Three-point Functions in $\mathcal{N} = 4$ SYM at strong coupling

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Based on [arXiv:1312.3727] with Yoichi Kazama

JSPS/RFBR workshop

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$\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence:

$$\mathcal{N} = 4 \text{ SYM} \quad \overset{?}{\longleftrightarrow} \quad \begin{array}{l}\text{String theory} \\ \text{on } AdS_5 \times S^5\end{array}$$

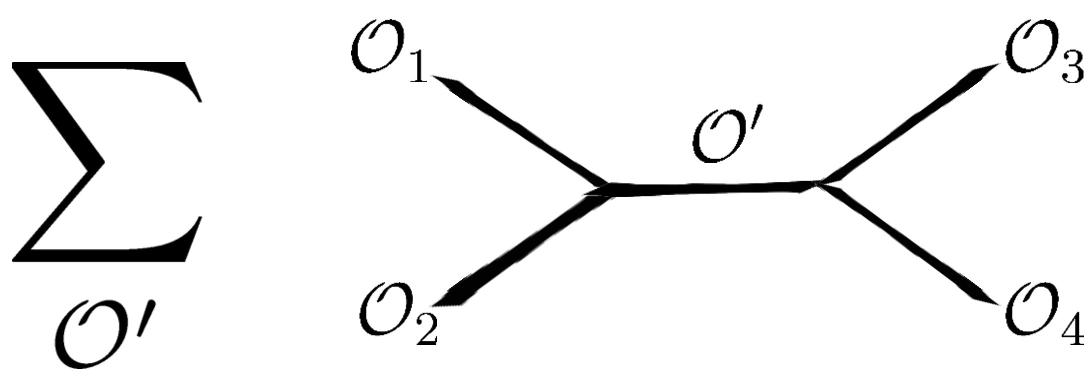
Goal of this talk:

Compute **non-BPS** three-point functions **at strong coupling** using the **(semi-)classical string** theory and **integrability**

"Heavy-Heavy-Heavy"

Introduction and Motivation

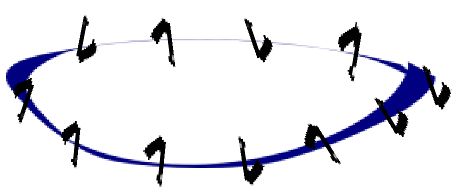
- Currently, we have no satisfactory understanding of AdS/CFT.
- It would be instructive to study in detail how the building blocks of the two theories are related with each other.
- The building blocks of $\mathcal{N}=4$ SYM are **two-** and **three-**point functions. (One can decompose any higher-point functions using the OPE.)



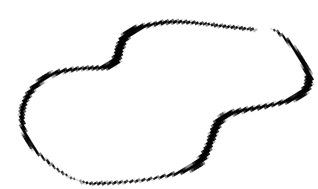
Two-point functions

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\rangle = \frac{\delta_{ij}}{|x_1 - x_2|^{2\Delta_i}} \,.$$

- $\Delta \longrightarrow$ Energy of a spin chain at weak coupling ($\lambda \ll 1$).
- $\Delta \longrightarrow$ Energy of a string at strong coupling ($\lambda \gg 1$).
- TBA, quantum spectral curve etc.



$\lambda \ll 1$

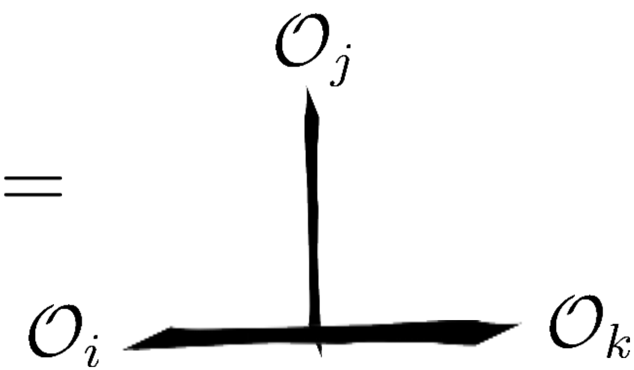


$\lambda \gg 1$

Three-point functions

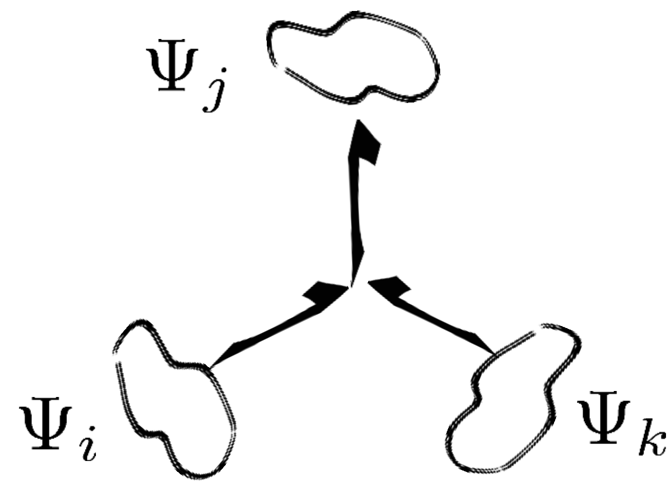
$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle = \frac{C_{ijk}}{|x_1-x_2|^{\Delta_1+\Delta_2-\Delta_3}|x_2-x_3|^{\Delta_2+\Delta_3-\Delta_1}|x_3-x_1|^{\Delta_3+\Delta_1-\Delta_2}}\,.$$

