AGT

1. Zamolodchikov
2. $\epsilon_2 \to 0 \Rightarrow$ QIS — using both sides of AGT!
3. Int SW $\overset{AGT}{\Longleftrightarrow}$ Int CFT | $\boxed{\text{Spectral duality}}$

Spin chains	Gaudin
$SU(k)^{n-1} \Leftarrow$ GL_n on k sites (of minimal rank) $\Leftrightarrow$	$(n+2)$-points GL_k Gaudin
$\uparrow$ old result (GGM)	$\nwarrow$ AGT (MMRZZ) $\swarrow$
5d $SU(k)^{n-1} \leftarrow GL_n$ on ~~k~~ sites	GL_k on n ~~k~~ sites

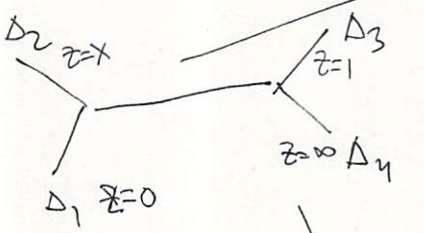
spin matrices of minimal rank

5d $\overset{\shortparallel}{SU(n)^{k-1}}$

4. Modular invariance $\Leftrightarrow$ S-duality

$\boxed{\text{next page}}$ P2

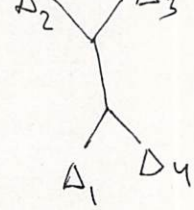
$$B_4(z_1, z_2, z_3, z_4) \sim \cancel{z_{12}\, z_{13} \dots} \prod_{i<j} z_{ij}^{\gamma_{ij}} \cdot B(x) \qquad x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$



S-transform

$1 \leftrightarrow 3$

$z_1 = 0$
$z_2 = x$
$z_3 = 1$
$z_4 = \infty$

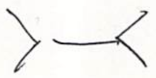


$x \to 1-x$

Modular transform — permutations of points S_3 generated by two generators S & T.

~~S-tran~~ T-transform: $1 \leftrightarrow 2$: $x \to \frac{x}{1-x}$

5-point modular inv. – S_4

Correlator  =  $= \sum_{\Delta} B_{\Delta}(x)\bar{B}_{\Delta}(\bar{x})$

$$B_{\Delta}(x)\, B(\cancel{\Delta}) \to \sum_{\Delta'} B'_{\Delta'}(x) M_{\Delta'\Delta}$$

↑ does not depend on x

Racah coeff. of Virasoro algebra

‖

6j symbols:

$$M_{\Delta\Delta'} = \left(\begin{array}{cc|c} \Delta_1 & \Delta_2 & \Delta \\ \Delta_3 & \Delta_4 & \Delta' \end{array}\right)$$

6-point  = 

Sicilian quiver (?)

Seiberg - Witten

S-duality:
(EM-duality)

Σ, dS

$\uparrow_{RS}$, A_i, B_i

$$\frac{\partial\, dS}{\partial\, \text{moduli}} = \text{hol}$$

$$\oint_{A_i} dS = a_i$$

$$\oint_{B_i} dS = \frac{\partial \mathcal{F}}{\partial a_i} \Leftarrow \text{prepotential} \; ; \quad \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} = \tau_{ij}$$

$M = |n_i a_i + m_i a_D^i|$

(n_i, m_i) - dyon

$n \to m$ interchanges electric & magnetic charges

$g=1$ - $SU(2)$

$\cancel{A_i \Leftrightarrow B_i}$

$$\begin{pmatrix} A_i \\ B_i \end{pmatrix} \to SL(2, \mathbb{Z})$$

$$\tau \to \tau + 1$$
$$\tau \to -\frac{1}{\tau}$$

$A \Leftrightarrow B$: $\cancel{\mathcal{F}^D = a a_D - \mathcal{F}}$ $\qquad \frac{\partial \mathcal{F}}{\partial a} = b$

$$\frac{\partial \mathcal{F}^*}{\partial b} = -a \qquad \mathcal{F}^* = -ab + \mathcal{F}$$

