change if we introduce the dynamical exponent z in the 2d theory as $\log l^{z} = z \log l$

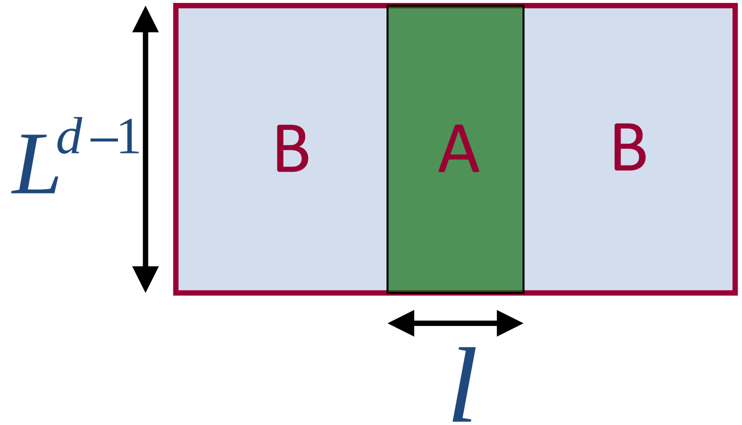
Therefore we find the characterization:

∃ Fermi surface ⇔ Logarithmic behavior of EE

To apply the AdS/CFT, we will embed the Fermi surface in a CFT. In this case, the leading divergence still satisfies the area law.
But the subleading finite term has the logarithmic behavior:

$$S_A = \gamma \cdot \frac{L^{d-1}}{a^{d-1}} + \eta \cdot (L \cdot k_F)^{d-1} \log(l \cdot k_F) + \cdots .$$

if we assume $l \cdot k_F >> 1$.



.

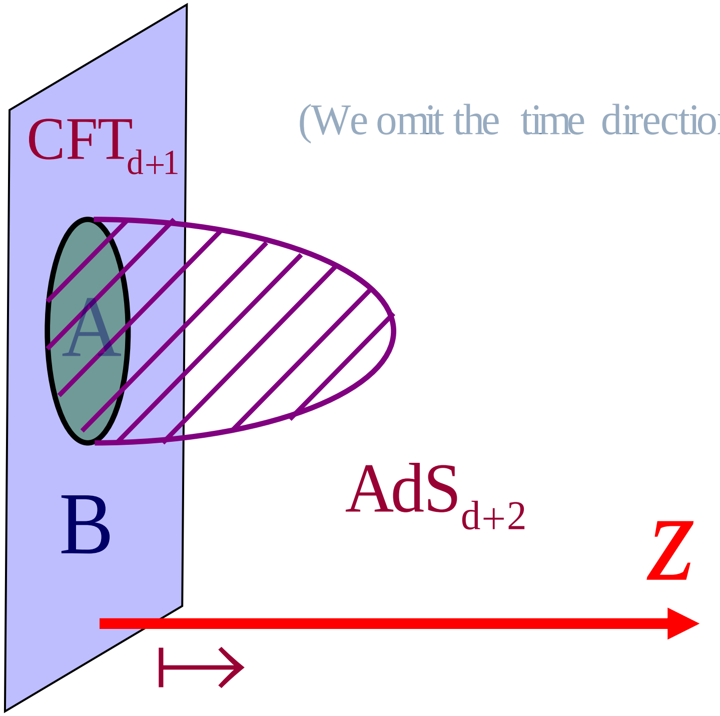
So, we will concentrate on the gravity dual whose entanglement entropy has this behavior in our arguments below.

③ Holographic Entanglement Entropy

Holographic Entanglement Entropy Formula [Ryu-TT 06]

(We omit the time direction.)

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_N}$$



γ_A is the minimal area surface (codim.=2) such that

$$\partial A = \partial \gamma_A \ \text{and} \ A \sim \gamma_A \ .$$

homologous

$z > a$ (UV cut off)

$$ds^2_{AdS} = R^2_{AdS} \frac{-dt^2 + \sum_{i=1}^{d-1} dx_i^2 + dz^2}{z^2} \ .$$

- In spite of a heuristic argument [Fursaev, 06] , there has been no complete proof. But, so many evidences and no counter examples.

