Gauge instantons in noncommutative space

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1. Introduction

Moyal space

The Moyal space $\mathbb{R}^4$ is a four-dimensional non-commutative space which coordinate ring is non-commutative associative, endowed with the Moyal product

$$(f \star g)(x) = \exp\left(\frac{\sqrt{-1}}{2}\sum_{\mu,\nu=1}^{4}\theta^{\mu\nu}\partial_\mu^{x'}\partial_\nu^{x''}\right)f(x')g(x'')\Big|_{x'=x''=x}$$
$$= f(x)g(x) + \frac{\sqrt{-1}}{2}\sum_{\mu\nu}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + O(\theta^2),$$

where $\partial_\mu^{x'} = \partial/\partial x'^{\mu}$ *etc* and $\theta^{\mu\nu}$ is a real anti-symmetric tensor of the form

$$(\,\theta^{\mu\nu}\,) = \begin{bmatrix} 0 & -\theta_1 & & \\ \theta_1 & 0 & & \\ & & 0 & -\theta_2 \\ & & \theta_2 & 0 \end{bmatrix}.$$

Non-commutativity is measured by $\theta^{\mu\nu}$ as seen from

$$\left[\,x^\mu,\,x^\nu\right]_\star \equiv x^\mu \star x^\nu - x^\nu \star x^\mu = \sqrt{-1}\theta^{\mu\nu}.$$

Non-commutative gauge instanton

Non-commutative gauge instanton A is a solution of Aniti-Seld-Dual (ASD) Yang-Mills equation on the Moyal space $\mathbb{R}^4$

$$*F = -F$$

where F is the curvature two-form

$$F = \frac{1}{2}\sum_{\mu,\nu} F_{\mu\nu} dx^\mu \wedge dx^\nu,$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \star A_\nu - A_\nu \star A_\mu,$$

and the symbol $*$ means taking the Hodge star

$$(*F)_{\mu\nu} = \frac{1}{2}\sum_{\rho,\sigma} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \qquad \left(\epsilon_{1234} = 1\right)$$

ADHM construction

The Atiyah-Drinfeld-Hitchin-Manin (ADHM) construction is one of the most useful method to generate all instanton solutions just by solving matrix equations.

It is based on a duality between an instanton moduli space specified by ASD Yang-Mills equation (PDE) and a moduli space specified by ADHM equation (matrix equation).

Contents

In this talk, we outline a non-commutative version of the ADHM construction and its inverse construction.

2. Non-commutative ADHM construction

Let $V \equiv \mathbb{C}^k$ and $W \equiv \mathbb{C}^N$ ($k, N = 1, 2, \cdots$).

Non-commutative ADHM data

Quadruple of matrices (B_1, B_2, I, J), where

$$B_1,\ B_2 \in \text{End}\ V,$$

$$I \in \text{Hom}(W,\ V), \qquad J \in \text{Hom}(V,\ W)$$

are called the non-commutative ADHM data when the matrices satisfy the non-commutative ADHM equations:

$$i)\ \ [B_1,\ B_2] + IJ = \mathbf{0},$$

$$ii)\ \ [B_1,\ B_1^\dagger] + [B_2,\ B_2^\dagger] + II^\dagger - J^\dagger J = \zeta \mathbf{1}_V,$$

where $\zeta \equiv -2(\theta_1 + \theta_2)$ is proportional to the seld-dual component of $\theta^{\mu\nu}$. When $\zeta = 0$, the matrices are the standard ADHM data of $U(N)$ k-instanton on $\mathbb{R}^4$.

In what follows, the condition $\zeta > 0$ is imposed.