$C_{ijk} =$



gauge

$\leftrightarrow$



string

Simplest class of 3-pt functions at weak coupling

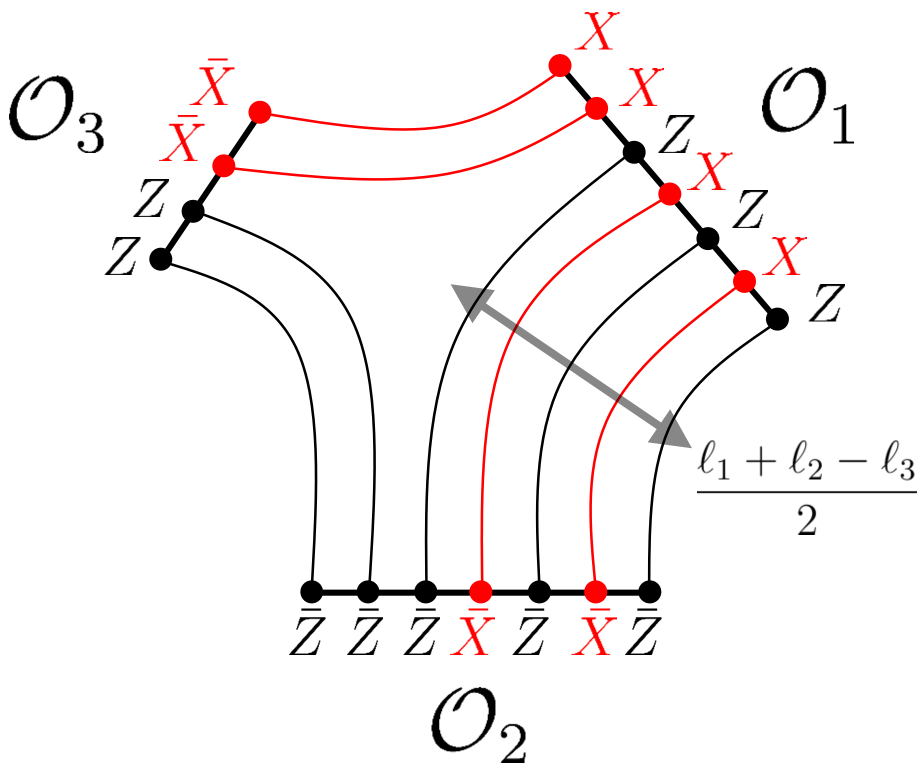
"SU(2)-sector"

$$\mathcal{O}_1 = \mathrm{tr}\,(Z Z X Z \cdots) + \cdots$$

$$\mathcal{O}_2 = \mathrm{tr}\,(\bar{X} \bar{Z} \bar{Z} \bar{Z} \cdots) + \cdots$$

$$\mathcal{O}_3 = \mathrm{tr}\,(Z \bar{X} Z Z \cdots) + \cdots$$

Structure constant from the Wick contraction



- No crossed contractions. (Suppressed by $1/N$)
- Simple in principle, Complicated combinatorics.
- Efficiently computed by the overlaps of the spin-chain states.

[Okuyama, Tseng] [Roiban, Volovich] [Alday, David, Gava, Narrain]
[Escobedo, Gromov, Sever, Vieira] [Kostov] [Foda] [Serban]

Three long operators [Kostov]

$$\ln C_{123} \sim \oint \frac{dx}{2\pi} \mathrm{Li}_2\left(e^{ip_1(x)+p_2(x)+i\ell_3/2x}\right) + \oint \frac{dx}{2\pi} \mathrm{Li}_2\left(e^{ip_3(x)+i(\ell_2-\ell_1)/2x}\right)$$

$$- \frac{1}{2} \sum_{j=1}^{3} \oint \frac{dx}{2\pi} \mathrm{Li}_2\left(e^{2ip_j(x)}\right)$$

The main point of the talk:

Similar dilogarithm expressions appear also from the computation at strong coupling using classical strings.

Three-point functions at strong coupling

$$\langle \mathcal{O}_i(x_1)\mathcal{O}_j(x_2)\mathcal{O}_k(x_3)\rangle_{\text{gauge}} = \langle \mathcal{V}_i\mathcal{V}_j\mathcal{V}_k\rangle_{\text{worldsheet}}$$

Three-point functions at strong coupling

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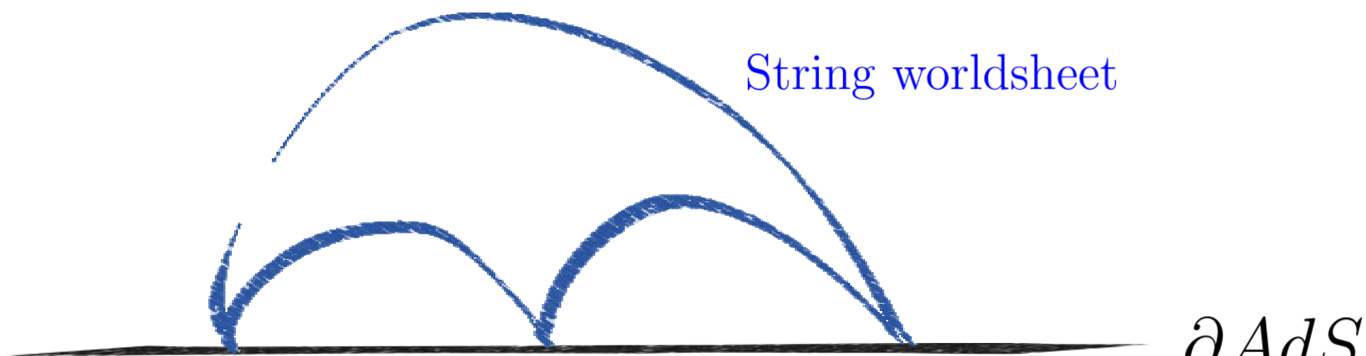
at strong coupling $(\lambda \to \infty)$

$$\mathcal{V}_i[X_*]\mathcal{V}_j[X_*]\mathcal{V}_k[X_*]e^{-\sqrt{\lambda}S_{\text{action}}[X_*]}$$