[A Partial List of Evidences]

- Area law follows straightforwardly [Ryu-TT 06]
- Agreements with analytical 2d CFT results for AdS3 [Ryu-TT 06]
- Holographic proof of strong subadditivity [Headrick-TT 07]
- Consistency of 2d CFT results for disconnected subsystems [Calabrese-Cardy-Tonni 09] with our holographic formula [Headrick 10]
- Agreement on the coefficient of log term in 4d CFT (~a+c) [Ryu-TT 06, Solodukhin 08,10, Lohmayer-Neuberger-Schwimmer-Theisen 09, Dowker 10, Casini-Huerta, 10, Myers-Sinha 10]
- A direct proof when A = round ball [Casini-Hueta-Myers 11]
- Holographic proof of Cadney-Linden-Winter inequality [Hayden-Headrick-Maloney 11]

④ Fermi Surfaces and HEE [Ogawa-Ugajin-TT 11]

(4-1) Setup of gravity dual

For simplicity, we consider a general gravity dual of 2+1 dim. systems. The general metric can be written as follows (up to diff.)

$$ds^2 = \frac{R^2}{z^2}\left(-f(z)dt^2 + g(z)dz^2 + dx^2 + dy^2\right),$$

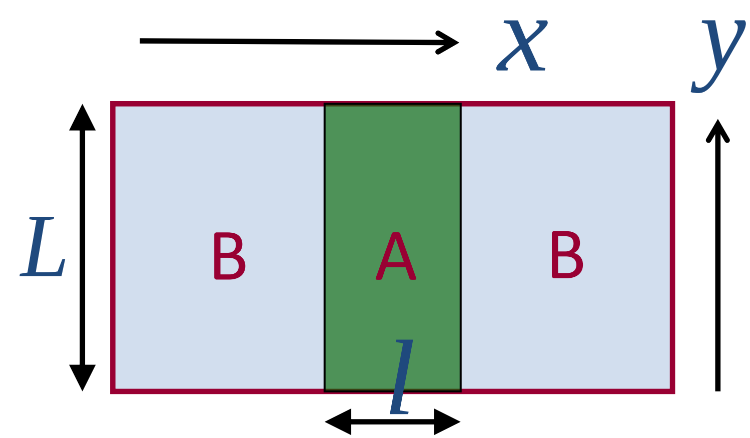
where f(z) and g(z) are arbitrary functions.
We impose that it is asymptotically AdS4 i.e.

$$f(z) \to 1 \quad \text{and} \quad g(z) \to 1 \quad \text{when} \quad z \to 0.$$

(4-2) Holographic EE

Now we would like to calculate the HEE for this gravity dual.

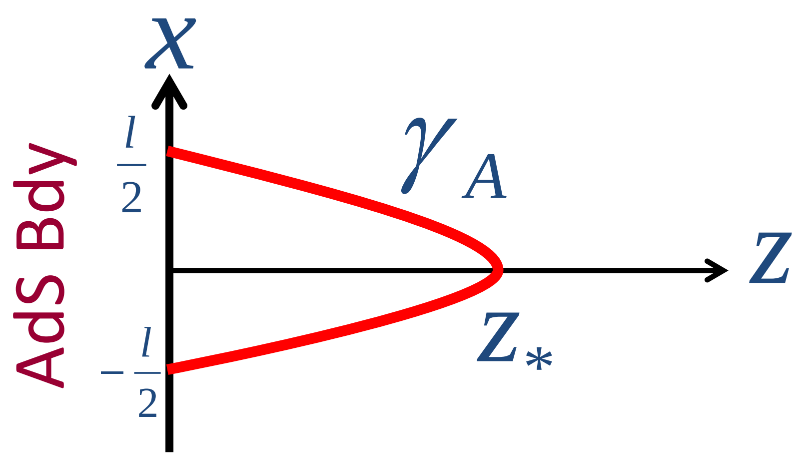
We choose the subsystem as the strip width l as before



$$A = \left\{ (x, y) \,|\, -\frac{l}{2} \le x \le \frac{l}{2},\ 0 \le y \le L \right\}$$

$$\text{Area} = 2R^2 L \int_a^{z^*} \frac{dz}{z^2} \sqrt{g(z) + x'(z)^2}.$$

the minimal surface condition reads



$$x'(z) = \frac{z^2}{z_*^2} \sqrt{\frac{g(z)}{1 - z^4 / z_*^4}}.$$

In the end, we obtain

$$S_A \approx \frac{R^2 L}{2G_N a} + k_n \frac{R^2 L}{G_N z_F} \cdot \left(\frac{l}{z_F} \right)^{\frac{n-1}{n+1}} + \ldots,$$

when the size of subsystem A is large $l >> z_F$.

