JSPS/RFBR collaboration
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Instanton partition function, DDAHA and recursion formula

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arXiv:????.????, 1306.1523

Mostly a review of
Schiffmann and Vasserot, arXiv:1202.2756

Introduction

AGT conjecture

Alday-Gaiotto-Tachikawa (2009)

N=2 SYM

Toda

Nekrasov partition function = Correlation function

$$Z^{\rm Nek} = \sum_{\vec{Y}} q^{|\vec{Y}|} Z_{\vec{Y}} \qquad \langle V_\Delta(z_1)\cdots V_\Delta(z_L)\rangle$$

From the viewpoint of M5 brane, it may be understood as the consequence of two types of compactification,

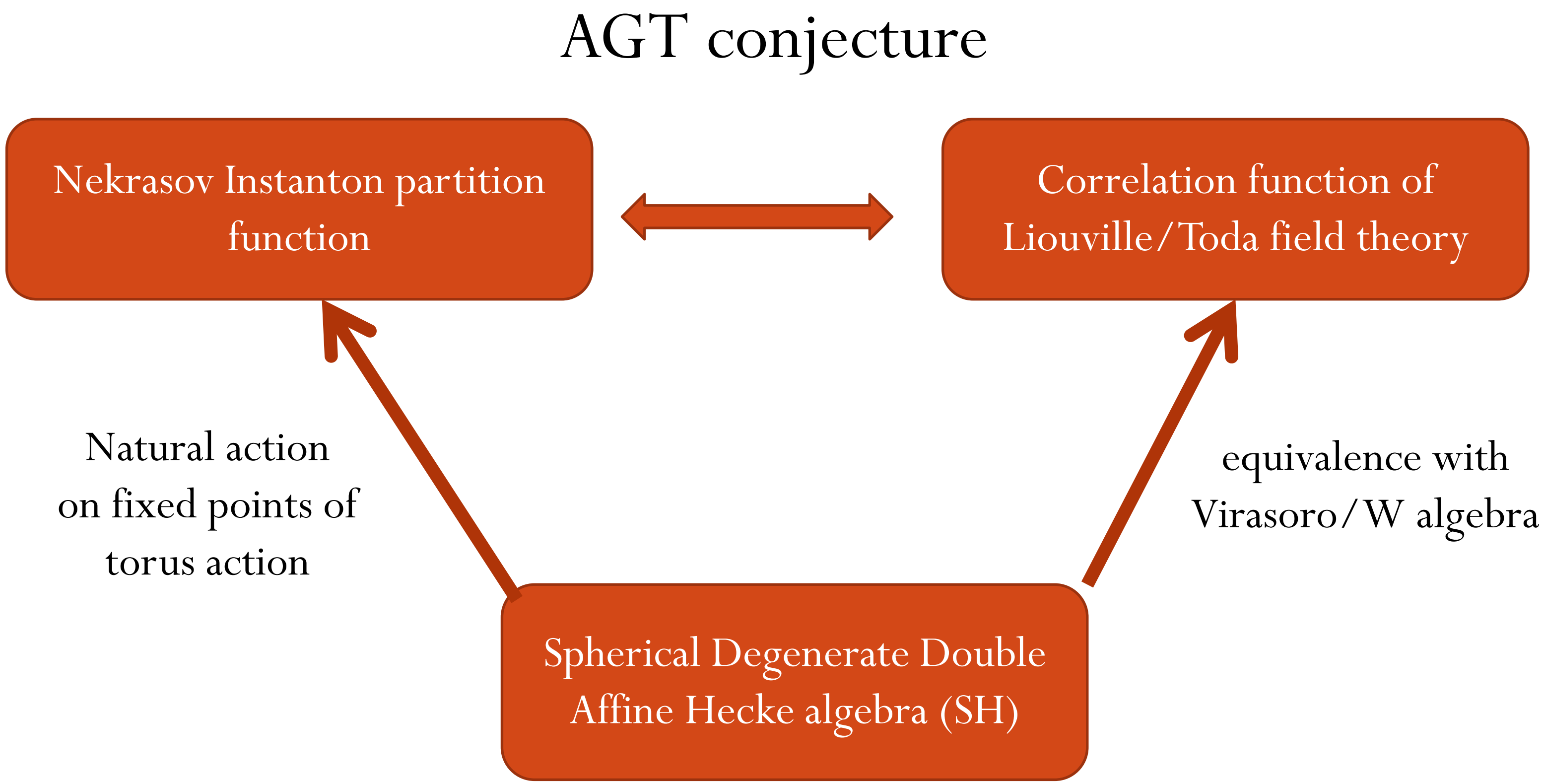
Riemann surface with punctures Σ → N=2 SYM

Four sphere → Toda/Liouville field theory

"Geometric Langrands program" (Witten)

Here we pursue more direct correspondence through the symmetry behind the scene.