$*\quad \langle \mathcal{V}_i\mathcal{V}_j\mathcal{V}_k\rangle \overset{\lambda\to\infty}{\sim} F_{\text{AdS}} \cdot F_{\text{S}}$ [This talk]

[Janik, Wereszczynski]

Two difficulties

$$\boxed{\mathcal{V}_i[X_*]\mathcal{V}_j[X_*]\mathcal{V}_k[X_*]e^{-S_{\rm action}[X_*]}}$$

1. It is very hard to construct the saddle point X_*.

2. We do not know the form of the vertex operators $\mathcal{V}_i$.

Two difficulties

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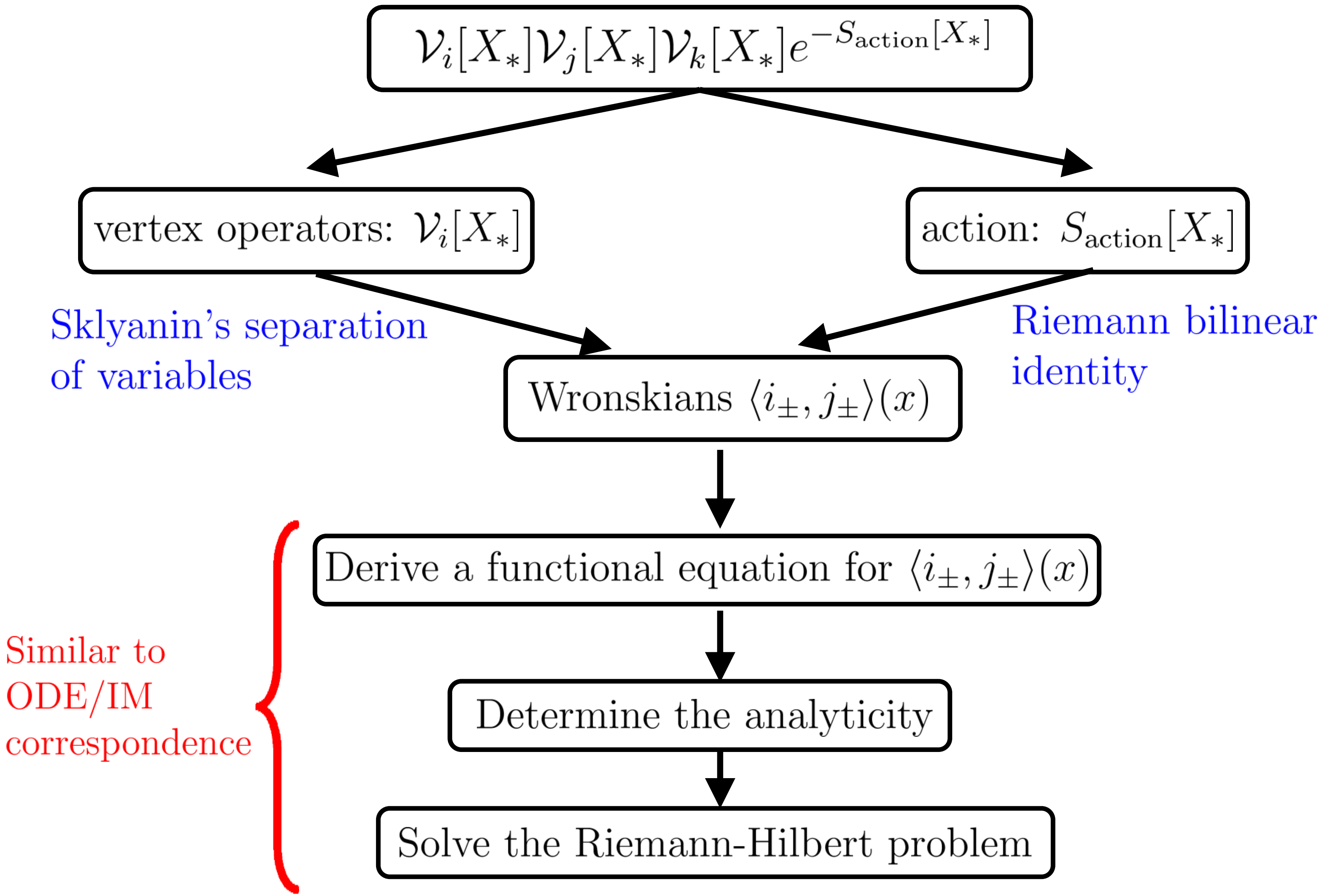
1. It is very hard to construct the saddle point X_*.

2. We do not know the form of the vertex operators $\mathcal{V}_i$.

Both can be overcome by the use of integrability.