In this case, the minimal surface extends to the IR region deeply.

⇒ The logarithmic behavior of EE is realized just when n=1.

$$i.e. \quad \exists \, \text{Fermi Surface} \quad \Leftrightarrow \quad g(z) \to \left(\frac{z}{z_F} \right)^2 \quad (z \to \infty) \ .$$

We identify z_F as a characteristic scale of the Fermi energy.

Note: f(z) does not affect the HEE and is still arbitrary.

(4-3) Null Energy Condition

To have a sensible holographic dual, a necessary condition is known as the null energy condition:

$$T_{\mu\nu}N^{\mu}N^{\nu} = \left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}\right)N^{\mu}N^{\nu} \geq 0$$

for any null vector N^{μ}.

In the IR region, the null energy condition argues

$$g(z) \propto z^{2}, \quad f(z) \propto z^{-2m} \quad \Rightarrow \quad m \geq 1.$$

At finite temperature, we expect that the solution is given by a black brane extension of our background:

$$ds^2 = R^2\left(-z^{-2(m+1)}h(z)dt^2 + \frac{dz^2}{\tilde{h}(z)} + \frac{dx^2 + dy^2}{z^2}\right).$$

The `non-extremal factors' behave near the horizon $z = z_H$

$$h(z) \approx \frac{z_H - z}{z_H}, \qquad \tilde{h}(z) \approx \frac{z_H - z}{z_H}.$$

From this, we can easily find the behavior of specific heat:

$$C \propto S \propto T^{\frac{2}{m+2}}.$$

Combined with the null energy condition: $m \geq 1$, we obtain

$$C \propto T^{\alpha} \quad \text{with} \quad \alpha \leq \frac{2}{3}.$$

Notice that this excludes the Landau's Fermi liquids (α=1).

In summary, we find that ***classical gravity duals only allow non-fermi liquids*.**

Comments:

(i) This result might not be so unnatural as the non-Fermi liquids are expected in strongly correlated systems.

(ii) Even in the presence of perturbative higher derivative corrections, the result does not seem to be changed.

(iii) *Some miracle coincidences ?*

AdS: No curvature singularity in the gravity dual

$\Rightarrow$ α=2/3 [11]

Shaghoulian

CMT: *Spin fluctuations:*

[Moriya, Hertz, Millis …. 70'-90']

N Fermions + U(1) gauge:

$\Rightarrow$ α=2/3 (i.e. z=3)

[Lee 09, Metlitski, and S. Sachdev 10,
Mross-McFreevy-Liu-Senthil 10,
Lawler-Barci-Fernandez-Fradkin-Oxman 06]

Experiment: $YbRh_2(Si_{1-x}Ge_x)_2$

$\Rightarrow$ α=2/3

Examples of heavy fermions

[Pepin 11, talk at KITP]