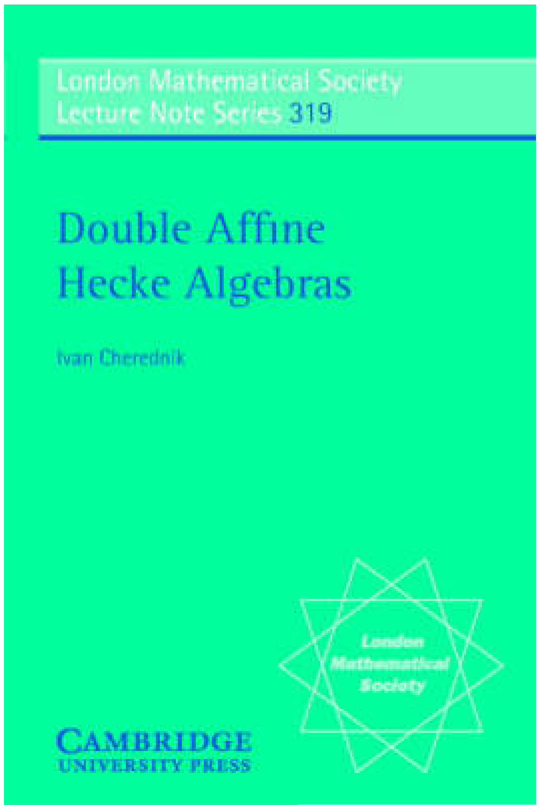
Mathematical foundation of AGT



Reference: Schiffmann and Vasserot
arXiv:1202.2756

Spherical Degenerate Double Affine Hecke Algebra

Double Affine Hecke Algebra (DAHA)



Introduced by Cherednik to investigate properties of Macdonald polynomial

DAHA for GL_n Generators:

$$X_1^{\pm 1}, \dots, X_n^{\pm 1}, Y_1^{\pm 1}, \dots, Y_n^{\pm 1}, T_1, \dots, T_{n-1}$$

Braid Relation: $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$

Hecke Relation: $(T_i + t^{1/2})(T_i - t^{-1/2}) = 0$

Extra generators: $X_i X_j = X_j X_i \qquad Y_i Y_j = Y_j Y_i$

$$T_i X_i T_i = X_{i+1}, \quad T_i^{-1} Y_i T_i^{-1} = Y_{i+1},$$

$$T_i X_j = X_j T_i, \quad T_i Y_j = Y_j T_i, \quad j \neq i, i+1$$

$$P X_i = X_{i+1} P, \quad P X_n = q^{-1} X_1 P, \quad P = Y_1^{-1} T_1 \cdots T_{n-1}, \quad i \neq n,$$

Graphical representation: Burella, Watts, Pasquier, Vala, Annals of Henri Poincarre 2013

Degenerate DAHA

Take a limit: $q = e^h, \quad t = e^{\beta h}, \quad h \to 0$

$$D_i = \lim_{h\to 0}(Y_i - 1)/h, \quad z_i = X_i, \quad T_i = \sigma_{i,i+1}$$

A representation: $D_i = z_i \nabla_i + \beta \sum_{j<i} \sigma_{ij}$

$$\nabla_i = \frac{\partial}{\partial z_i} + \beta \sum_{j(\neq i)} \frac{1}{z_i - z_j}(1 - \sigma_{ij})$$ Dunkl operator

$$[z_i, z_j] = 0, \quad [D_i, D_j] = 0, \quad z_i \sigma_{ij} = \sigma_{ij} z_j$$

$$[D_i, z_j] = \begin{cases} -\beta z_i \sigma_{ij} & i < j \\ z_i + \beta(\sum_{k<i} z_k \sigma_{ik} + \sum_{k>i} z_i \sigma_{ik} & i = j \\ -\beta z_j \sigma_{ij} & i > j \end{cases}$$

SH (Spherical DDAHA)

DDAHA: generated by $z_i^{\pm 1},\ D_i,\ \sigma \in S_N$

SH **restriction on the symmetric part**

generators: $\hat{D}_{rs} \sim \mathcal{S}\left(\sum_i (z_i)^r (D_i)^s\right)\mathcal{S}\,, \quad r \in \mathbf{Z}, s \in \mathbf{Z}_{\geq 0}$

$$\mathcal{S} = \frac{1}{n!}\sum_{\sigma \in S_n} \sigma$$: projection to totally symmetric part

$\beta \to 1,$ reduction to $\mathcal{W}_{1+\infty}$

$$D_{rs} \sim \oint d\zeta \bar{\psi}(\zeta)\zeta^r D^s \psi(\zeta), \quad D = \zeta\partial_\zeta$$

