Factorization of orbifold partition functions

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arXiv:1208.1404 Y.I., D.Yokoyama

arXiv:1311.2371 Y.I., D.Yokoyama, and H.Matsuno

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Partition function

$$Z = \int \mathcal{D}\Phi \, e^{-\int_M \mathcal{L}}$$

M : background manifold

exactly calculable by localization

②

These are useful to check dualities.

$$Z(\text{theory A}) = Z(\text{theory B})$$

- strong evidence for the duality.
- numerical factors are important.

Q. To what extent can we fix the numerical factor?

(Today we consider 3d $\mathcal{N}=2$ SUSY theories)

③

A simple way to fix the normalization of Z_{S^3}

= dimensional reduction

4d theory in $S^1(\text{time}) \times S^3$

$\downarrow$ $\beta \to 0$ limit

3d theory in S^3

Dolan, Spiridonov, Vartanov 1104.1787

Gadde, Yan 1104.2592

Y.I. 1104.4882

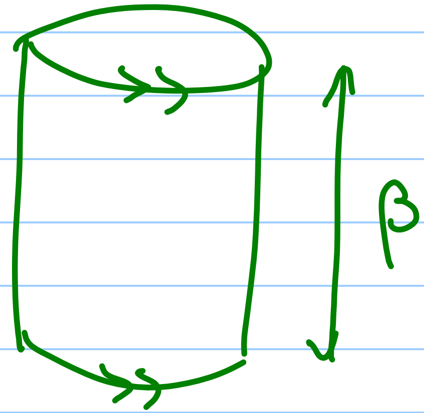
This determines Z_{S^3} unambiguously if

? no ambiguity in $Z_{S^3 \times S^1}$

✓ no divergences in $\beta \to 0$

Superconformal index

$$Z_{S^3\times S^1} = I = \int \mathcal{D}\Phi \, e^{-\int_{S^1\times S^3} \mathcal{L}\, d^4x}$$



$$= \mathrm{tr}\left[(-1)^F e^{-\beta H + \cdots}\right]$$

$$= \sum_n \langle n| (-1)^F e^{-\beta H + \cdots} |n\rangle$$

natural normalization : $\langle n|n\rangle = 1$

We still have an ambiguity

$$(-1)^F |0\rangle = \pm |0\rangle$$

In general, we cannot uniquely determine the statistics of $|0\rangle$.

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In the literature, the sign of Z is often neglected, and only the absolute value is focused on.

However,

There is a case we need to fix the sign.
If many sectors contribute to Z (or I)

$$Z_{tot} = Z_1 + Z_2 + \cdots + Z_n$$

we need to fix relative signs of Z_i.

⑥

This is the case if $\mathcal{M} = S^3/\mathbb{Z}_n$

This space has the fundamental group

$$\pi_1(S^3/\mathbb{Z}_n) = \mathbb{Z}_n$$

U(1) gauge theory on $\underset{\substack{\parallel \\ \hphantom{x}}}{\mathcal{M}}$ ($= S^3/\mathbb{Z}_n$) has n degenerate vacua labeled by

holonomy $h \equiv \frac{n}{2\pi} \oint_\gamma A = 0, 1, \cdots, n-1$

γ : generator of $\pi_1(\mathcal{M})$

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partition function for $\mathcal{M} = S^3/\mathbb{Z}_n$ is

$$Z = \underbrace{Z_{h=0} + Z_{h=1} + \cdots + Z_{h=n-1}}_{\text{relative signs are important.}}$$

How can we fix the relative signs?

We need an extra criterion.

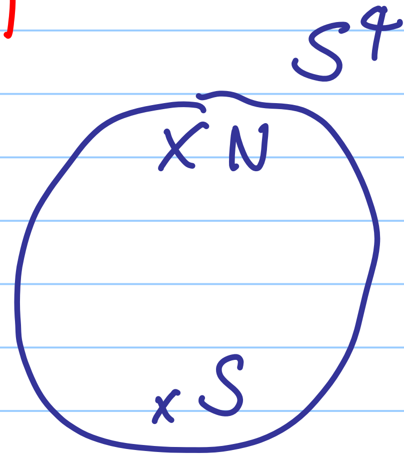
Remember, in the case of S^4, ...

Coulomb moduli

$$Z = \sum_{n_N, n_S} \underbrace{c(n_N, n_S)}_{\substack{\text{potential}\\ \text{relative}\\ \text{weights}}} \int da\, Z_N(n_N, a)\, Z_S(n_S, a)$$

n_N : inst at N

n_S : anti-inst at S



$\downarrow$ assume "factorization"

$$Z = \int da \left(\sum_{n_N} c_N(n_N)\, Z_N(n_N, a)\right)\left(\sum_{n_S} c_S(n_S)\, Z_S(n_S, a)\right)$$

$$c(n_N, n_S) = c_N(n_N)\, c_S(n_S)$$

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$\leftarrow$ Total inst. number $n_N - n_S$.