Outline of the computation



Brief review of classical integrability of string

[Kazakov-Marshakov-Minahan-Zarembo 2004]

$\mathcal{O}$ in SU(2)-sector $\longleftrightarrow$ String rotating in S^3

$$S^3 : \sum_{I=1}^{4} Y_I Y_I = 1 \qquad \mathbb{Y} = \begin{pmatrix} Z & X \\ -\bar{X} & \bar{Z} \end{pmatrix} \qquad \begin{cases} Z := Y_1 + iY_2 \\ X := Y_3 + iY_4 \end{cases}$$

Eq. of motion:

$$\boxed{\text{EOM: } \partial\bar{\partial} Y_I + \left(\sum_J Y_J Y_J\right) Y_I = 0}$$ nonlinear, difficult

$$\rightarrow \quad \boxed{\left[\partial + \frac{j_z}{1-x}, \bar{\partial} + \frac{j_{\bar{z}}}{1+x}\right] = 0\,.}$$ x: spectral parameter

$$j := \mathbb{Y}^{-1} d\mathbb{Y}$$

Brief review of classical integrability of string

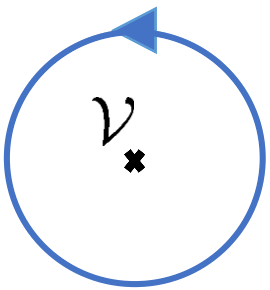
$$\left[\partial + \frac{j_z}{1-x}\,,\bar{\partial} + \frac{j_{\bar{z}}}{1+x}\right] = 0\,.$$

Auxiliary linear problem:

$$\left(\partial + \frac{j_z}{1-x}\right)\psi(z,\bar{z};x) = 0\,,\qquad \left(\partial + \frac{j_{\bar{z}}}{1+x}\right)\psi(z,\bar{z};x) = 0$$

Monodromy matrix:

$\Omega(x)$



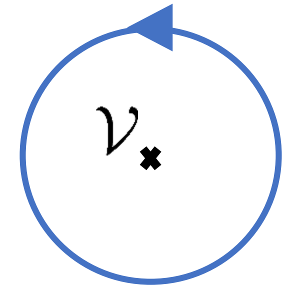
$$\psi(e^{2\pi i}z, e^{-2\pi i}\bar{z};x) = \Omega(x)\psi(z,\bar{z};x)$$

$$\Omega(x) := \mathrm{P}\exp\left(-\oint \frac{j_z dz}{1-x} + \frac{j_{\bar{z}}d\bar{z}}{1+x}\right)$$

Quasi-momentum:

$$\Omega(x) \sim \begin{pmatrix} e^{+ip(x)} & 0 \\ 0 & e^{-ip(x)} \end{pmatrix}$$

$\Omega(x)$

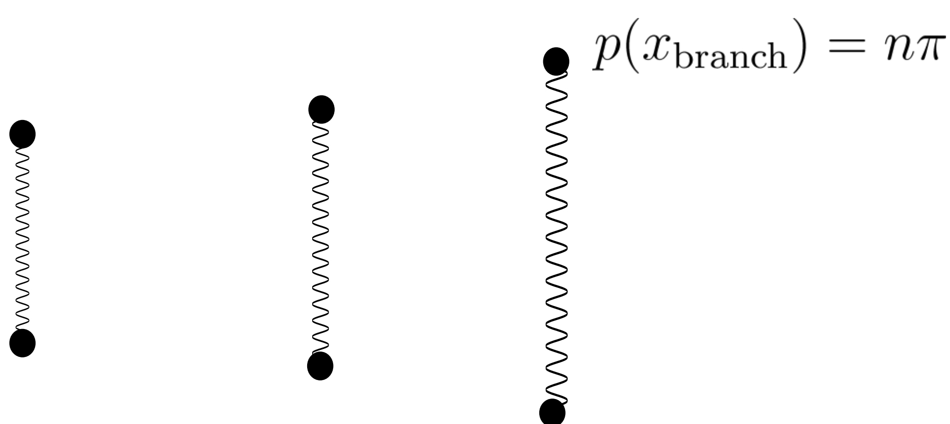


$p(x)$: Generating function of charges

Spectral curve:

$$\det\left(y \cdot \mathbf{1} - \Omega(x)\right) = 0$$

$$\Longleftrightarrow \; (y - e^{ip(x)})(y - e^{-ip(x)}) = 0$$



Square-root branch cuts

- We use quasi-momenta/spectral curve to characterize semi-classical string states.

$$p_1(x),\ p_2(x),\ p_3(x) \quad \text{(+ ``polarization'')}$$

$$\Big\downarrow \quad \text{Determine}$$

$$C_{123}$$

Action in terms of Wronskians

$$\text{Action: } S_{S^3} \propto \int d^2z \sum_I \partial Y_I \bar{\partial} Y_I = \frac{i}{2} \int \varpi \wedge \eta$$

$$\begin{cases} \varpi = \sqrt{T} dz \\ \eta = \dfrac{\sum_I \partial Y_I \bar{\partial} Y_I}{\sqrt{T}} d\bar{z} + \bullet dz \,, \quad \boldsymbol{d\eta = 0} \end{cases}$$