Calogero-Sutherland and Dunkl operator

The algebra SH (spherical degenerate double affine Hecke algebra) is most naturally understood in the context of Calogero-Sutherland system with the Hamiltonian:

$$\begin{aligned} H &= \sum_{i=1}^{n} \left(z_i \frac{\partial}{\partial z_i} \right)^2 + \beta \sum_{j<k} \left(\frac{z_j + z_k}{z_j - z_k} \right) \left(z_j \frac{\partial}{\partial z_j} - z_k \frac{\partial}{\partial z_k} \right) \\ &= \sum_i ((D_i)^2 - (n-1)\beta D_i) + \text{const.} \end{aligned}$$

$z_i = e^{i\theta_i}$: coordinate of a circle

The eigenstate of this Hamiltonian is called Jack polynomial which is labeled by Young diagram

$$H J_Y(z) = \epsilon_Y J_Y(z) \qquad \epsilon_Y = \sum_i ((Y_i)^2 + \beta(n - 2i + 1)Y_i)$$

Higher conserved charges: $H_l = \sum_i (D_i)^l \sim \hat{D}_{0,l}$

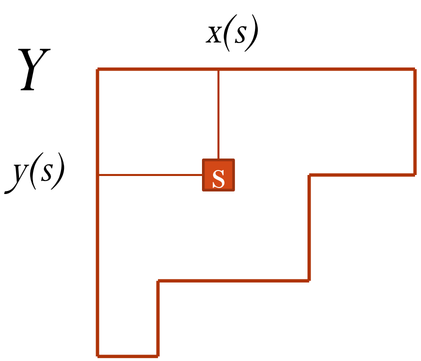
$$[H_n, H_m] = 0$$

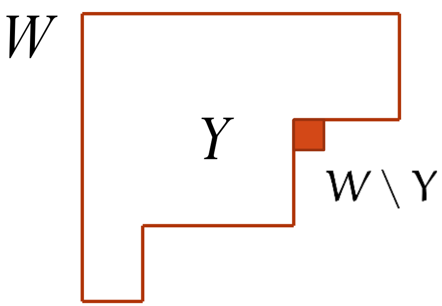
Recursion relation for Jack polynomials

The operators z_i and D_i represent Dunkl operators of Calogero-Sutherland system.
A natural action is defined for their eigen-function – Jack polynomials.
They can generate every Jack polynomial through recursion relations.
In particular, the action of $D_{1,n}$ and $D_{0,n}$ takes a simple form,

$$\hat{D}_{0,n} J_Y = \sum_{s \in Y} (c(s))^n J_Y$$

$$\hat{D}_{1,n} J_Y = \sum_{W \supset Y} c(W \setminus Y)^n \psi_{W \setminus Y} J_W \qquad c(s) = \beta x(s) - y(s)$$





SH^c : Central extension of SH

$$
\begin{aligned}
[D_{0,l}, D_{1,k}] &= D_{1,l+k-1}, \quad l \geq 1\,, \\
[D_{0,l}, D_{-1,k}] &= -D_{-1,l+k-1}, \quad l \geq 1\,, \\
[D_{-1,k}, D_{1,l}] &= E_{k+l} \quad l,k \geq 1\,, \\
[D_{0,l}, D_{0,k}] &= 0\,, \; k,l \geq 0\,,
\end{aligned}
$$

Nonlinear part of the algebra:

$$1 + \xi \sum_{l \geq 0} E_l s^{l+1} = \exp\left((-1)^{l+1} c_l \pi_l(s)\right) \exp\left(\sum_{l \geq 0} D_{0,l+1} \omega_l(s)\right)$$

$$\pi_l(s) = s^l G_l(1+\xi s) \omega_l(s) = \sum_{q=1,-\beta,\beta-1} s^l \left(G_l(1-qs) - G_l(1+qs)\right)$$

$$\xi = 1 - \beta \qquad \sum_{l=0}^{\infty} G_l(s) x^l = \log \frac{1-x}{s-x}$$

$c_l \quad (l = 0, 1, 2, \cdots)$ arbitrary parameters (central charges)

Action of SH on cohomology of instanton moduli space

Instanton moduli space and 2d CFT
An Intuitive Picture (1)

Rough illustration of the correspondence between
the cohomology of instanton moduli space and 2D CFT and SH

We start from a simplified version of moduli space where we keep the location of instantons as a moduli: Hilbert scheme of k points on euclidian 4D space C^2. $(C^2)^k$ is divided by S_k since these points are not distinguishable.