(n_N, n_S)
(n_N+1, n_S+1) $\Big\}$ in the same component of configuration space.

$\rightarrow$ the same weight factor (if we appropriately define $Z_{N/S}$)

$$C_N(n_N)C_S(n_S) = C_N(n_N+1)C_S(n_S+1)$$

$$\rightarrow \frac{C_N(n_N+1)}{C_N(n_N)} = \frac{C_S(n_S)}{C_S(n_S+1)} = a$$

$$\rightarrow \begin{cases} C_N(n_N) = C_N(0)\,a^{n_N} \\ C_S(n_S) = C_S(0)\,a^{-n_S} \end{cases}$$

$$Z = c(0,0) \sum_{n_N, n_S} a^{n_N - n_S} \int da\, Z_N(n_N, a)\, Z_S(n_B, a)$$

The relative phase is identified with the θ term.

$$a^{n_N - n_S} = \exp \frac{i\theta}{4\pi} \int_{S^4} \mathrm{tr}\, F \wedge F$$

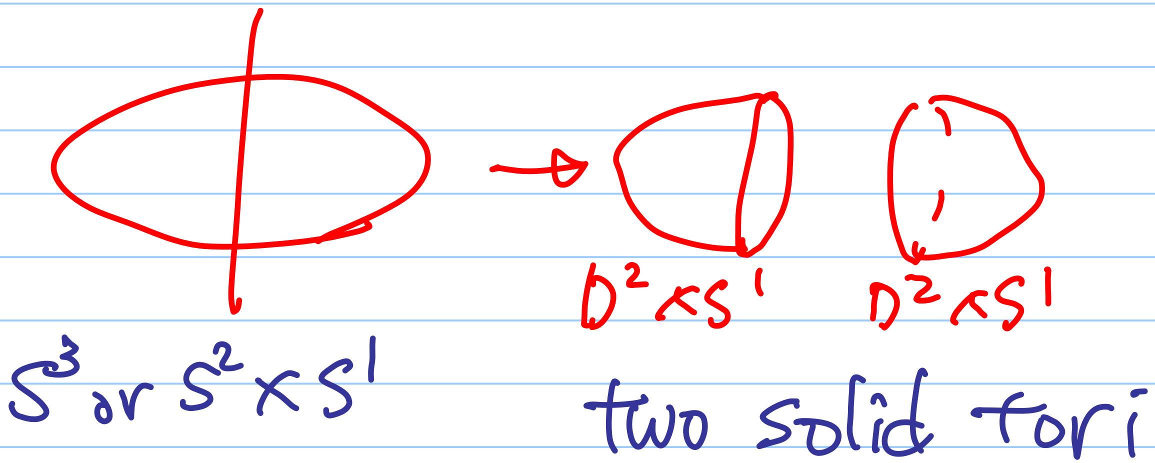
Lesson

$$\boxed{\text{The factorization strongly restricts the weight factor.}}$$

Let us use the factorization to determine relative weights in 3d partition function. (signs)

Factorization in 3d

Both S^3 and $S^2 \times S^1$ are decomposed into a pair of $D^2 \times S^1$. (Heegaard decomposition)



Correspondingly, both Z_{S^3} and $Z_{S^3\times S^1} = I$ are factorized into the same building blocks. (holomorphic blocks)

S^3

$$Z_{S^3}(\mu_i; b) = B(x_i; q)\, B(\tilde{x}_i; \tilde{q})$$

Pasquetti 1111.6905

$q = e^{2\pi i b^2}$ $\quad \tilde{q} = e^{2\pi i b^{-2}}$

$x_i = e^{2\pi b \mu_i}$ $\quad \tilde{x}_i = e^{2\pi b^{-1}\mu_i}$

$S^1\times S^2$

$$I(z_i, m_i; q) = B(x_i; q)\, B(\tilde{x}_i; \tilde{q})$$

Dimofte, Gaiotto, Gukov 1112.5179

$\tilde{q} = q^{-1}$

$x_i = q^{\frac{m_i}{2}} z$ $\quad \tilde{x}_i = q^{\frac{m_i}{2}} z^{-1}$

The different gluing $\to$ relations among variables
= gluing relations

B : partition func. on a solid torus

Remark

In general, Z_{S^3} and I are given by

$$Z_{S^3} = \sum_\alpha B^\alpha B^\alpha \qquad I = \sum_\alpha B^\alpha B^\alpha$$

where α labels the Higgs vacua of the theory.

