1

Tau-functions for quiver gauge theories (S-duality class of)

- AM, JHEP-2013 ("old paper")
- Pasha Gavrylenko + A.M., hep-th

JSPS-RFBR workshop (closing?)
(started due to questions on previous workshop in Japan)

$$\mathcal{F} = \mathcal{F}(a_1 \dots a_g, q_1, \dots q_g, \dots) = \log \tau$$

(in japanese sense)
Krichever - 1992

↓
effective action in $N=2$ SUSY
(quiver gauge theory $SU(2)^{\otimes g}$)
in terms of family of curves Σ (with extra structure)

- originally (I.K.): function on Teichmüller moduli space of curves : $\simeq 3g$ variables
- SW context : $\simeq g$-variables, g-parametric family of curves for integrable systems
- extra parameters?
SU(2) quiv. gauge theories: $2g = \underset{\{\vec{a}\}}{g} + \underset{\{\vec{q}\}}{g}$
SW periods bare couplings

span cotangent bundle
$T^*(\mathcal{T})$ $g = \dim \mathcal{T} = 3g_0 - 3 + n$

$g_0 = \text{genus}(\Sigma_0)$ - UV or Gaiotto curve X

$g_0 = 0, 1$ ↑ explicit ↖ clear $g_0 > 1$ not clear gauge/string duality

②

Construction of the SU(2)-quiver τ-function

$x^2 = t(z)$ SW curve for a SU(2)-g.g theory in Gaiotto form

$t(z)\,dz^2$ 2-differential on UV curve Σ_0 with fixed residues

$$\oint_{P_i} \sqrt{t(z)}\,dz = m_i = \text{fixed}$$

linear combinations of masses of g.t.

Krichever data:

2 meromorphic differentials with fixed periods dx, dz

$$\oint dx = 0, \quad \oint dz = 0$$

$dS = x\,dz$ SW 1-form

$$\left.\begin{aligned} \vec{a} &= \oint_{\vec{A}} x\,dz \\ \frac{\partial \mathcal{F}}{\partial \vec{a}} &= \oint_{\vec{B}} x\,dz \end{aligned}\right\} \text{ on } \Sigma$$

$\Sigma_0 \sim$ sphere ($g_0 = 0$) with n punctures

$$\dim \mathcal{J}_{\Sigma_0} = 3g_0 - 3 + n \underset{g_0=0}{=} n-3$$

fix $\{z_i\}^n = \{0, 1, \infty, q_i\}^{n-3}$ $\quad q_i \quad d\Omega = \frac{dz}{z}$

$$\log q_i = \oint_{B_i^{(0)}} d\Omega = \int_1^{q_i} \frac{dz}{z}$$

$\sim$ degenerate "B_0"-periods

Then:

$$q_i \frac{\partial \mathcal{F}}{\partial q_i} = \frac{1}{2} \oint_{A_i^{(0)}} \frac{dS \cdot dS}{d\Omega} = \operatorname*{res}_{q_i} \left(z x^2 dz \right)$$

or

$$\boxed{\frac{\partial \mathcal{F}}{\partial q_i} = \operatorname*{res}_{q_i} \left(x^2 dz \right)}$$

(3)

Connection to: ~~[illegible]~~ AGT-relation and isomonodromic deformation

$$x^2 = t(z) = \sum_{i=1}^{n}\left[\frac{m_i^2}{(z-z_i)^2} + \frac{u_i}{z-z_i}\right]$$

$$\operatorname*{res}_{z_i} x^2 dz = u_i = \left.\frac{\partial \mathcal{F}}{\partial z_i}\right|_{\vec{a}}$$

$\mathcal{F} = \mathcal{F}(\vec{a}, \vec{q})$ so that $\frac{\partial u_i}{\partial z_j} = \frac{\partial u_j}{\partial z_i}$

is a <u>nontrivial</u> relation, coming from RBI

$$\sum u_i = 0, \quad \sum (z_i u_i + m_i^2) = 0 \quad \sum(z_i^2 u_i + 2 z_i m_i^2) = 0$$