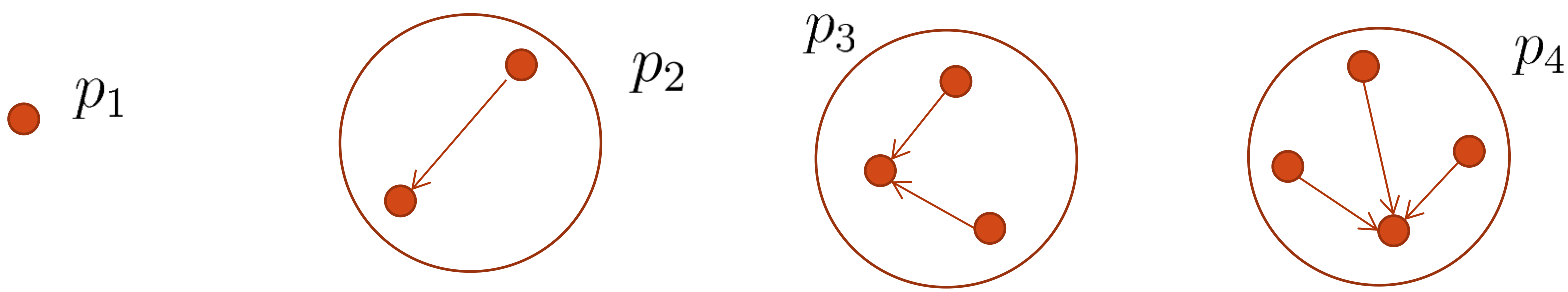
$$\mathrm{Hilb}_k = (\mathbf{C}^2)^k / S_k$$

S_k: permutation group

When the location of some instantons overlaps, we meet orbifold singularities. We need blow up such singularity in order to have a smooth space.

Instanton moduli space and 2d CFT An Intuitive Picture (2)

When two instantons overlap, we need attach two cocycle to resolve the singularity, which represents the direction where two instantons collides.
For three instanton case, we have four cycle.



In general, *2(n-1)* cycle is needed to resolve the singularity from *n*-points getting together. This resembles the Fock space of free boson if we sum over the moduli space of arbitrary number of instantons

$$H_*(\mathrm{Hilb}) = H_*(\sqcup_{n=0}^{\infty}\mathrm{Hilb}_n) = C[p_1, p_2, \cdots] = \mathcal{F}_{\mathrm{boson}}$$

Torus action and equivariant cohomology

More precise definition of Hilbert scheme

$$\mathrm{Hilb}_n = \{I \in \mathbf{C}[\![X,Y]\!], I: \text{ ideal of codim } n\}$$

Torus action:

$$(z_1, z_2)\cdot I = \left\{p((z_1)^{-1}X, (z_2)^{-1}Y; p(X,Y) \in I\right\}$$

fixed points of torus action: parameterized by a Young table Y

$$I_Y = \oplus_{s\notin Y} \mathbf{C} X^{x(s)-1} Y^{y(s)-1}$$

Action of DDAHA on fixed points

$$\begin{aligned}
\hat{D}_{0,n}[I_Y] &= \sum_{s \in Y} (c(s))^n [I_Y] \\
\hat{D}_{1,n}[I_Y] &= \sum_{W \supset Y} c(W \setminus Y)^n \psi_{W \setminus Y} [I_W] \\
\hat{D}_{-1,n}[I_Y] &= \sum_{W \subset Y} c(Y \setminus W)^n \tilde{\psi}_{Y \setminus W} [I_W]
\end{aligned}$$

which implies a correspondence:

$$J_Y \leftrightarrow [I_Y]$$

DDAHA is a natural symmetry acting on the cohomology of instanton moduli space

Coproduct of DDAHA and W-algebra

DDAHA is a natural symmetry acting on the cohomology of instanton moduli space.
It does not, however, describe the instanton moduli space.
For SU(N) case, it is parametrized by,

Location of instanton
size of each instanton
mutual SU(N) orientation

In order to describe the extra moduli, one needs to take the coproduct of DDAHA and pick up the irreducible representation out of it, just as composition of spins

$$[2] \otimes \cdots \otimes [2] = [l+1] \oplus \cdots$$

For SH, the "fundamental representation" is described by
a free boson Fock space *F* since it is parameterized by Young diagam Y

One may assign (in the large N limit) $p_m(z) = \sum_i (z_i)^m$

$$p_n(z) \leftrightarrow a_{-n}, \quad n\frac{\partial}{\partial p_n} \leftrightarrow a_n \qquad [a_n, a_m] = n\delta_{n+m,0}$$

Representation of SH

$$D_{1,0} = a_{-1}, \quad D_{-1,0} = a_1$$

$$D_{0,2} = \frac{1}{2}\sum_{n,m=1}^{\infty}(a_{-n}a_{-m}a_{n+m} + a_{-n-m}a_n a_m) - \frac{Q}{2}\sum_{n=1}^{\infty}(n-1)a_{-n}a_n$$

$$Q = \beta^{1/2} - \beta^{-1/2}$$

Awata, Odake, M, Shiraishi, 1995
in the context of Calogero-Sutherland

It is possible to generate all other generators from

$$D_{\pm 1,0}, D_{0,2}$$

by taking commutation relations

Representation by *n* bosons

We have derived a representation of SH on free boson Fock space $\mathcal{F}$
Representation on the direct product

$$\mathcal{F} \otimes \cdots \otimes \mathcal{F}$$

is nontrivial since the algebra is nonlinear, i.e. the coproduct is nontrivial.
In particular, the generator D_{rs} is not sum of operators acting on each F

$$D_{\pm 1,0} = \sum_{A=1}^{N} D^{(A)}_{\pm 1,0}$$

$$D_{0,2} = \sum_{A=1}^{N} D^{(A)}_{0,2} - Q \sum_{A<B} \sum_{r=1}^{\infty} r D^{(A)}_{r,0} D^{(B)}_{-r,0}$$