$$\tau \to -\frac{1}{\tau}$$

$B \to A + B$: $\tau \to \tau + 1$

Problem: lifting to Nekrasov functions

Manifestly ex. of SW ~~pure gauge~~ with $N_f = 4$, $m_i = 0$

$$dS = \frac{u\,dz}{\sqrt{z(z-x)(z-1)}}$$

$$\oint_A dS = K(x) \qquad \oint_B dS = K(1-x)$$

$$\mathcal{F}(a) = i\frac{a^2}{2} \frac{K(1-x)}{K(x)}$$

$$F^*(a, x) = F(a, 1-x)$$

CFT: $\Delta = \frac{Q^2}{4} - a^2$ $\quad c = 1 + 6Q^2 \quad Q = b + \frac{1}{b}$

$\Delta_i = \alpha(Q-\alpha)$

$b^2 = -\frac{\epsilon_1}{\epsilon_2}$

$g_s^2 = -\epsilon_1\epsilon_2 \; : \; \Delta \to \frac{\Delta}{g_s^2}$

SW limit: $g_s \to 0$: Mamo - planar limit

$(b = i\hbar, c = 0)$

$$\boxed{e^{\mathcal{F}_{SW}/g_s^2} \simeq B} \approx \int dz_i \prod_{i<j} z_{ij}^{\mp 2b^2} \prod z_i^{\mp 2b\alpha_1/g_s} \prod (z_i - x)^{\mp 2b\alpha_2/g_s} \prod (z_i - 1)^{\mp 2b\alpha_3/g_s}$$

point 0 — point x — point 1

$N = N_1 + N_2$ $\quad b N_1 = \alpha - \alpha_1 - \alpha_2$

$\int_0^x \quad \int_0^1$ $\quad b N_2 = Q - \alpha - \alpha_3 - \alpha_4$

$$\mathcal{F}^*(b, x) = \mathcal{F}(b, 1-x) = -ab + \mathcal{F}(a, x)$$

$$e^{\mathcal{F}^*(b)/g_s^2} = \int da \, e^{-\frac{ab}{g_s^2}} \, e^{\mathcal{F}(a)/g_s^2}$$

$M_{ab} = e^{-\frac{ab}{g_s^2}}$ - Fourier

GMMN: Claim It is exact (perturbatively in g_s^2)

$\boxed{\text{To check it - Mamo}}$

Exact case

1) $\Delta_i = \frac{1}{16}$ $\quad c=1$

$$B(a,x) = \frac{e^{\pi i / g_s^2 \, \tau a^2}}{[x(1-x)]^{1/8}\, \theta_3(\tau)} \qquad \tau = i\frac{K(1-x)}{K(x)}$$

$$B(a,x) = \int \frac{db}{g}\, e^{\frac{2\pi i ab}{g^2}}\, B(b,1-x)$$

2) SU(2) with adjoint : toric conformal block

$$\Delta = \frac{a^2}{g^2} \quad \Delta_{ext} = 0 \quad c = 1 :$$

$$B(a,\tau) = \frac{e^{\frac{2\pi i}{g_s^2}\tau a^2}}{\eta(\tau)} \qquad \tau - \text{torus}$$

$$B(a,\tau) = \int \frac{db}{g}\, e^{\frac{4\pi i\, ab}{g_s^2}}\, B(b,-\tfrac{1}{\tau})$$

Mariño technique:

$$\rho_1(z) = \left\langle \sum_i \frac{1}{z-z_i} \right\rangle \equiv \hat{\nabla}(z)\, F$$

$$\rho_2(z_1,z_2) = \left\langle \sum_{i,j} \frac{1}{(z_1-z_i)(z_2-z_j)} \right\rangle_c \equiv \hat{\nabla}(z_1)\hat{\nabla}(z_2)\, F$$

etc

$$g_s^2 b^2 \rho_2(z,z) + g_s Q \frac{\partial \rho_1(z)}{\partial z} + \rho_1^2(z) + W'(z)\rho_1(z) + \sum_i \frac{c_i}{z-x_i} = 0$$

$x_0 = 0$
$x_1 = x$
$x_2 = 1$
$x_3 = \infty$

ρ_2 includes ρ_3 etc.