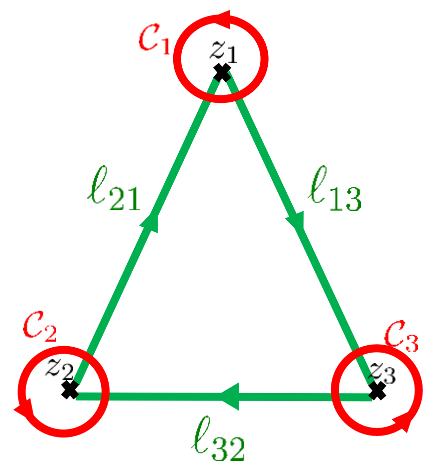
$T:$ stress-energy tensor for S^3

$$\propto \left(\sum_I \partial Y_I \partial Y_I \right)$$

↓ Stokes thm + Riemann bilinear identity

$$\text{Action} \propto \left(\oint_{\mathcal{C}_1+\mathcal{C}_2-\mathcal{C}_3} \varpi \int_{\ell_{21}} \eta + (\text{permutation}) \right) - (\varpi \longleftrightarrow \eta) + \cdots$$

easy difficult



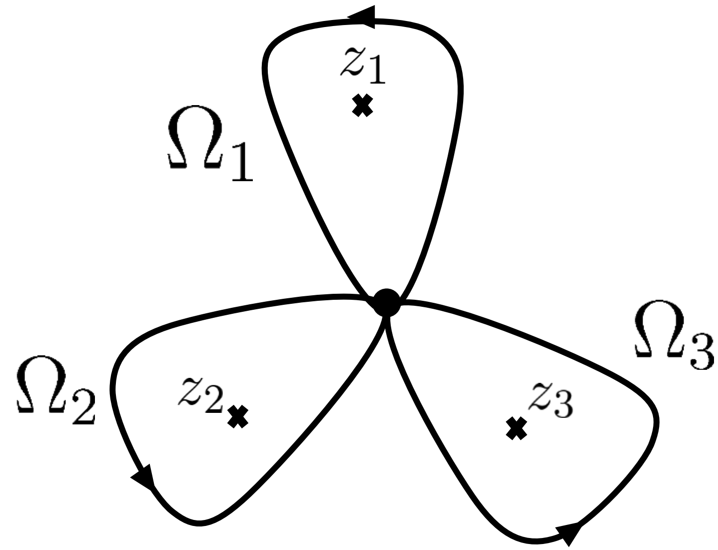
Action in terms of Wronskians

$$\left(\oint_{\mathcal{C}_1+\mathcal{C}_2-\mathcal{C}_3} \varpi \int_{\ell_{21}} \eta + (\text{permutation})\right) - (\varpi \longleftrightarrow \eta)$$

Wronskian:

$$\Omega_i\, i_\pm = e^{\pm i p_i(x)} i_\pm$$

$$\langle i_\pm\,, j_\pm\rangle(x) \equiv \det\left(i_\pm\,, j_\pm\right)$$



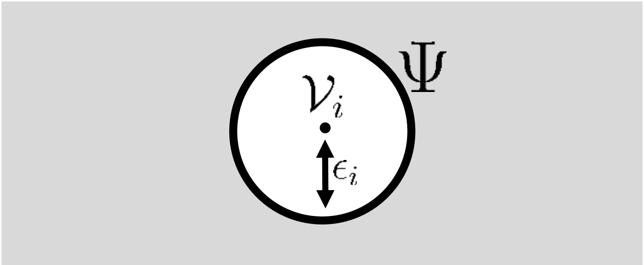
WKB-expansion of Wronskian:

$$\ln\langle 2_+\,, 1_+\rangle(x) \sim \frac{1}{x-1}\int_{\ell_{21}} \varpi + \quad \bullet \quad + \frac{x-1}{4}\int_{\ell_{21}} \eta + \cdots$$

The action can be computed from the Wronskians.

Vertex operators in terms of Wronskians

- State-operator correspondence:

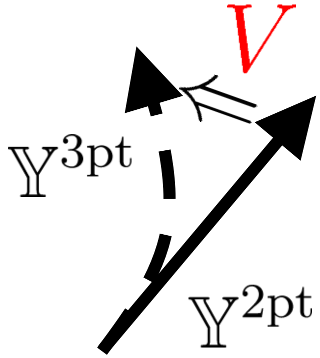


- Wave function in terms of action-angle variables

$$\Psi \sim \exp\left(i \sum_j S_j \phi_j\right)$$

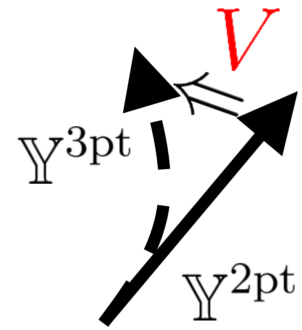
Sklyanin's separation of variables

- Near the vertex operator, the 3-pt sol. and the 2-pt sol. are related by the (complexified) global transformation.



Vertex operators in terms of Wronskians

- The global transformation V determines the shift of the angle variables:



$$\delta\phi_i = \phi_i^{3\text{pt}} - \phi_i^{2\text{pt}}$$

- It turns out that V can be computed by the Wronskians.

- As a result, we get

$\sim$(kinematical factor)

$$\times\left(\frac{1}{\langle 1_+\,,2_+\rangle|_{x=0}}\right)^{J_1+J_2-J_3}\left(\frac{1}{\langle 2_+\,,3_+\rangle|_{x=0}}\right)^{J_2+J_3-J_1}\left(\frac{1}{\langle 3_+\,,1_+\rangle|_{x=0}}\right)^{J_3+J_1-J_2}$$

$$\times\left(\frac{1}{\langle 1_-\,,2_-\rangle|_{x=\infty}}\right)^{J_1'+J_2'-J_3'}\left(\frac{1}{\langle 2_-\,,3_-\rangle|_{x=\infty}}\right)^{J_2'+J_3'-J_1'}\left(\frac{1}{\langle 3_-\,,1_-\rangle|_{x=\infty}}\right)^{J_3'+J_1'-J_2'}$$

The vertex ops. can also be computed from the Wronskians.

