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ODE/IM correpondence and modified affine Toda field equations: II

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Based on: [K.Ito-CL arXiv:1312.6759]

Introduction

- The ODE/IM correspondence is a relation between spectral analysis of ODEs, and the ″functional relations″ approach to 2d quantum integrable models (IM)
- Dr. Ito just gave an excellent introduction on this topic, so we will focus on only a specific part of this correspondence for the *modified affine Toda field* equation
- We will
  1. First, look at asymptotic behavior of solutions
  2. Then introduce the so called ψ-system associated with the algebra
  3. Use ψ-system to derive Q-functions and Bethe ansatz equations

Affine Toda equation

- We will use complex coordinates

$$z = \frac{1}{2}(x^0 + ix^1), \quad \bar{z} = \frac{1}{2}(x^0 - ix^1)$$

- Motivated by recent papers [$A_1^{(1)}$: Lukyanov-Zamolodchikov 2010 and $A_2^{(2)}$: Dorey-Faldella-Negro-Tateo 2012], we define

$$p(z) = z^{hM} - s^{hM}, \ \bar{p}(\bar{z}) = \bar{z}^{hM} - \bar{s}^{hM}$$

with h the Coxeter number, and M a positive real parameter

- Using this, the *modified* affine Toda field equation is

$$\beta\partial\bar{\partial}\phi + m^2 \left[\sum_{i=1}^{r} n_i\alpha_i \exp(\beta\alpha_i\phi) + p(z)\bar{p}(\bar{z})n_0\alpha_0 \exp(\beta\alpha_0\phi)\right] = 0$$

Asymptotic behavior of ϕ

- In the **large** |z| limit, we can simply assume that the asymptotic solution to the modified affine Toda equation is
$$\phi(z,\bar{z}) = \frac{M\rho^\vee}{\beta}\log(z\bar{z}) + \mathcal{O}(1)$$
- For **small** z near 0, we assume logarithmic divergence, with expansion
$$\phi(z,\bar{z}) = g\log(z\bar{z}) + \cdots$$

  Here, g is a vector that controls the small-z behavior
- Substituting this expansion into the modified affine Toda equation, one can show that $1+\beta\alpha_i\cdot g>0$

Note about representations

- In our paper [1312.6759], we focused on specific a specific matrix representation, the *fundamental representation*;

- however, in this presentation we will present the same basic results using a *general* highest weight representation, and arrive at the Bethe ansatz equations much faster

Linear equation formulation

- The modified affine Toda equation can be written as a linear problem as well for an appropriate one-form connection

$$(\mathrm{d} + \mathbf{A})\Psi = 0$$

$$A_z = \beta\partial\phi \cdot H + m e^{\lambda} \left\{ \sum_{i=1}^{r} \sqrt{n_i^\vee} E_{\alpha_i} + p(z) \sqrt{n_0^\vee} E_{\alpha_0} \right\}$$

- $\mathbf{A}$ can exist in any highest weight representation
- Denote fundamental modules with highest weight ω_a as $V^{(a)}$ with set of weight vectors $\{h_1^{(a)}, \ldots, h_{\dim(V^{(a)})}^{(a)}\}$
- In this notation, Ψ is a wavefunction in this representation space,

$$\Psi^{(a)}(z) = \sum_{i=1}^{\dim(V^{(a)})} \psi^{(a;i)}(z) |h_i^{(a)}\rangle$$

Linear equation formulation

- The **small z** limit of the linear problem is

$$\left(\partial + \frac{\beta g \cdot H}{z}\right)\Psi^{(a)} = 0$$

- This has a basis of non-trivial solutions given by

$$\xi_i^{(a)} \sim z^{\mu_i^{(a)}} |h_i^{(a)}\rangle, \quad \text{where} \;\; \mu_i^{(a)} + \beta g \cdot h_i^{(a)} = 0$$

- This basis will be used to expand a preferred solution that comes from looking at the **large z** behavior