This time the algebra SH has nontrivial central extension

$$1 + (1-\beta)\sum_{l\geq 0} E_l s^{l+1} = \exp\Big(\sum_{l\geq 0}(-1)^{l+1} c_l \pi_l(s)\Big)\exp\Big(\sum_{l\geq 0} D_{0,l+1}\omega_l(s)\Big),$$

$$c_l = \sum_A (a_A - \xi)^l \qquad a_A : \text{ factor from torus action on SU(N)}$$

Representation by orthonormal basis

We introduce an orthonormal basis for the representation of DDAHA $|\vec{b}, \vec{W}\rangle$

$$D_{-1,l}|\vec{b},\vec{W}> \;=\; (-1)^l \sum_{q=1}^{N} \sum_{t=1}^{\tilde{f}_q} (b_q + B_t(W_q))^l \Lambda_q^{(t,-)}(\vec{W}) |\vec{b}, \vec{W}^{(t,-),q}> ,$$

$$D_{1,l}|\vec{b},\vec{W}> \;=\; (-1)^l \sum_{q=1}^{N} \sum_{t=1}^{\tilde{f}_q+1} (b_q + A_t(W_q))^l \Lambda_q^{(t,+)}(\vec{W}) |\vec{b}, \vec{W}^{(t,+),q}> ,$$

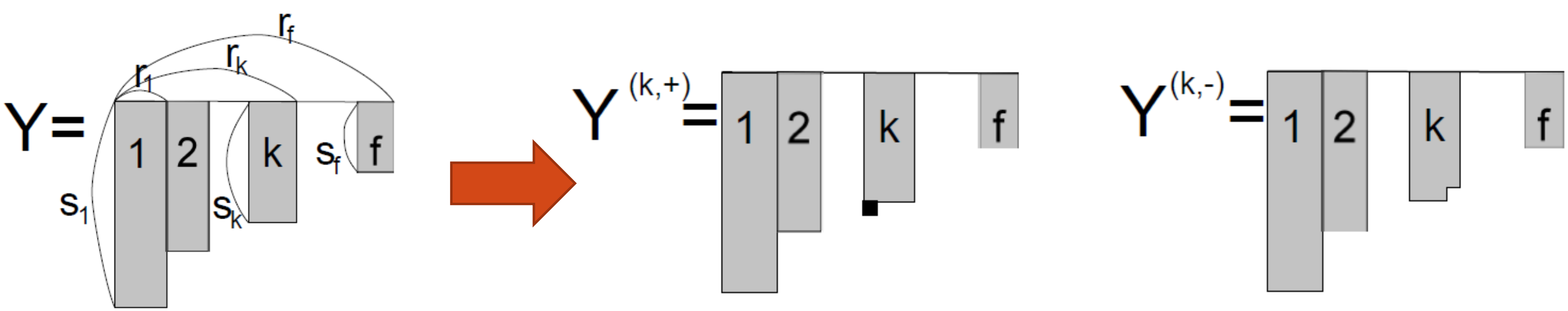
$$D_{0,l+1}|\vec{b},\vec{W}> \;=\; (-1)^l \sum_{q=1}^{N} \sum_{\mu \in W_q} (b_q + c(\mu))^l |\vec{b}, \vec{W}> ,$$

which gives a representation of DDAHA with central charges

$$c_l = \sum_{q=1}^{N} (b_q - \xi)^l, \quad \xi := 1 - \beta$$

$$\Lambda_p^{(k,+)}(\vec{a},\vec{Y}) = \left(\prod_{q=1}^{N}\left(\prod_{\ell=1}^{f_q}\frac{a_p-a_q+A_k(Y_p)-B_\ell(Y_q)+\xi}{a_p-a_q+A_k(Y_p)-B_\ell(Y_q)}\prod{}'^{f_q+1}_{\ell=1}\frac{a_p-a_q+A_k(Y_p)-A_\ell(Y_q)-\xi}{a_p-a_q+A_k(Y_p)-A_\ell(Y_q)}\right)\right)^{1/2}$$