$\left(\sum_\alpha \sim \int da \text{ in 4d} \right)$ $\Rightarrow$ Honda-san's talk.

$$\boxed{\text{Here we only consider theories with unique Higgs vacuum.}}$$

Example

The S^3 partition function of Δ is

theory of a single chiral mult.

real mass

$Q = b + b^{-1}$ (squashing param.)

$$Z_\Delta = \exp\left(\frac{\pi i}{2}\left(\mu - \frac{iQ}{2}\right)^2\right) S_b\left(-\mu + \frac{iQ}{2}\right)$$

$$= \underbrace{(qx^{-1}; q)}\underbrace{(\tilde{q}\tilde{x}^{-1}; \tilde{q})}$$

holomorphic blocks

q-Pochhammer symbol

$$(z; q) = \prod_{k=0}^{\infty}(1 - q^k z)$$

where we introduced the variables

$$q = e^{2\pi i b^2} \quad \tilde{q} = e^{2\pi i b^{-2}} \quad x = e^{2\pi b\mu} \quad \tilde{x} = e^{2\pi b^{-1}\mu}$$

$S^3/\mathbb{Z}_n$ can also be decomposed into two solid tori.

$\Rightarrow$ Factorization is expected.

Indeed, we showed

$$B(x, q)\, B(\tilde{x}; \tilde{q}) = \pm\, Z_{S^3/\mathbb{Z}_n}$$

$$q = \omega\, e^{\frac{2\pi i b^2}{n}}$$

$$x = \omega^{h}\, e^{\frac{2\pi b \mu}{n}}$$

$$\tilde{q} = \omega\, e^{\frac{2\pi i b^{-2}}{n}}$$

$$\tilde{x} = \omega^{-h}\, e^{\frac{2\pi b^{-1} \mu}{n}}$$

factorization requires extra sign.

computed by a known formula in the literature

Extra sign factor for each chiral mult.

$$\rho(h) = \exp\left[\frac{\pi i}{2n}\left([h](n-[h]) - (n-1)h^2\right)\right] = \pm 1$$

$$0 \leq [h] < n, \quad [h] \equiv h \bmod n$$

h : holonomy of the gauge field coupling to the chiral mult.

$\boxed{\text{Main result}}$

The factorization determines the relative weights of holonomy sectors.

Remarks.

- I do not claim that this is the only choice of $\mathcal{I}$ that gives a consistent theory.
- I just claim that if we assume the factorization the relative sines are uniquely determined.
- There may be other "consistent" choices of $\mathcal{I}$.
- However, it is interesting that this choice is compatible with some dualities.

Duality.

Hori et.al. 9702154

XYZ-model		SQED $(N_f=1)$
no gauge group	dual $\Longleftrightarrow$	$G=U(1)$
chiral multiplets X, Y, Z		chiral multiplets $Q, \tilde{Q}$
$W = XYZ$		$W=0$

$$Z_{XYZ} = Z_{SQED}$$

$$\equiv Z_{h=1} + \cdots + Z_{h=n-1}$$

Numerical analysis

parameters

$$\boxed{b = e^{0.2i}, \quad \Delta = 0.3 \\ h_{top} = 0, \ h_A = 1}$$

$S^3/\mathbb{Z}_3 \quad (n=3)$

Z^{SQED}	$h=0$	$0.32172 + 0.02704\,i$
	$h=1$	$0.40491 - 0.02862\,i$
	$h=2$	$0.40491 - 0.02862\,i$

Z^{XYZ} $\quad 0.48809 - 0.08429\,i$

Not agree

Numerical analysis

parameters
$b = e^{0.2i}$, $\Delta = 0.3$
$h_{top} = 0$, $h_A = 1$

$S^3/\mathbb{Z}_3$ $(n=3)$

Z^{SQED}

$h=0$ $-$ $0.32172 + 0.02704\,i$

$h=1$ $+$ $0.40491 - 0.02862\,i$

$h=2$ $+$ $0.40491 - 0.02862\,i$

Z^{XYZ} $0.48809 - 0.08429\,i$

agree!!

same as the signs required by the factorization.

Conclusions

- Factorization of $Z_{S^3/\mathbb{Z}_n} \Rightarrow$ relative signs of Z_h determined

  (We have checked only for non-gauge theories and a few examples of gauge theories.)

- We have checked compatibility of the I with some dualities.

$\Rightarrow$ There is a "special choice" of I.

(But I'm not sure if we must always take it.)