Compound	$H_c/P_c/x_c$	$\frac{C_v}{T} \to \infty ?$	$\rho \sim T^\alpha$	Reference
$YbRh_2(Si_{1-x}Ge_x)_2$	$x_c = 0.05$, $H_c^{\parallel c} = 0.66T$, $H_c^{\perp c} = 0.06T$	$T^{-0.34}$	T	Dresden, Grenoble
$CeCoIn_5$	$H_c = 5T$	$T^{-\alpha}$	T	Los Alamos, Grenoble
$Ce(Cu_{1-x}Au_x)_6$	$x_c = 0.016$	$Log\left(\frac{T_0}{T}\right)$	T	Karlsruhe
$CeCu_{6-x}Ag_x$	$x_c = 0.2$	$Log\left(\frac{T_0}{T}\right)$	$T^{1.1}$	Gainesville
$CeNi_2Ge_2$	$P_c = 0$	$Log\left(\frac{T_0}{T}\right)$	$T^{1.4}$	Karlsruhe, Cambridge
U_2Pt_2In	$P_c = 0$	$Log\left(\frac{T_0}{T}\right)$	T	Leiden
$CeCu_2Si_2$	$Pc = 0$	$Log\left(\frac{T_0}{T}\right)$	$T^{1.5}$	Dresden, Grenoble
$Ce(Ni_{1-x}Pd_x)Ge_2$	$x = 0.065$	$\gamma_0 - T^{1/2}$	$\rho_0 + T^{3/2}$	Los Alamos
$YbAgGe$	$H = 4T$	$Log\left(\frac{T_0}{T}\right)$	T	Ames, Grenoble
$CeIn_{3-x}Sn_x$	$p_c = 26kbar$	?	$T^{1.6}$	Dresden
U_2Pd_2In	$P_c < 0$	?	T	Leiden
$CePd_2Si_2$	$P_c > 0$	?	$T^{1.2}$	Karlsruhe, Dresden
$CeRhIn_5$	$P_c \sim 1.6GPa$	?	T	Los Alamos, Grenoble
$CeIn_3$	$P_c > 0$	?	$T^{1.5}$	Dresden
$Ce_{1-x}La_xRu_2Si_2$	$x_c = 0.1$	no	?	Grenoble
$U_3Ni_3Sn_4$	$P_c > 0$	no	?	Leiden

(iv) We can embed this background in an effective gravity theory:

$$S_{EMS} = \frac{1}{16G_N}\int dx^{d+2}\sqrt{-g}\left[R-2\Lambda-W(\phi)F_{\mu\nu}F^{\mu\nu}-\partial_\mu\phi\partial^\mu\phi-V(\phi)\right].$$

[This theory was already extensively studied in Charmousis et.al. 10]

if W and V behave in the large ϕ limit as follows [Ogawa-Ugajin-TT 11]

$$V(\phi)+2\Lambda \approx -\frac{(p^2+12p+32)}{4R_{AdS}^2}\cdot e^{-\sqrt{\frac{2}{p-2}}\phi},$$

$$W(\phi) \approx \frac{8A^2}{z_F^2\, p(8+p)R^2} e^{3\sqrt{\frac{2}{(p-2)}}\phi},$$

$$\Rightarrow f(z) \propto z^{-p}, \quad g(z) \propto z^2, \qquad (p>2).$$

[A solution in the p=2 case has been found in Shaghoulian 11]

(v) This metric can also be regarded as a generalization of Lifshitz backgrounds so that it violates the hyperscaling.

[Huijse-Sachdev-Swingle 11, Dong-Harrison-Kachru-Torroba-Wang 12]

$$ds^2_{(d+2)} = r^{-(d-\theta)}\left(-r^{-2(z-1)}dt^2 + dr^2 + \sum_{i=1}^{d} dx_i^2\right) \quad .$$

$$\Rightarrow C \propto S \propto T^{(d-\theta)/z}.$$

$$d-1 < \theta < d: \quad S_A \sim \mathrm{L}^q, \quad d\text{-}1 < q < d \quad \rightarrow \text{ Violation of Area law}$$

$$\theta = d\text{-}1 \quad : \quad S_A \sim (L)^{d\text{-}1}\log L \quad \text{Fermi surface}$$

$$0 < \theta < d\text{-}1 \quad : \quad S_A \sim \mathrm{L}^q, \quad 0 < q < d-1$$

⑤ Conclusions

- The entanglement entropy (EE) is a useful bridge between gravity (string theory) and cond-mat physics.

Gravity ⟺ Entanglement ⟺ Cond-mat. systems

$g_{\mu\nu}$ $\quad S_A \approx \text{Area}$ $\quad |\Psi\rangle$

- Classical gravity duals + Null energy condition
  ⇒ a constraint on specific heat $C \propto T^{\alpha}$ with $\alpha \leq \frac{2}{3}$.
  ⇒ Non-fermi liquids !

- Questions: Any string theory embeddings of the NFL b.g. ?
  Can we see Fermi surfaces more directly ? (Maybe smeared ?)

[see also Hartnoll- Shaghoulian 12]