$$\Lambda_p^{(k,-)}(\vec{a},\vec{Y}) = \left(\prod_{q=1}^{N}\left(\prod_{\ell=1}^{f_q+1}\frac{a_p-a_q+B_k(Y_p)-A_\ell(Y_q)-\xi}{a_p-a_q+B_k(p)-A_\ell(q)}\prod{}'^{f_q}_{\ell=1}\frac{a_p-a_q+B_k(Y_p)-B_\ell(Y_q)+\xi}{a_p-a_q+B_k(Y_p)-B_\ell(Y_q)}\right)\right)^{1/2}$$



$$A_k(Y) = \beta r_{k-1} - s_k - \xi, \quad (k=1,\cdots,f+1),$$
$$B_k(Y) = \beta r_k - s_k, \quad (k=1,\cdots,f),$$

$$\xi = 1-\beta$$

From the expression of coproduct, one may derive the stress energy tensor:

$$
\begin{aligned}
J(z) &= \sum_n J_n z^{-n-1} = \beta^{-1/2} \sum_{i=1}^{N} \partial_z \varphi^{(i)}(z)\,, \\
T(z) &= \sum_n L_n z^{-n-2} = \sum_{i=1}^{N} \left(\frac{1}{2} (\partial \varphi^{(i)}(z))^2 - Q \rho_i \partial^2 \varphi^{(i)}(z) \right)\,,
\end{aligned}
$$

where

$$
\varphi^{(i)}(z) = q^{(i)} + \alpha_0^{(i)} \log z - \sum_{n \neq 0} \frac{\alpha_n^{(i)}}{n} z^{-n}\,,
$$

$$
[\alpha_n^{(i)}, \alpha_m^{(j)}] = n \delta_{n+m,0} \delta_{ij}\,, \quad [\alpha_m^{(i)}, q^{(j)}] = \delta_{m,0} \delta_{ij}\,,
$$

$$
\rho_i = \frac{N+1}{2} - i, \qquad i, j = 1, \cdots, N\,.
$$

This is the standard representation of W_N -algebra in terms of N free bosons with central charge

$$
c = 1 + (N-1)(1 - Q^2 N(N+1)), \quad Q = \sqrt{\beta} - \sqrt{\beta}^{-1}
$$

Further confirmation of correspondence

Structure of Hilbert space is determined by the singular vector.

For *W*-algebra it is determined by the nonvanishing action of the screening operator.

For SH, one may find it by the condition.

$$D_{1,r}|\vec{a},\vec{Y}\rangle = 0 \quad r = 0,1,2,\cdots$$

which is implemented by

$$\Lambda_p^{(k,+)}(\vec{a},\vec{Y}) = 0 \qquad \text{for all } k, p$$

It gives a consistent result.

Recursive proof of AGT

Based on algebra SH, one may give a direct proof of AGT conjecture.

Proof for pure super Yang-Mills

Schiffmann and Vassero('12), Maulik and Okounkov ('13)

Proof for N=2*

Fateev-Litvinov ('09)

Proof for linear quiver gauge theories

Alba-Fateev-Litvinov-Tarnopolskiy ('10)
Belavin-Belavin ('11)
Fateev-Litvinov ('11)
Morosov-Mironov-Shakirov ('11)
Zhang-M ('11), Kanno-M-Zhang ('12, '13)
Morosov-Smirnov ('13)

Basic strategy

Typical form of Nekrasov partition function

$$Z = \sum_{\vec{Y}} f(\vec{Y}) q^{|\vec{Y}|}$$

q: moduli space parameter
Sum over fixed point of localization parameterized by Young diagram *Y*
f(Y): contribution of each fixed point

SH acts on the fixed points, thus gives recursion relations among coefficients which typically are the products of factors of the form:

$$g_{Y,W}(x) = \prod_{(i,j)\in Y} (x + \beta(Y'_j - i + 1) + W_i - j) \prod_{(i,j)\in W} (-x + \beta(W'_j - i) + Y_i - j + 1)$$

Recursion relation among factors

For the factor from bifundamental matter:

$$\frac{g_{Y_p^{(k,+)}W_q}(a_p-b_q-\mu)}{g_{Y_pW_q}(a_p-b_q-\mu)} = \frac{\prod_{\ell=1}^{\tilde{f}_q+1}(a_p-b_q-\mu+A_k(Y_p)-A_\ell(W_q)-\xi)}{\prod_{\ell=1}^{\tilde{f}_q}(a_p-b_q-\mu+A_k(Y_p)-B_\ell(W_q))},$$

$$\frac{g_{Y_p^{(k,-)}W_q}(a_p-b_q-\mu)}{g_{Y_pW_q}(a_p-b_q-\mu)} = \frac{\prod_{\ell=1}^{\tilde{f}_q}(a_p-b_q-\mu+B_k(Y_p)-B_\ell(W_q))}{\prod_{\ell=1}^{\tilde{f}_q+1}(a_p-b_q-\mu+B_k(Y_p)-A_\ell(W_q)-\xi)},$$