Fock representation of the Moyal space

We associate complex coordinates $z_1 \equiv x^2+\sqrt{-1}x^1$, $z_2 \equiv x^4+\sqrt{-1}x^3$ with operators

$$\widehat{z}_i = \sqrt{-2\theta_i}\,\widehat{a}_i^\dagger, \qquad \widehat{\overline{z}}_i = \widehat{z}_i^\dagger = \sqrt{-2\theta_i}\,\widehat{a}_i$$

where $\widehat{a}_i$, $\widehat{a}_i^\dagger$ $(i = 1, 2)$ are annihilation and creation operators of harmonic oscillators with commutation relations $\left[\,\widehat{a}_i, \widehat{a}_j^\dagger\,\right] = \delta_{ij}$. These operators satisfy the same commutation relations as in the Lie algebra derived from the Moyal product:

$$\left[\,\widehat{\overline{z}}_i, \widehat{z}_j\,\right] = -2\theta_i\delta_{ij}, \qquad \left[\,\widehat{z}_i, \widehat{z}_j\,\right] = \left[\,\widehat{\overline{z}}_i, \widehat{\overline{z}}_j\,\right] = 0.$$

- We interpret $\widehat{z}_i$, $\widehat{\overline{z}}_i$ as endomorphisms over the Fock space $\mathcal{F}$

$$\mathcal{F} = \bigoplus_{m_1,m_2\geq 0} \mathbb{C}(\widehat{a}_1^\dagger)^{m_1}|0\rangle^{(1)} \otimes (\widehat{a}_2^\dagger)^{m_2}|0\rangle^{(2)}$$

This yields an identification of the Moyal space $\mathbb{R}^4$ (plus the complex structure) with End$\mathcal{F}$. In particular, left- and right-multiplications of the Moyal product are converted respectively to the left- and right-multiplications on End$\mathcal{F}$.

Non-commutative ADHM complex

Let $\widehat{V} \equiv V \otimes \mathrm{End}\mathcal{F}$ and $\widehat{W} \equiv W \otimes \mathrm{End}\mathcal{F}$, where $\mathcal{F}$ is the Fock space. For the non-commutative data (B_1, B_2, I, J), we associate a chain of linear mappings as

$$\widehat{V} \xrightarrow{\ \widehat{\beta}\ } \begin{array}{c} \widehat{W} \\ \bigoplus \\ \widehat{V} \\ \bigoplus \\ \widehat{V} \end{array} \xrightarrow{\ \widehat{\alpha}\ } \widehat{V} \tag{1}$$

where linear operators $\widehat{\alpha}$ and $\widehat{\beta}$ are given by

$$\widehat{\alpha} = \big(I,\, \widehat{z}_2 - B_2,\, \widehat{z}_1 - B_1 \big) \quad {}^t(\,\hat{w},\, \hat{v}_1,\, \hat{v}_2\,) \mapsto I\hat{w} + (\widehat{z}_2 - B_2)\hat{v}_1 + (\widehat{z}_1 - B_1)\hat{v}_2,$$

$$\widehat{\beta} = {}^t\big(J,\, -(\widehat{z}_1 - B_1),\, \widehat{z}_2 - B_2 \big) \quad \hat{v} \mapsto {}^t(\, J\hat{v},\, -(\widehat{z}_1 - B_1)\hat{v},\, (\widehat{z}_2 - B_2)\hat{v}\,).$$

- The ADHM equation $i)$ yields $\widehat{\alpha}\widehat{\beta} = \mathbf{0}$. Thus, the chain (1) is a complex.

Dirac operator

The preceeding complex gives rise to the following Dirac operator:

$$\widehat{\nabla} \equiv (\widehat{\alpha}^\dagger, \widehat{\beta}) = \begin{pmatrix} I^\dagger & J \\ \widehat{\overline{z}}_2 - B_2^\dagger & -(\widehat{z}_1 - B_1) \\ \widehat{\overline{z}}_1 - B_1^\dagger & \widehat{z}_2 - B_2 \end{pmatrix} \quad \begin{matrix} \widehat{V} \\ \bigoplus \\ \widehat{V} \end{matrix} \quad \longrightarrow \quad \begin{matrix} \widehat{W} \\ \bigoplus \\ \widehat{V} \\ \bigoplus \\ \widehat{V} \end{matrix} \qquad (2)$$

