Residue Formulas for Prepotentials, Instanton Expansions and Conformal Blocks

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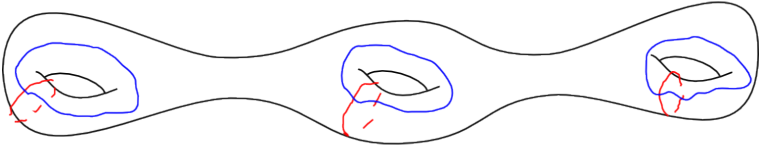
Generalized Seiberg-Witten system

- g-parametric family of the genus-g curves Σ: $F(x, z; u_1, ..., u_g) = 0$ with marked cycles $\{A_i, B_i\} \in H_1(\Sigma, \mathbb{Z})$



- Seiberg-Witten 1-form $dS = xdz$
- Such connection ∇ that $\nabla_{\frac{\partial}{\partial u_i}} dS$ are holomorphic

Generalized Seiberg-Witten equations (Krichever(92), Seiberg-Witten(94),...)

$$a_i = \frac{1}{2\pi i} \int_{A_i} dS, \qquad a_i^D = \int_{B_i} dS = \frac{\partial \mathcal{F}}{\partial a_i} \qquad i = 1, \ldots, g$$

Generalized Seiberg-Witten system

$$\frac{\partial \mathcal{F}}{\partial q_i} = \frac{1}{2} \sum_{p^{-1}(q_i)} \mathrm{Res} \frac{(dS)^2}{dz}$$

Where q_i are the poles of 2-differential in the curve definition in the Gaiotto description of the quiver gauge theories.

$$x^2 = \phi_2(z)$$

$$dS = xdz$$

Where

$$\phi_2(z)dz^2 \approx \frac{m_i^2 dz^2}{(z - z_i)^2} + \frac{c_i(u_1, ..., u_g)dz^2}{z - z_i} + reg.$$

m_i are mass parameters

$$\{z_i\} = \{z_1, ..., z_{g+3}\} \to \{0, 1, \infty, q_1, ..., q_g\}$$

3-rd dervative residue formula and WDVV equation

Theorem:(Krichever, P.G.-A.M.) Extended SW prepotential satisfies an identity

$$\boxed{\frac{\partial^3\mathcal{F}}{\partial X_I \partial X_J \partial X_K} = \sum_{dx=0} \mathrm{Res} \frac{d\varpi_I d\varpi_J d\varpi_K}{dx dz}}$$

where $\{X_I\} = \{a_i\} \cup \{q_k\}$ and $\{d\varpi_I\} = \{d\omega_i\} \cup \{d\Omega_k\}$

Here $d\omega_i = \frac{\partial dS}{\partial a_i}$ are holomorphic, and $d\Omega_i = \frac{\partial dS}{\partial q_i}$ are meromorphic.

Theorem: (Consequence of the previous one): Prepotential of the quiver gauge theory satisfies WDVV equation (Marshakov-Mironov-Morozov, 96)

$$\mathcal{F}_I \mathcal{F}_J^{-1} \mathcal{F}_K = \mathcal{F}_K \mathcal{F}_J^{-1} \mathcal{F}_I$$

Where matrices $(\mathcal{F}_I)_{JK} = \frac{\partial^3\mathcal{F}}{\partial X_I \partial X_J \partial X_K}$

Proof of the WDVV relation (point-counting argument)

Suppose that we have the formula

$$\mathcal{F}_{IJK} = \sum_{f(z)=0} \mathrm{res} \frac{r_I(z) r_J(z) r_K(z)}{f(z)} R(z) dz$$

Where **the number of points** $f(\lambda_i) = 0$ **equals to the number of indices** (Marshakov '00)
Then introduce (commutative) algebra H_S with the multiplication $\underset{S}{*}$

$$(r_I \underset{S}{*} r_J)(\lambda_i) = S(\lambda_i) r_I(\lambda_i) r_J(\lambda_i)$$

The homomorphism

$$I_S(r) = \sum_{i=1}^{N} \frac{R(\lambda_i)}{f'(\lambda_i) S(\lambda_i)^2} r(\lambda_i)$$

Proof of the WDVV relation (point-counting argument)

And the non-degenerate bilinear form

$$\eta_{IJ} = I_S(r_I * r_J)$$

Then

$$\mathcal{F}_{IJK} = (\mathcal{F}_I)_{JK} = I_S(r_I * r_J * r_K)$$

Associativity of H_S:

$$\mathcal{F}_I \eta^{-1} \mathcal{F}_K = \mathcal{F}_K \eta^{-1} \mathcal{F}_I$$

We choose $S(\lambda_i) = \frac{1}{r_J(\lambda_i)}$, then $r_J * = id$ and $\eta = \mathcal{F}_J$

So

$$\boxed{\mathcal{F}_I \mathcal{F}_J^{-1} \mathcal{F}_K = \mathcal{F}_K \mathcal{F}_J^{-1} \mathcal{F}_I}$$

WDVV equation in the quiver gauge theories

- General SU(2) quiver theory

$$x^2 = \frac{P_{g-1}(z)}{z(z-1)\prod_{i=1}^{g}(z-q_i)}$$

The set

$$\{X_I\} = \{a_1, ..., a_g, q_1, ..., q_g\}$$

The number of critical poinrs is $2g$.