Taking one more derivative

$$\frac{\partial^2 \mathcal{F}}{\partial z_i \partial z_j} = 2\operatorname*{res}_{z_i} x\,dz\,\frac{\partial x}{\partial z_j} = 2\operatorname*{res}_{z_i} \frac{dS\, d\Omega_j}{dz} = \left(= \oint_{A_i^{(0)}} \frac{dS\, d\Omega_j}{dz}\right.$$

$$d\Omega_j = \frac{\partial x}{\partial z_j} dz \ ; \quad \oint_{\vec{A}} d\Omega_j = 0$$

normalized meromorphic differential with 2nd-order pole at $z \to z_j$

Remembering, that Σ can be presented as a curve for the Gaudin model

$$\det_{2\times 2}\left(L(z) - x\right) = 0 \qquad \text{with } L(z) = \sum_{j=1}^{n} \frac{A_j}{z - z_j}$$

one gets for ($\operatorname{Tr} A_j = 0$) traceless

$$u_j = \sum_{k \neq j} \frac{\operatorname{Tr}(A_j A_k)}{z_j - z_k} = \frac{\partial \mathcal{F}}{\partial z_j}$$

{a well-know relation from isomonodromy problem}

④ Third derivatives & WDVV equations:

$\exists$ a universal residue formula
(written by I.K., and in fact proven) $\longrightarrow$ many proofs for particular cases, most generic Chekhov, Mironov, Vasiliev & AM
Gavrylenko & AM

Here:
$$\frac{\partial^3 \mathcal{F}}{\partial q_i \partial q_j \partial q_k} = \sum_{dx=0} \mathrm{Res}\left(\frac{d\Omega_i\, d\Omega_j\, d\Omega_k}{dx\, dz}\right)$$
$$\frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k} = \sum_{dx=0} \mathrm{Res}\left(\frac{d\omega_i\, d\omega_j\, d\omega_k}{dx\, dz}\right)$$
+ mixed cases

$$\{\vec{X}\} = \{\vec{a}\} \cup \{\vec{q}\}, \quad \{d\varpi\} = \{d\omega\} \cup \{d\Omega\}$$

$$\frac{\partial^3 \mathcal{F}}{\partial X_I \partial X_J \partial X_K} = \sum_{dx=0} \mathrm{Res}\left(\frac{d\varpi_I\, d\varpi_J\, d\varpi_K}{dx\, dz}\right)$$

Proof: diff. f-la for 2nd derivatives

Matrices: F_I : $(F_I)_{JK} = \mathcal{F}_{IJK}$

then: $F_I F_J^{-1} F_K = F_K F_J^{-1} F_I \quad \forall\ I, J, K$

first written in this form by AM^3 in about 1996.

The most easy proof is based on "counting argument" ONCE you have the residue formula

$$\frac{\partial^3 \mathcal{F}}{\partial X_I \partial X_J \partial X_K} = -2 \sum_{dz=0} \mathrm{res}\left(\frac{d\varpi_I\, d\varpi_J\, d\varpi_K}{dx\, dz}\right) =$$
projected to Σ_0

$$\# \lambda_I = \# X_I$$
$$\Downarrow$$

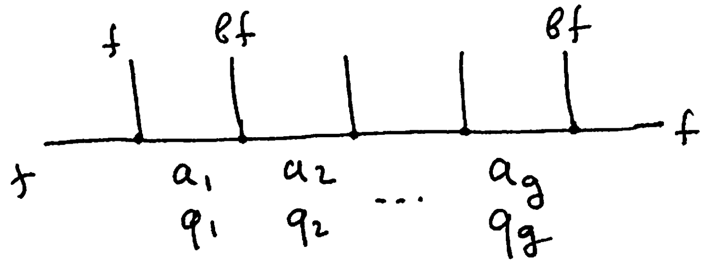
add $C_{IJ}^{L}(I_0)$ str. constants of finite-dimensional commutative ring

$$= \sum_{z=\lambda_I} \left(\ldots \right)$$

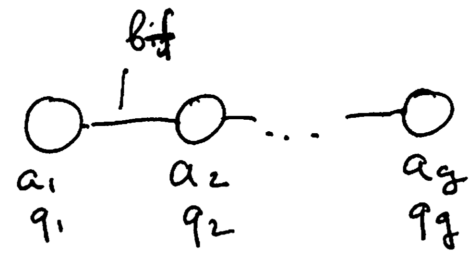
since $\dfrac{\partial^3 \mathcal{F}}{\partial X_I \partial X_J \partial X_K} \overset{\text{WDVV}}{=} C_{IJ}^{L}(I_0) \dfrac{\partial^3 \mathcal{F}}{\partial X_K \partial X_L \partial X_{I_0}}$