Computation of Wronskians

Functional relation for the Wronskians

- Take the basis which diagonaizes Ω_1: $\Omega_1 = \begin{pmatrix} e^{ip_1} & 0 \\ 0 & e^{-ip_1} \end{pmatrix}$

- Then, Ω_2 is constrained by the following conditions:

$$\Omega_2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \begin{aligned} \mathrm{tr}\,\Omega_2 &= a + d = e^{ip_2} + e^{-ip_2} \\ \det \Omega_2 &= ad - bc = 1 \end{aligned}$$

- Owing to the relation, $\Omega_1 \Omega_2 \Omega_3 = \mathbf{1}$, we get

$$\Omega_3 = \Omega_2^{-1}\Omega_1^{-1} = \begin{pmatrix} e^{-ip_1} d & -e^{ip_1} b \\ -e^{-ip_1} c & e^{ip_1} a \end{pmatrix} \qquad \mathrm{tr}\,\Omega_3 = e^{-ip_1} d + e^{ip_1} a = e^{ip_3} + e^{-ip_3}$$

- a, d and the product bc can be determined explicitly.

- Using the explicit form of Ω's we get

$$\langle 1_+, 2_+ \rangle \langle 1_-, 2_- \rangle = \frac{\sin \frac{p_1+p_2+p_3}{2} \sin \frac{p_1+p_2-p_3}{2}}{\sin p_1 \sin p_2} \qquad \text{etc.}$$

Functional relation for the Wronskians

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Analytic properties

$$\langle 1_+, 2_+ \rangle \langle 1_-, 2_- \rangle = \boxed{\frac{\sin \frac{p_1+p_2+p_3}{2} \sin \frac{p_1+p_2-p_3}{2}}{\sin p_1 \sin p_2}}$$

- Poles at $\sin p_i = 0$.
- Zeros at $\sin \frac{p_1+p_2\pm p_3}{2} = 0$

Once we understand
which poles and zeros belong to which Wronskian,
we can uniquely determine individual Wronskians.

Poles

$$\langle 1_+, 2_+\rangle\langle 1_-, 2_-\rangle = \frac{\sin\frac{p_1+p_2+p_3}{2}\sin\frac{p_1+p_2-p_3}{2}}{\sin p_1 \sin p_2}$$

- Consider a pole, $\sin p_1 = 0$.
- Although one may expect that

$$\Omega_1 = \begin{pmatrix} e^{ip_1} & 0 \\ 0 & e^{-ip_1} \end{pmatrix} = \pm\mathbf{1} \qquad \text{at } \sin p_1 = 0,$$

this is wrong.

$\begin{cases} \text{If } \Omega_1 = \pm\mathbf{1},\ \Omega_2 = \pm\Omega_3^{-1} \text{ owing to } \Omega_1\Omega_2\Omega_3 = \mathbf{1}. \\ \text{But, for generic } p_2 \text{ and } p_3, \text{ this is not satisfied.} \end{cases}$

- The only remaining possibility is

$$\Omega_1 = \pm\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}. \qquad \text{Jordan block}$$

$\longrightarrow$ Two eigenvectors degenerate at $\sin p_1 = 0$.

Behavior around the vertex operator

- Near the vertex operator, the 3-pt solution $\sim$ the 2-pt solution.

$$1_+ = 1_+^{\text{2pt}} + \cdots$$

$$1_- = 1_-^{\text{2pt}} + \cdots$$

- **But** $1_\pm^{\text{2pt}}$ are **non-degenerate** at $\sin p_1 = 0$.
- This apparent contradiction can be resolved iff one of $1_\pm$ appears in the expansion of the other with a coefficient divergent at $\sin p_1 = 0$:

$$(a)\ \ 1_+ = 1_+^{\text{2pt}} + \cdots + C(x) 1_- + \cdots \qquad C(x) \to \infty \text{ at } \sin p_1 = 0$$

or

$$(b)\ \ 1_- = 1_-^{\text{2pt}} + \cdots + \hat{C}(x) 1_+ + \cdots \qquad \hat{C}(x) \to \infty \text{ at } \sin p_1 = 0$$

- (a) occurs only when $1_+^{\text{2pt}} \gg 1_-^{\text{2pt}}$. (b) occurs only when $1_-^{\text{2pt}} \gg 1_+^{\text{2pt}}$.

$$\longrightarrow \quad \boxed{\text{Big} = \infty \cdot \text{Small} \quad \text{at } \sin p_1 = 0,}$$

- For a simple class of non-BPS string states called *one-cut solutions*,

$$1_{\pm}^{\rm 2pt} \sim e^{\pm q_1(x)\tau} \quad \tau := \ln|z - z_1| \qquad q_1 : \text{ quasi-energy}$$

$$\longrightarrow \begin{cases} 1_{-}^{\rm 2pt} \gg 1_{+}^{\rm 2pt} & \text{for } \operatorname{Re} q_1 > 0 \\ 1_{+}^{\rm 2pt} \gg 1_{-}^{\rm 2pt} & \text{for } \operatorname{Re} q_1 < 0 \end{cases}$$