$$\frac{g_{Y_pW_q^{(\ell,+)}}(a_p-b_q-\mu)}{g_{Y_pW_q}(a_p-b_q-\mu)} = \frac{\prod_{k=1}^{f_p+1}(b_q-a_p+\mu+A_\ell(W_q)-A_k(Y_p))}{\prod_{k=1}^{f_p}(b_q-a_p+\mu+A_\ell(W_q)-B_k(Y_p)+\xi)},$$

$$\frac{g_{Y_pW_q^{(\ell,-)}}(a_p-b_q-\mu)}{g_{Y_pW_q}(a_p-b_q-\mu)} = \frac{\prod_{k=1}^{f_p}(b_q-a_p+\mu+B_\ell(W_q)-B_k(Y_p)+\xi)}{\prod_{k=1}^{f_p+1}(b_q-a_p+\mu+B_\ell(W_q)-A_k(Y_p))}.$$

One may derive the recursion relation for other factors from them. Other relation which we use is the following algebraic identities.

$$\sum_I \frac{1}{\zeta-x_I}\frac{\prod_J(x_I-y_J)}{\prod_J'(x_I-x_J)} = \prod_I \frac{\zeta-y_I}{\zeta-x_I}-1$$

Proof for pure SYM

AGT conjecture:

$$Z(\vec{a}) = \sum_{\vec{Y}} q^{|\vec{Y}|} f(\vec{a}, \vec{Y}) : \quad f(\vec{a}, \vec{Y}) = \prod_{p,q=1}^{N} \frac{1}{g_{Y_p, Y_q}(a_p - a_q)}$$

$$Z(\vec{a}) = \langle G | q^{L_0} | G \rangle$$

$|G\rangle$: Gaiotto state or Whittaker vector

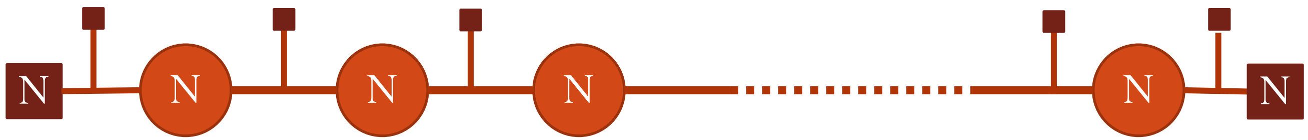
$$\begin{aligned} W_1^{(N)} |G\rangle &= (-1)^{1-N} \beta^{-N/2} |G\rangle \\ W_1^{(d)} |G\rangle &= 0 \qquad d = 1, 2, \cdots, N-1 \\ W_l^{(d)} |G\rangle &= 0 \qquad l \geq 2 \end{aligned}$$

One may map such conditions to similar ones in SH. One may directly check the following state satisfies them. They have the desired norm.

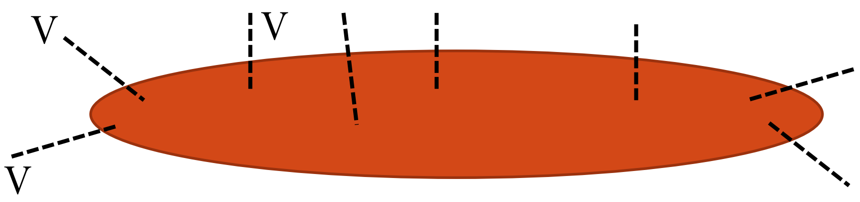
$$|G\rangle = \sum_{\vec{Y}} (f(\vec{a}, \vec{Y}))^{1/2} |\vec{a}, \vec{Y}\rangle$$

Proof for linear quiver

Nekrasov Partition function for conformal inv. SU(N)x…xSU(N) (*L* factors) quiver



is identical to conformal block function of 2D CFT described by
SU(N) Toda field theory with L+3 vertex operator insertions



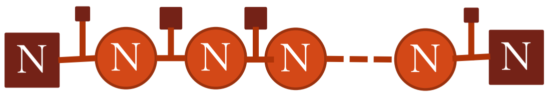
with identification of parameters

$$q_i \text{ : coupling const} \leftrightarrow \text{location of vertex}$$

$$\mu_i \text{ : mass for hyper} \leftrightarrow \text{momentum of vertex}$$

$$\vec{a}_i \text{ : VEV for vect. mult.} \leftrightarrow \text{weight of intermediate state}$$

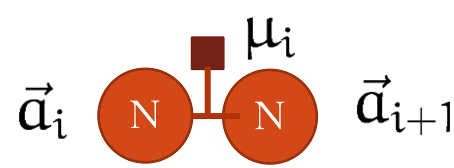
Partition function takes the form of matrix multiplication