- Constrained Zamolodchikov case

$$x^2 = \frac{Q_{\tilde{g}-1}^2(z)}{z(z-1)\prod_{i=1}^{2\tilde{g}-1}(z-q_i)}$$

The set

$$\{X_I\} = \{a_1, ..., a_{\tilde{g}}, q_1, ..., q_{2\tilde{g}-1}\}$$

The number of critical points is $3\tilde{g}-1$

- WDVV is satisfied in these both cases!

Relations for the period matrix

In the constrained Zamolodchikov case

$$\mathcal{F}(\mathbf{a},\mathbf{q}) = \frac{1}{2}\sum_{i,j=1}^{g} a_i \mathcal{T}_{ij}(\mathbf{q}) a_j$$

All relations — relations for the period matrix (**a** are parameters):

- WDVV
- Rauch formula
- 3-rd derivative residue
- Their consequences

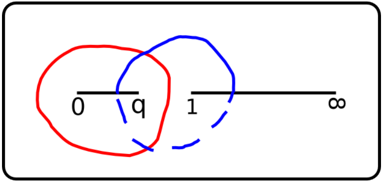
Further directions

- Field-theoretic calculations for the half-trifundamental
- $SU(N)$ quiver theories
- New exact conformal blocks (for W_N algebras) and simple check of the AGT relation in Zamolodchikov-like limit.
- By the Gamayun-Iorgov-Lisovy hypothesis (theorem for $N = 2$) each exact conformal block can give the corresponding isomonodromic τ-function. So we can try to generalize this hypothesis to $N > 2$: $SL(N)$ case at the isomonodromic side and W_N case at the CFT side.

Thank you for your attention!

Equation: $N_c = 2$, $N_f = 4$ theory

$$x^2 = \frac{q(q-1)\frac{\partial \mathcal{F}}{\partial q}}{z(z-1)(z-q)}$$



$$\boxed{\{\mathcal{F}, q\} = \frac{1-q+q^2}{2q^2(q-1)^2}}$$

$$\mathcal{F}(a,q) = -\pi a^2 \frac{K(1-q)}{K(q)}$$

Equation: $N_c = 2$, $N_f = 4$, 1 mass deformation

$$x^2 = \frac{zq(q-1)\frac{\partial \mathcal{F}}{\partial q} - (z-q)m^2}{z^2(z-1)(z-q)}$$

$$\mathcal{F}''' +$$
$$+\frac{m^2\mathcal{F}'[3q(2-3q)\mathcal{F}'' + 2(1-3q)\mathcal{F}'] - [3q^4(q-1)^2\mathcal{F}'^2\mathcal{F}''^2 + q^2(q^2-q+1)\mathcal{F}'^4]}{2q^2(q-1)\mathcal{F}'[q^2(q-1)\mathcal{F}'^2 + m^2q(q-2)\mathcal{F}' - m^4]} -$$
$$-\frac{m^4[(q-1)^2q^2\mathcal{F}''^3 + 6q(q-1)^2\mathcal{F}'\mathcal{F}''^2 + 3(q^2+q-1)\mathcal{F}''\mathcal{F}'^2 + (3+2q)\mathcal{F}'^3]}{2(q-1)\mathcal{F}'[q^2(q-1)\mathcal{F}'^2 + m^2q(q-2)\mathcal{F}' - m^4]} = 0$$

Equation: $N_c = 2$, $N_f = 4$, 1 mass deformation

Computations ising the non-linear equation:

$$\mathcal{F} = \mathcal{F}_{\mathrm{pert}}(a;m) + (a^2 - m^2)\log q + \frac{a^2-m^2}{2}q + \frac{13a^4-14a^2m^2+m^4}{64a^2}q^2 + \ldots$$

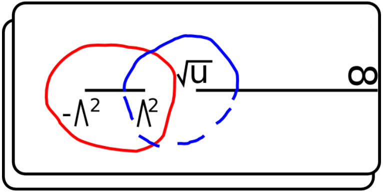
Nekrasov-Lukyanov-Litvinov-Zamolodchikov computation of the quasiclassical conformal block:

$$\mathcal{F}_{\epsilon_1=0} = a^2\log q + \frac{a^2+m_1^2-m_\infty^2}{2}q +$$
$$\left(\frac{a^2+m_1^2-m_\infty^2}{4} + + \frac{a^4+2a^2(m_1^2+m_\infty^2)-3(m_1^2-m_\infty^2)^2}{64(a^2+\frac{3}{4})} - \frac{a^4-(m_1^2-m_\infty^2)^2}{16a^2}\right)q^2 + \ldots$$

Answers differes by the integer numbers like $\frac{3}{4}$ and coincide in the $a \to \infty$ limit.

Equation: $N_c = 2$, $N_f = 0$

$$dS = \sqrt{\frac{2(u-x)}{x^2 - \Lambda^4}}dx$$



$$2\Lambda^2\left(\left(\frac{\partial \mathcal{F}}{\partial \Lambda}\right)^2 - 16\Lambda^2\right)\frac{\partial^3 \mathcal{F}}{\partial \Lambda^3} + \left(\Lambda\frac{\partial^2 \mathcal{F}}{\partial \Lambda^2} - \frac{\partial \mathcal{F}}{\partial \Lambda}\right)^3 = 0$$

The known equation (M. Matone), equivalent to the hypergeometric one.