Linear equation formulation

- In the **large z** limit, it is convenient to gauge transform the connection into the form

$$z^{M\rho^\vee \cdot H}(\partial + A_z)z^{-M\rho^\vee \cdot H} \sim \partial + me^{\lambda}z^M\Lambda^+ + \cdots$$

where

$$\Lambda^+ = \sum_{i=0}^{r} \sqrt{n_i^\vee} E_{\alpha_i}$$

- Denoting the unique eigenvalue of Λ^+ in representation $V^{(a)}$ with largest real part as $\lambda^{(a)}$, we have the unique solution which decays fastest along the positive real axis [Sun-2012]

$$\psi^{(a)} \sim \vec{\lambda}^{(a)} \exp\left(-\lambda^{(a)} \frac{me^{\lambda} z^{M+1}}{M+1} + g(\bar{z})\right)$$

Linear equation formulation

- The previous asymptotically decaying unique solution for **large z** can now be expanded in terms of our basis of solutions defined by their behavior around **small z**:

$$\psi^{(a)}(z) = \sum_{i=1}^{\dim(V^{(a)})} Q^{(a;i)}(\lambda)\xi_i^{(a)}(z)$$

- These Q-coefficients do not depend on the coordinates z
- As per previous work with sinh-Gordon and Bullough-Dodd cases, we expect that in the ODE/IM correspondence, that these will coincide with the vacuum expectation values of the Q-functions of a 2D *massive* QFT

Conformal limit

- To study the *massless* limit, we can define the following

$$x = (me^{\lambda})^{1/(M+1)} z, \quad E = s^{hM} (me^{\lambda})^{hM/(M+1)}$$

- The conformal limit is defined as the limit where we take $\bar{z}$ to zero, then take z to 0 and λ to infinity such that x and E remain finite values
- This converts the affine Toda field equation to the massless Toda field equation

- To handle the massive case, we need the full connection, but when looking at the massless case just the holomorphic connection is enough

Symanzik rotation

- Before proceeding, we need to define a Symanzik rotation
- Our linear problem is symmetric under the transformation for integers k

$$\hat{\Omega}_k : \left\{ \begin{array}{c} z \to z \ e^{2\pi k i/hM} \\ s \to s \ e^{2\pi k i/hM} \\ \lambda \to \lambda - \frac{2\pi k i}{hM} \end{array} \right.$$

- We denote a Symanzik rotated function with a subscript real number, as

$$f_k(z,\lambda) \equiv \hat{\Omega}_k f(z,\lambda)$$

ψ-system

- To obtain the Bethe ansatz equations (BAE) for this, we need to determine the so-called ψ-system, which are relations between the different $\psi^{(a)}$
  - Focus on just the case of **A-type** Lie algebra
- It can be shown that the following highest weights are equivalent by showing they have the same eigenvalues with respect to H, and are annihilated by each raising operator

$$|h_1^{(a)}\rangle \wedge |h_2^{(a)}\rangle = |\omega_{a-1}\rangle \otimes |\omega_{a+1}\rangle, \quad \text{where} \quad |\omega_0\rangle \equiv 1 \equiv |\omega_{r+1}\rangle$$

- This can be written down as

$$\psi_{-k}^{(a)} \wedge \psi_k^{(a)} = \psi^{(a-1)} \otimes \psi^{(a+1)}$$

where the Symanzik rotated ψ on the left ensures that the wedge product is not zero

ψ-system

$$\psi^{(a)}_{-k} \wedge \psi^{(a)}_{k} = \psi^{(a-1)} \otimes \psi^{(a+1)}$$

- By applying $(\partial + A_z)$ to both sides, one can see that each side is a solution to the linear problem
- Furthermore, each side has the same **large z** behavior if k=½, so the A-type ψ-system is

$$\psi^{(a)}_{-1/2} \wedge \psi^{(a)}_{1/2} = \psi^{(a-1)} \otimes \psi^{(a+1)}$$