The ADHM equation $ii)$ yields $\widehat{\alpha}\widehat{\alpha}^\dagger = \widehat{\beta}^\dagger\widehat{\beta} := \widehat{\Box}$. Together with the closedness $\widehat{\alpha}\widehat{\beta} = 0$, the Laplacian becomes a diagonal matrix of the form

$$\widehat{\nabla}^\dagger\widehat{\nabla} = \mathsf{diag}\big(\widehat{\Box}, \widehat{\Box}\big) \quad \widehat{V} \oplus \widehat{V} \longrightarrow \widehat{V} \oplus \widehat{V}.$$

- When $\zeta \neq 0$, $\widehat{\Box} \in \mathrm{End}\widehat{V}$ is bijective for $\forall(B_1, B_2, I, J)$ and therefore invertible:

$$\exists\ \widehat{\Box}^{-1} \in \mathrm{End}\widehat{V}.$$

Solutions of the Dirac equation

- When $\zeta \neq 0$, $\quad \mathrm{Ker}\widehat{\nabla} = \emptyset, \quad \mathrm{Ker}\widehat{\nabla}^\dagger \simeq \bigoplus^N \mathrm{End}\widehat{V} \quad$ for $\forall(B_1, B_2, I, J)$.

On $\mathrm{Ker}\widehat{\nabla}^\dagger \subset \widehat{W} \oplus \widehat{V} \oplus \widehat{V}$, End$\mathcal{F}$ acts freely by the right multilplication $\widehat{\mathsf{v}} \mapsto \widehat{\mathsf{v}}\widehat{\Omega}$, where $\widehat{\Omega} \in \mathrm{GL}(N) \otimes \mathrm{End}\mathcal{F}$. By taking a suitable $\widehat{\Omega}$, the following basis $\widehat{\mathsf{v}}^{(1)}, \cdots, \widehat{\mathsf{v}}^{(N)}$ of $\mathrm{Ker}\widehat{\nabla}^\dagger$ is chosen.

$$\widehat{\nabla}^\dagger \widehat{\mathsf{v}}^{(a)} = 0, \qquad \widehat{\mathsf{v}}^{(a)\dagger} \widehat{\mathsf{v}}^{(b)} = \delta^{ab} \mathbf{1}_{\mathcal{F}} \qquad (a, b = 1, \cdots, N)$$

We introduce the matrix

$$\widehat{\mathsf{V}} = \left(\widehat{\mathsf{v}}^{(1)}, \cdots, \widehat{\mathsf{v}}^{(N)} \right)$$

The orthonormality of the basis implies

$$\widehat{\mathsf{V}}^\dagger \widehat{\mathsf{V}} = \mathrm{id}$$

Non-commutative ADHM construction

The trivial connection on $\widehat{W} \oplus \widehat{V} \oplus \widehat{V}$ induces an $U(N)$ connection $\widehat{A}$ on $\bigoplus^{N} \mathrm{End}\mathcal{F}$.
It is given by using the matrix $\widehat{\mathsf{V}}$ as

$$\widehat{A}_\mu \; = \; \widehat{\mathsf{V}}^\dagger \, \hat{\partial}_\mu \cdot \widehat{\mathsf{V}} \quad \in \; \mathrm{u(N)} \otimes \mathrm{End}\mathcal{F}$$

where $\hat{\partial}_\mu$ ($\mu = 1, \cdots, 4$) are differential operators of the form

$$\hat{\partial}_\mu \; \equiv \; -\sqrt{-1}\theta^{-1}_{\mu\nu}\widehat{x}^\nu, \qquad \hat{\partial}_\mu \cdot \widehat{\mathsf{V}} \; \equiv \; [\, \hat{\partial}_\mu, \; \widehat{\mathsf{V}} \,].$$