(5) Physics & some explicit calculations

S-duality class of quiver gauge theories



dual to

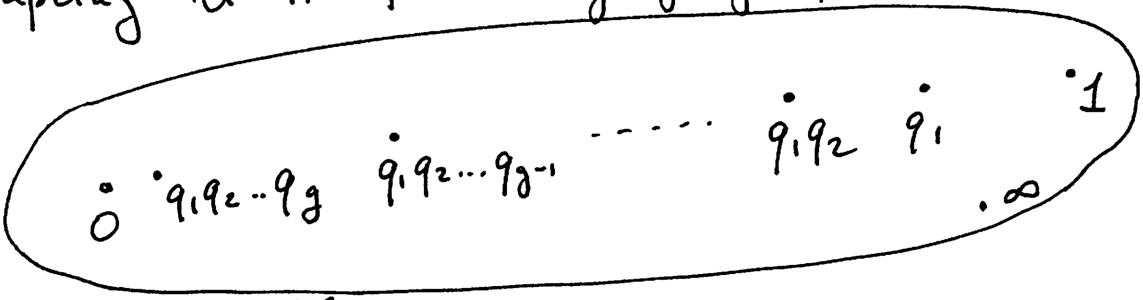


$n = g+3$ conformal block on sphere in appropriate channel

"quiver diagram"

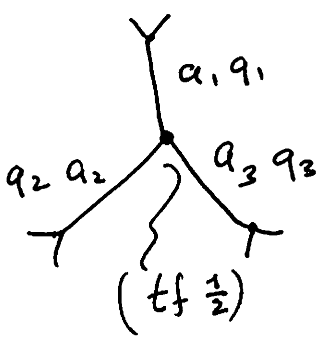
weak-coupling limit for all gauge groups

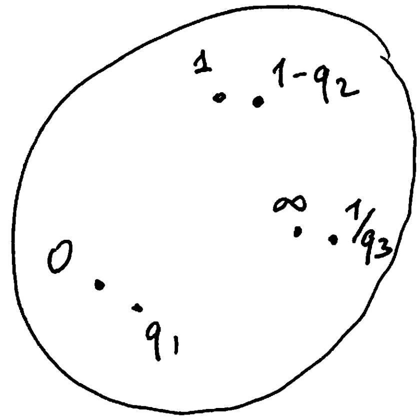
Linear quiver



$$\int_1^{q_1} \frac{dz}{z} = \log q_1 \, ; \qquad \int_{q_1}^{q_1 q_2} \frac{dz}{z} = \log q_2 \, , \ \ldots$$

Sicilian quiver:



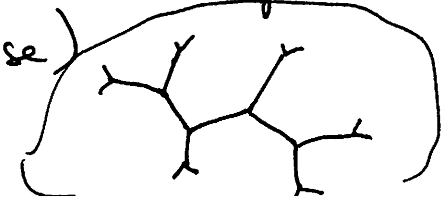


weak coupling regime

- "no quiver diagram"
- "no Nekrasov function exists" (for some restricted values of bare couplings?)

(+ no Lagrangian beyond SU(2) !!!)

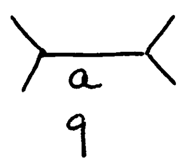
- there exists some version of 4d/2d duality (at least in the restricted case)



⑥ Put all masses $= 0$

(bad physically, but easy to compute)

Simplest example: $n = 4$, $n-3 = 1$



$$x^2 = \frac{q(1-q)\,u}{z(z-1)(z-q)}, \quad u = \frac{\partial \mathcal{F}}{\partial q}$$

Elliptic curve:

$$a = \oint_A x\,dz = \sqrt{u}\,K(q)$$

$$\mathcal{F} = \frac{1}{2}\tau a^2 \quad \Leftarrow \quad a_D = \oint_B x\,dz = \sqrt{u}\,\underline{\underline{i}}\,K(1-q) = a\tau$$

$$\tau = i\,\frac{K(1-q)}{K(q)} = \frac{1}{i\pi}\Big(\underbrace{\log q}_{\text{UV coupling}} - \underbrace{\log 16}_{\text{perturbative finite shift}} + \underbrace{\tfrac{1}{2}q + \tfrac{13}{64}q^2 + \dots}_{\text{instanton renormalization in 4d CFT with } \beta = 0}\Big)$$

Follows from the Zamolodchikov's conformal block in $c=1$ Ashkin-Teller (AM $\frac{3}{4}$)

$$B\left(\Delta_i = \tfrac{1}{16}, c = 1, \Delta = a^2; q\right) = \frac{e^{i\pi\tau a^2}}{\theta_3(0|q)\,q^{1/8}(1-q)^{1/8}}$$