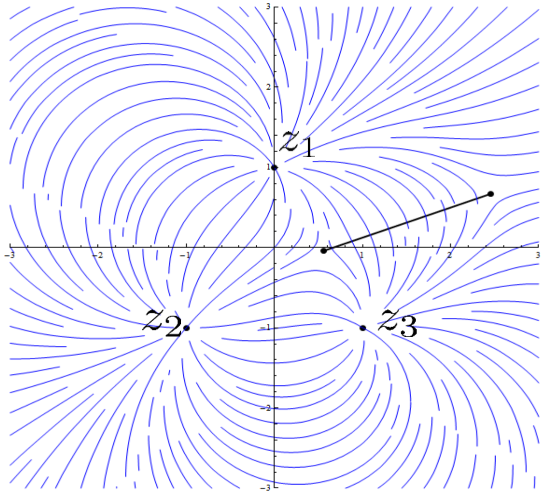
- Thus the analyticity of the Wronskian changes depending on $\operatorname{Re} q_1(x)$

$$\langle 1_+, \bullet \rangle \neq \infty\,, \quad \langle 1_-, \bullet \rangle = \infty \qquad \text{for } \operatorname{Re} q_1 > 0$$

$$\langle 1_+, \bullet \rangle = \infty\,, \quad \langle 1_-, \bullet \rangle \neq \infty \qquad \text{for } \operatorname{Re} q_1 < 0$$

Evaluation of Wronskians in terms of convolution integrals

- Zeros can also be determined (although far more complicated) using the generalization of the WKB-curves `cf. [Gaiotto-Moore-Neitzke]`



- From the anlalyticity, one can determine each Wronskian by the Riemann-Hilbert analysis.

Evaluation of Wronskians in terms of convolution integrals

- As a result, we get

$$\langle 1_+, 2_+ \rangle = \int_\Gamma \mathcal{K}_1 * \ln \sin p_1 + \cdots$$

$$\left(\mathcal{K}_1(x'; x) = \frac{1}{x' - x} \sqrt{\frac{(x - u_1)(x - \bar{u}_1)}{(x' - u_1)(x' - \bar{u}_1)}} \right)$$

u_1 : branch point of p_1

- Expressions in the intermediate stages are complicated.

- However, if we sum the contribution from the action and the contribution from the vertex operators, an unexpected simplification occurs.

Final result

$$
\begin{aligned}
&\ln\left(S^3\text{-contribution}\right) = \\
&\int_{\mathcal{M}^{uuu}_{---}} \frac{z(x)\,(dp_1+dp_2+dp_3)}{2\pi i}\ln\sin\left(\frac{p_1+p_2+p_3}{2}\right) \\
&+\int_{\mathcal{M}^{uuu}_{--+}} \frac{z(x)\,(dp_1+dp_2-dp_3)}{2\pi i}\ln\sin\left(\frac{p_1+p_2-p_3}{2}\right) \\
&+\int_{\mathcal{M}^{uuu}_{-+-}} \frac{z(x)\,(dp_1-dp_2+dp_3)}{2\pi i}\ln\sin\left(\frac{p_1-p_2+p_3}{2}\right) \\
&+\int_{\mathcal{M}^{uuu}_{+--}} \frac{z(x)\,(-dp_1+dp_2+dp_3)}{2\pi i}\ln\sin\left(\frac{-p_1+p_2+p_3}{2}\right) \\
&-2\sum_{j=1}^{3}\int_{\Gamma^u_{j_-}} \frac{z(x)\,dp_j}{2\pi i}\ln\sin p_j\,.
\end{aligned}
$$

$$
z = \frac{\sqrt{\lambda}}{4\pi}\left(x+\frac{1}{x}\right)
$$

Zhukovsky variable

- The integration contours are determined by $(\operatorname{Re} q_i = 0)$ + (several other conditions).
- The explicit shapes of the contours depend strongly on the details of the operators.

Discussion and consistency checks

- The integrands provide a natural generalization of the weak-coupling result.

$$-\int \frac{z dp_1}{2\pi i} \ln \sin p_1 \quad \xrightarrow[\text{(and neglect unimportant terms)}]{\text{Integrate by parts}} \quad \int dz\, \mathrm{Li}_2\left(e^{2ip_1}\right)$$

cf. [Serban]