- Now, substituting in the expansion $\psi^{(a)}(x) = \sum_i Q^{(a;i)}\xi^{(a)}_i(x)$ on both sides, and extracting just the coefficient of the most singular term near x = 0, we get the relation $(\omega = e^{2\pi i/h(M+1)})$

$$\omega^{(\mu_1^{(a)}-\mu_2^{(a)})/2} Q^{(a;1)}_{-1/2} Q^{(a;2)}_{1/2} - \omega^{(\mu_2^{(a)}-\mu_1^{(a)})/2} Q^{(a;2)}_{-1/2} Q^{(a;1)}_{1/2} = Q^{(a-1;1)} Q^{(a+1;1)}$$

ψ-system

- Denote the zeros of $Q^{(a;1)}(E)$ as $E_i^{(a)}$
- Then evaluating the previous equation at this point after Symanzik rotating the whole equation by ±½, then dividing the two resulting equations gives

$$\omega^{\mu_2^{(a)}-\mu_1^{(a)}} \left. \frac{Q_{-1/2}^{(a-1)} Q_1^{(a)} Q_{-1/2}^{(a+1)}}{Q_{1/2}^{(a-1)} Q_{-1}^{(a)} Q_{1/2}^{(a+1)}} \right|_{E_i^{(a)}} = -1$$

- This is the BAE for A-type Lie algebras, and we can arrive at it without the need for complex Plücker relations of determinants

Matrix representation and ODE

- Quick aside on (pseudo-)ODEs:
  - these highest weight representations can be written out as a matrix
  - by looking at each component equation of the linear problem, one can arrive at a (pseudo-)differential equation that characterizes the behavior of solutions
  - for instance in A-type case we have in the vector representation,

$$(\partial + \beta\partial\phi \cdot h_{r+1}^{(1)}) \cdots (\partial + \beta\partial\phi \cdot h_1^{(1)})\psi^{(1;1)} = (-me^{\lambda})^h p(z)\psi^{(1;1)}$$

  - in previous papers, these (pseudo-)differential equations were the objects of study, but our work shows that one can focus entirely on just the linear problem and get the same information *(asymptotic behaviors, Q-coefficients, T-Q relations, BAE, etc.)*

Other algebras

- The above analysis was carried out for all but the E- and F-type affine Lie algebras
- In the case of non-simply laced algebras, we found something different than previous papers:
  - Our results were in agreement only if we took the *Langlands dual*
- Our calculated (pseudo-)ODEs had the following relation

modified affine Toda	Dorey et al. (2007)
$\mathrm{A}_r^{(1)}$	A_r
$(\mathrm{B}_r^{(1)})^\vee = A_{2r-1}^{(2)}$	B_r
$(\mathrm{C}_r^{(1)})^\vee = D_{r+1}^{(2)}$	C_r
$\mathrm{D}_r^{(1)}$	D_r

Other algebras

- Why is it that the Langlands dual $\hat{\mathfrak{g}}^\vee$ for our modified affine Toda equation correspond to the equations of Dorey et al.?

- Would like to check if there are differences for the dual pairs

$$\left(B_r^{(1)}, (B_r^{(1)})^\vee\right), \left(C_r^{(1)}, (C_r^{(1)})^\vee\right), \text{ and } \left(G_2^{(1)}, D_4^{(3)}\right)$$

in terms of, for instance, BAE, or T-Q relations

Conclusion

1) Looked at asymptotic behavior of modified affine Toda equation, and linear problem solutions

2) Found unique asympotically decaying solution to linear problem that leads naturally to ψ-system

3) Showed how ψ-system can give BAE

Future directions

1) Calculate Bethe ansatz equations for non-simply laced cases to see if there is any difference
2) Work on E- and F-type cases as well
3) Investigate effects of using $p(z) = \left(z^{hM/K} - s^{hM/K}\right)^K$ as in the Dorey et al. paper
4) Look into the *massive* ODE/IM correspondence for these Toda equations
5) Supersymmetric Toda equations?