Partition function looks like matrix multiplication

$$Z^{\text{inst}} = \sum_{\vec{Y}_1,\cdots,\vec{Y}_L} \left(\prod_i q_i^{|\vec{Y}_i|}\right) \bar{V}_{\vec{Y}_1} Z_{\vec{Y}_1 \vec{Y}_2} \cdots Z_{\vec{Y}_{L-1} \vec{Y}_L} V_{\vec{Y}_L}$$

$$Z_{\vec{Y},\vec{W}} = Z(\vec{a},\vec{Y};\vec{a},\vec{W};\mu) = \frac{\prod_{p,q} g_{Y_p,W_q}(a_p - b_q - \mu)}{\prod_{p,q} g_{Y_p Y_q}(a_p - a_q) g_{W_p,W_q}(b_p - b_q)}$$



This time, what we need to prove is

$$Z_{\vec{Y},\vec{W}} = \langle \vec{a}, \vec{Y} | V_\kappa | \vec{b}, \vec{W} \rangle$$

with a vertex operator insertion.

Recursion for quiver from SH

$$\delta_{\pm 1,n} Z_{\vec{Y},\vec{W}} - U_{\pm 1,n} Z_{\vec{Y},\vec{W}} = 0$$

$$\begin{aligned}
\delta_{-1,n} Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu) &= \sum_{p=1}^{N}\left(\sum_{k=1}^{f_p+1}(a_p+\nu+A_k(Y_p))^n \Lambda_p^{(k,+)}(\vec{a},\vec{Y}) Z(\vec{a},\vec{Y}^{(k,+),p};\vec{b},\vec{W};\mu)\right. \\
&\quad \left. - \sum_{k=1}^{\tilde{f}_p}(b_p+\mu+\nu+B_k(W_p))^n \Lambda_p^{(k,-)}(\vec{b},\vec{W}) Z(\vec{a},\vec{Y};\vec{b},\vec{W}^{(k,-),p};\mu)\right), \\
\delta_{1,n} Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu) &= \sum_{p=1}^{N}\left(-\sum_{k=1}^{f_p}(a_p+\nu+B_k(Y_p))^n \Lambda_p^{(k,-)}(\vec{a},\vec{Y}) Z(\vec{a},\vec{Y}^{(k,-),p};\vec{b},\vec{W};\mu)\right. \\
&\quad \left. + \sum_{k=1}^{\tilde{f}_p}(b_p+\nu+\mu+A_k(W_p)+\xi)^n \Lambda_p^{(k,+)}(\vec{b},\vec{W}) Z(\vec{a},\vec{Y};\vec{b},\vec{W}^{(k,+),p};\mu)\right),
\end{aligned}$$

The coefficients $\delta_{-1,n}$ coincide with $D_{-1,n}$ action on the bra and ket.
This recursion formula uniquely fix the partition function thus proves AGT.
On the other hand the coefficient appearing in $\delta_{1,n}$ is slightly shifted by ξ.
This small mismatch will be explained by an exotic choice of vertex operator.

Anomaly in vertex operator

The vertex operator is written as a product of U(1) (Heisenberg) part and W_N part:

$$V = \tilde{V}^H V^W$$

The standard definition in terms of free boson is written as,

$$\tilde{V}^H_\kappa = e^{-\frac{\kappa}{N}\vec{e}\cdot\vec{\varphi}}, \quad V^W_\kappa = e^{-\kappa(\vec{e}_N - \frac{\vec{e}}{N})\vec{\varphi}},$$

$$\vec{e} := (1, \cdots, 1) \qquad \vec{e}_N = (0, \cdots, 0, 1)$$

We need modify the U(1) part of the vertex operator

Carlsson-Okounkov

$$V^H_\kappa = e^{\frac{1}{\sqrt{N}}(NQ-\kappa)\phi_-} e^{-\frac{\kappa}{\sqrt{N}}\phi_+}, \qquad \phi = \vec{e}\cdot\varphi$$

Anomaly in the commutator with Virasoro cancels those in recursion relation.

Summary

1. We reviewed Spherical DDAHA (SH) which describes the structure of instanton moduli space.

2. SH is equivalent to W_N-algebra + U(1) factor when it is realized by irreducible "symmetric part" in the tensor product of N "fundamental" representation. In principle, it is much larger symmetry such as $W(1+\infty)$

3. SH is much simpler to be treated in the instanton partition function. Indeed, the recursive proof of AGT is given in variety of cases.

Future prospects

- The symmetry SH is quite interesting mathematical object and should be pursued further.
- Connection between coproduct of SH and integrable models (Maulick-Okounkov): R-matrix and Yangian
- Generalization of SH for other gauge groups
- Treatment of nonlocal objects in SYM.
- Application of SH to higher spin gauge theory and/or fractional Hall effects
- Nekrasov-Shatashvilli limit: action on noncommutative Riemann surface, Toda system, Calogero system