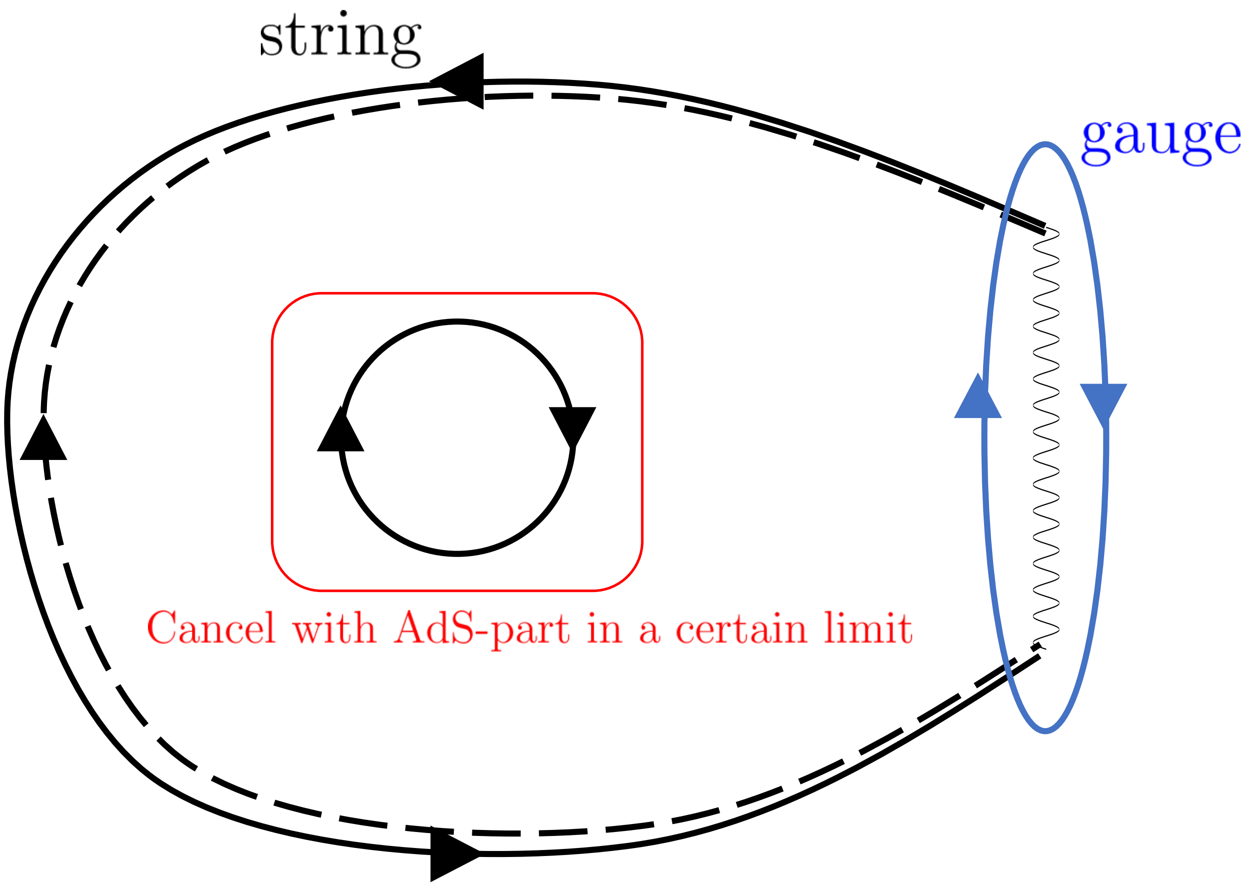
Weak-coupling result: $\int dx\, \mathrm{Li}_2\left(e^{2ip_1}\right)$

- Reduces to the two-point function when one of the operators vanishes.

- The method reproduces the classical limit of DOZZ 3-pt function correctly when applied to Liouville theory. [Honda,SK]

However, the integration contours are quite different.

Integration contour for (BPS)-(BPS)-(non-BPS) (for S^3 part)



Discussion on the mismatch

- There is no a priori reason to expect the match between strong/weak results. (It may be the consequence of the notorious "order-of-limits" problem.)

- In the computation, we implicitly assumed that the saddle-point configuration is smooth except at the positions of the vertex operators. If a different saddle which contains additional cusps/spikes contribute, the contours may change.

The results are unexpectedly simple
but the issue of the integration contours is still to be clarified.

Summary and Prospect

- Same expression from weak and strong sides.
- What is the physics behind the dilog-expressions?
- Full $AdS_5 \times S^5$, higher-point functions?

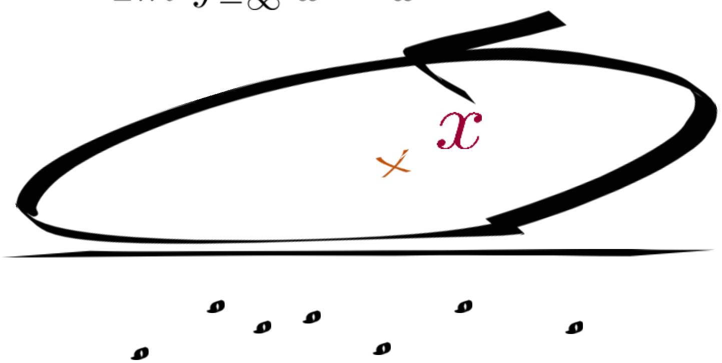
Example of Riemann-Hilbert technique

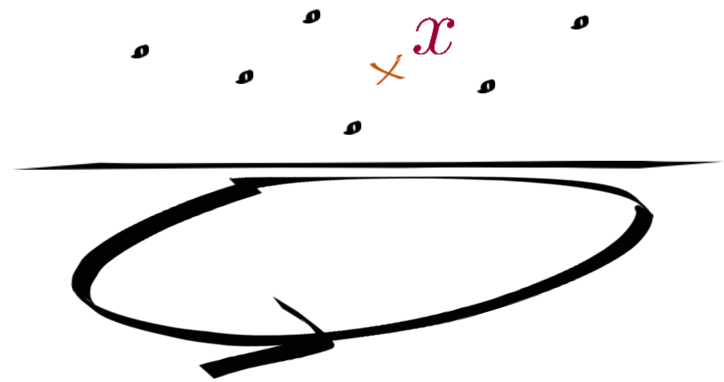
$$F_{\text{tot}}(x) = F_{\uparrow}(x) + F_{\downarrow}(x)$$

$F_{\uparrow}$: regular on $\text{Im}\, x > 0$ $\qquad$ $F_{\downarrow}$: regular on $\text{Im}\, x < 0$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dx'}{x'-x}F_{\text{tot}}(x') = \begin{cases} F_{\uparrow}(x) & (\text{Im}\, x > 0) \\ -F_{\downarrow}(x) & (\text{Im}\, x < 0) \end{cases}$$

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dx'}{x'-x}F_{\uparrow}(x') \qquad \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{dx'}{x'-x}F_{\downarrow}(x')$$





- Apply it to $\log\langle 1_+, 2_+\rangle + \log\langle 1_-, 2_-\rangle = \log \frac{\sin\frac{p_1+p_2+p_3}{2}\,\sin\frac{p_1+p_2-p_3}{2}}{\sin p_1 \sin p_2}$