$g_s \to 0$: genus expansion

check: modular transform is Fourier

~~Inte~~ Torus: $\Delta_{ext} = \mu(Q-\mu)$:

$$M(a,a') = \int d\xi \frac{S_b(\xi+\frac{\mu}{2}-a')\, S_b(\xi+\frac{\mu}{2}+a')}{S_b(\xi+Q-\frac{\mu}{2}-a')\, S_b(\xi+Q-\frac{\mu}{2}+a')} e^{4\pi i a\xi}$$

There are non-perturbative corrections

Interpretation

Dual partition functions: $\cancel{Z_a(Q)}$

$$Z_a(q) \quad ; \quad \bar{Z}_b(q)$$

Operators $\hat{A}$ & $\hat{B}$ such that

$$\left.\begin{aligned} \hat{A}\, Z_a(q) &= \lambda_a \cancel{e^{ia}}\, Z_a(\cancel{x}q) \\ \hat{B}\, \bar{Z}_b(q) &= \lambda_b\, \bar{Z}_b(q) \end{aligned}\right\}$$

This is our way to define the partition functions

Suppose $\hat{A}\hat{B} = e^{i\hbar}\hat{B}\hat{A}$

$$\lambda_a = e^{ia} \qquad \lambda_b = e^{ib}$$

Duality is defined by comm. rels between $\hat{A}$ & $\hat{B}$

Then: $Z_a(q) = \int d\cancel{b}\, e^{iab/\hbar}\, \bar{Z}_b(q)\, db$

~~In terms Zh~~ In the eigenvalue representation:

$$\check{A} = e^{ia} \qquad \check{B} = e^{\hbar \frac{\partial}{\partial a}}$$

Then: $\check{A}(a)\, M(a,b) = \check{B}(b)\, M(a,b) \Rightarrow$

$$M(a,b) = e^{i\frac{ab}{\hbar}}.$$

Conformal block is an eigenfunction of some $\check{A}$ and ~~the~~ the modular transformed one of some $\check{B}$.

$$\rho_1(z) = \hat{\nabla}(z)\, F = \frac{1}{Z}\, \hat{\nabla}(z)\, Z = \frac{1}{Z}\, \check{\nabla}(z)\, Z$$

check operators in the ~~space of~~ moduli space of ~~sol~~ solutions of the loop equations

$$\left[b \oint_{A_i} \check{\nabla}(z),\ b \oint_{B_j} \check{\nabla}(z) \right] = g_s\, \delta_{ij}$$

$$L_\gamma = e^{b \oint_\gamma \check{\nabla}(z)} \quad , \quad L_A L_B = e^{g_s} L_B L_A$$

In the eigenvalue representation:

$$\oint_A \rho_1(z) = a$$

$$\oint_B \rho_1(z) = b \frac{\partial \log Z}{\partial a}$$

$$L_A Z = e^{ba} Z$$

$$L_B Z = e^{b \frac{\partial}{\partial a}} Z$$

This gives Fourier, Non-perturbative corrections:

there are two branches in ~~of~~ the planar limit : $\rho_1 = \pm\sqrt{}$

$$\Downarrow$$

$$\oint_A \rho_1^{\pm}(z) = \pm a$$

$$Z_\gamma^\pm = e^{b \oint_\gamma \check{\nabla}^\pm}$$

$$L_\gamma = L_\gamma^+ + L_\gamma^-$$

$$L_B = e^{b\partial_a/g_s} + e^{-b\partial_a/g_s}$$

$$L_A = e^{ba} + e^{-ba}$$

one exponential survives at small g_s, corrections are non-perturbative in g_s.

$B = \cancel{\frac{Z}{N(a)}}$

$$\frac{1}{N(a)} e^{b\partial_a} N(a) + \frac{1}{N(-a)} e^{-b\partial_a} N(-a)$$

normalization factor

$B \sim Z$

Eigen~~value~~ function is conformal block

$$L_B M(a,a') = L_A(a') M(a,a')$$

$$M(a,a') = \int d\xi\; C_1(\xi) C_2(a') \frac{S_b(\xi + \frac{\mu}{2} - a') S_b(\xi + \frac{\mu}{2} + a')}{S_b(\xi + Q - \frac{\mu}{2} - a') S_b(\xi + Q - \frac{\mu}{2} + a')} \times e^{4\pi i a \xi}$$