• The curvature $\widehat{F} = \frac{1}{2}\sum_{\mu,\nu} \widehat{F}_{\mu\nu} dx^\mu \wedge dx^\nu$, where $\widehat{F}_{\mu\nu} = \hat{\partial}_\mu \cdot \widehat{A}_\nu - \hat{\partial}_\nu \cdot \widehat{A}_\mu + [\, \widehat{A}_\mu, \; \widehat{A}_\nu \,]$ takes the form

$$\widehat{F} = \sum_{i,j,k=1}^{3} \widehat{C}_i \sqrt{-1}\epsilon_{ijk}(dx^j \wedge dx^k - *dx^j \wedge dx^k), \qquad \widehat{C}_i = \widehat{\mathsf{V}}^\dagger \begin{bmatrix} 0 & 0 \\ 0 & \sigma^i \otimes \widehat{\square}^{-1} \end{bmatrix} \widehat{\mathsf{V}}$$

where $\epsilon_{123} = 1$ and $\sigma^{i=1,2,3}$ denote Pauli's matrices. Clearly we see

$$*\widehat{F} \; = \; -\widehat{F}$$

3. Non-commutative topological charge

2nd Chern class

$$\widehat{c}_2 \equiv \frac{1}{32\pi^2}\mathrm{Tr}_{\mathbb{C}^N}\widehat{F}_{\mu\nu}\widehat{F}_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} \in \mathrm{End}\,\mathcal{F}$$

where $\epsilon^{1234} = 1$.

CGOT formula

The ADHM construction yields $(32\pi)^{-1}\widehat{F}_{\mu\nu}\widehat{F}_{\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = \pi^{-2}\sum_i \widehat{C}_i^2$. By using the expression we obtain the formula:

$$\widehat{c}_2 = \frac{1}{16\pi^2}\mathrm{Tr}_{\mathbb{C}^k}\,\hat{\partial}^2\cdot\left\{\hat{\partial}^\mu\cdot\left(\widehat{\Box}\,\hat{\partial}_\mu\cdot\widehat{\Box}^{-1}\right)\right\} \in \mathrm{End}\,\mathcal{F}$$

where

$$\widehat{\Box} = \sum_i(\widehat{z}_i - B_i)(\widehat{\overline{z}}_i - B_i^\dagger) + II^\dagger \;\left(= \widehat{\alpha}\widehat{\alpha}^\dagger\right)$$

$$\hat{\partial}^2\cdot\widehat{\mathcal{O}} = \hat{\partial}^\mu\cdot\left(\hat{\partial}_\mu\cdot\widehat{\mathcal{O}}\right), \quad \widehat{\mathcal{O}} \in \mathrm{End}\mathcal{F}$$

Topological charge

$$c_2 \equiv \langle \widehat{c}_2 \rangle$$

where

$$\langle \widehat{c}_2 \rangle \equiv 4\pi\theta_1\theta_2 \lim_{L\to+\infty} \sum_{m_1+m_2\leq L} \langle m_1, m_2| \widehat{c}_2 |m_1, m_2\rangle$$
$$\left(= 4\pi\theta_1\theta_2 \mathrm{Tr}_{\mathcal{F}} \widehat{c}_2\right)$$

c_2 is computed by applying the CGOT formula together with using formulas for the partial diagonal sum of $\widehat{\mathcal{O}}_{(m_1,m_2),(n_1,n_2)} \equiv \langle m_1, m_2| \widehat{\mathcal{O}} |n_1, n_2\rangle$

• It turns out that c_2 takes the value

$$c_2 = k, \qquad N = 1,\ 2,\ \cdots .$$

4. Inverse construction

Let $N, k = 1, 2, \cdots$ and specialize $\theta_1 = \theta_2 = -\zeta/4$.

Non-commutative ASD instanton

$$\widehat{A}_\mu \in u(N) \otimes \mathrm{End}\mathcal{F}$$

1. $\widehat{F}^{(+)}_{\mu\nu} = \left(\widehat{\partial}_\mu \cdot \widehat{A}_\nu - \widehat{\partial}_\nu \cdot \widehat{A}_\mu + [\widehat{A}_\mu, \widehat{A}_\nu]\right)^{(+)} = 0$