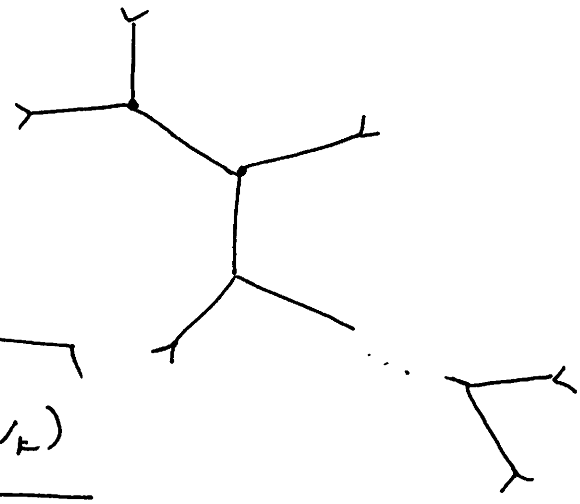
The easiest way to derive:

$$\frac{\partial \mathcal{F}}{\partial q} = \operatorname*{res}_{z=q} x^2 dz \quad \Rightarrow$$

$$\frac{\partial \log q}{\partial \tau} = \frac{1}{i\pi}\,\theta_4^4(0|q) = 4\,\frac{d}{d\tau}\log\frac{\theta_2(0|q)}{\theta_3(0|q)}$$

one of the simplest relation from Oyama? (Osaka)

Extension to the quiver theories $(m=0)$



$a_1 \ldots a_g$

$q_1 \ldots q_g$

$$x^2 = t(z)$$

$$dS = x\,dz = \sqrt{\alpha \frac{\prod_{k=1}^{g-1}(z-v_k)}{z(z-1)\prod_{j=1}^{g}(z-q_j)}}$$

$$\Big\{ \underbrace{\frac{\partial dS}{\partial \alpha}, \frac{\partial dS}{\partial v_k} \ldots}_{g} \Big\} \quad \text{holomorphic}$$

$$\vec{a} = \oint_A dS \quad \ldots$$

$$\frac{\partial dS}{\partial q_j} \sim \text{meromophic normalized}$$

Zamolodchikov's limit

- Take even n
  $g = n-3 \Rightarrow$ odd
- impose $\frac{g-1}{2} \equiv \tilde{g}-1$ constraints to the numerator

Then

1) $$dS = \sqrt{\alpha}\, \frac{Q_{\tilde{g}-1}(z)}{z(z-1)\prod_{j=1}^{2\tilde{g}-1}(z-q_j)}\, dz$$ becomes a holomorphic diff. on $\tilde{\Sigma}$ of genus $\tilde{g}$

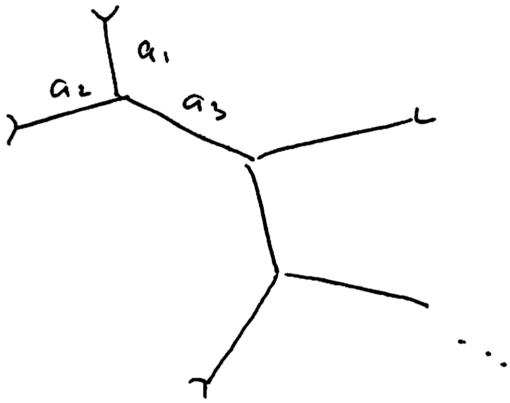
$$y^2 = z(z-1)\prod_{j=1}^{2\tilde{g}-1}(z-z_j)$$

2) $$dS = \sum_{\alpha=1}^{\tilde{g}} a_\alpha\, d\omega_\alpha$$

3) $$\mathcal{F} = \frac{1}{2}\sum_{\alpha,\beta=1}^{\tilde{g}} a_\alpha T_{\alpha\beta}(q)\, a_\beta$$

8

What does it mean physically!



At each vertex we have imposed

$$a_{i_1} + \varepsilon a_{i_2} + \varepsilon' a_{i_3} = 0$$

$$\varepsilon, \varepsilon' = \pm 1$$

$$V = \tilde{g} - 1 \quad \text{constraints}$$

One of the trifundamental masses vanishes!

From CFT $\Rightarrow$ conservation of the U(1) charges in all vertices

$$\frac{\partial \mathcal{F}}{\partial q_i} = \operatorname*{res}_{q_i} \frac{dS\, dS}{dz} = \sum_{\alpha,\beta=1}^{\tilde{g}} a_\alpha a_\beta \frac{\partial T_{\alpha\beta}}{\partial q_i}$$

$$\frac{\partial T_{\alpha\beta}}{\partial q_i} = \hat{\omega}_\alpha (q_i)\, \hat{\omega}_\beta (q_i)$$ one of the Rauch formulas

Inverse is given by the Thomae formulas:

$$q_k^2 = \pm \frac{\Theta_{\eta_1}(T)^4\, \Theta_{\eta_2}(T)^4}{\Theta_{\eta_3}(T)^4\, \Theta_{\eta_4}(T)^4}$$