2. $c_2 = k$

3. $\exists\, \widehat{g}_\infty \in \mathrm{End}\mathcal{F}^N$ such that

   - $\widehat{g}^\dagger_\infty \widehat{g}_\infty = 1_N$

   - when $m_1 + m_2, n_1 + n_2 \geq L$

$$\left(\widehat{A}_\mu - \widehat{g}_\infty \widehat{\partial}_\mu \cdot \widehat{g_\infty^{-1}}\right)_{(m_1,m_2),(n_1,n_2)} = O(L^{-\frac{3}{2}})$$

   as $L \to +\infty$

Dirac equation

$$\widehat{\overline{\mathcal{D}}}_A = \overline{e}^\mu(\widehat{\partial}_\mu \cdot + \widehat{A}_\mu) \qquad \widehat{\mathcal{D}}_A = e^\mu(\widehat{\partial}_\mu \cdot + \widehat{A}_\mu)$$

where $\overline{e}^\mu = (\overline{e}^{\mu\,\dot\alpha\beta})$, $e^\mu = (e^\mu_{\alpha\dot\beta})$ are 4d 2×2 γ-matrices.

<u>Assumption</u> we assume Ker $\widehat{\overline{\mathcal{D}}}_A \simeq \bigoplus^k \mathrm{End}\mathcal{F}$ and Ker $\widehat{\mathcal{D}}_A \simeq \emptyset$

Basis of Ker $\widehat{\overline{\mathcal{D}}}_A$: $\widehat{\psi}_1 = \left(\widehat{\psi}^a_{\alpha\,1}\right)_{a=1,\cdots,N}$, $\widehat{\psi}_2, \cdots, \widehat{\psi}_k$

$$\widehat{\overline{\mathcal{D}}}_A \cdot \widehat{\psi}_i = 0, \qquad \langle \widehat{\psi}_i^\dagger \widehat{\psi}_j \rangle = \delta_{ij}$$

Define the matrix

$$\widehat{\Psi} = \left(\widehat{\psi}_1, \cdots, \widehat{\psi}_k \right)$$

It is also convenient to express the spinor index explicitly as

$$\widehat{\Psi} = \begin{pmatrix} \widehat{\Psi}_1 \\ \widehat{\Psi}_2 \end{pmatrix} \qquad \widehat{\Psi}_\alpha = \left(\widehat{\psi}^a_{\alpha 1}, \cdots, \widehat{\psi}^a_{\alpha k} \right)_{a=1,\cdots,N}$$

Asymptotics

Asymptotic condition on $\widehat{A}_\mu$ implies

- $\exists$ constant matrix $\Omega = \begin{pmatrix} \Omega^1 \\ \Omega^2 \end{pmatrix}$, where $\Omega^{\dot\alpha} = \left(\Omega_i^{\dot\alpha a}\right)_{i=1,\cdots,k;\, a=1,\cdots,N}$, satisfying the condition that , for $m_1+m_2, n_1+n_2 \geq L$

$$\left(\widehat{\Psi} - \widehat{g}_\infty(\widehat{\Psi}_0 \otimes 1_N)\Omega\right)_{(m_1,m_2),(n_1,n_2)} = O(L^{-\frac{5}{2}})$$

as $L \to +\infty$.

$\widehat{\Psi}_0$ is a pair of solutions of the free Dirac equation $(\overline{e}^\mu \widehat{\partial}_\mu) \cdot \widehat{\Psi}_0 = 0$ and takes the form

$$\widehat{\Psi}_0 = \frac{1}{\pi}\left(\widehat{\Box}_0 - \frac{\zeta}{2}\right)^{-1} \sum_\mu e_\mu \widehat{x}^\mu \left(\widehat{\Box}_0 - \frac{\zeta}{2}\right)^{-1}$$

where $\widehat{\Box}_0 = \sum_i \widehat{\overline{z}}_i \widehat{z}_i = \sum_\mu (\widehat{x}^\mu)^2 + \frac{\zeta}{2}$.

Inverse construction

- Define

$$B_i \equiv \langle \, \widehat{z}_i \widehat{\Psi}^\dagger \widehat{\Psi} \rangle \; \in \; \mathsf{End}\, V,$$

$$I \equiv -\Omega^{2\dagger} \; \in \; \mathsf{Hom}(W, V), \quad J \equiv \Omega^1 \; \in \mathsf{Hom}(V, W)$$

$\Longrightarrow (B_1, B_2, I, J)$ satisfies the non-commutative ADHM equations $i), ii)$.

- The inverse construction is actually the inverse of the ADHM construction of non-commutative instatntons.