Exact Results in Supersymmetric Lattice Gauge Theories

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Introduction

The *localization* reduces the path integral to finite dimensional multiple integrals or sums, and solves exactly (non-perturbatively) some problems in field theories:

$$\begin{aligned} Z_{\mathrm{2dYM}} &= \int \mathcal{D}A\mathcal{D}\Phi\, e^{\mathrm{Tr}\int i\Phi F - \frac{1}{2}\Phi^2} \\ &= \int \mathcal{D}A\mathcal{D}\Phi\mathcal{D}\lambda\cdots\, e^{\mathrm{Tr}\int i\Phi F - \frac{1}{2}\Phi^2 + \lambda\lambda + \frac{1}{g^2}S_{\mathrm{SYM}}} \\ &= \left\langle e^{\mathrm{Tr}\int i\Phi F - \frac{1}{2}\Phi^2 + \lambda\lambda} \right\rangle_{N=(2,2)\mathrm{SYM}} \\ &= \sum_{\text{fixed points}} (\text{1-loop dets})\, e^{\mathrm{Tr}\int i\Phi F - \frac{1}{2}\Phi^2 + \lambda\lambda} \end{aligned}$$

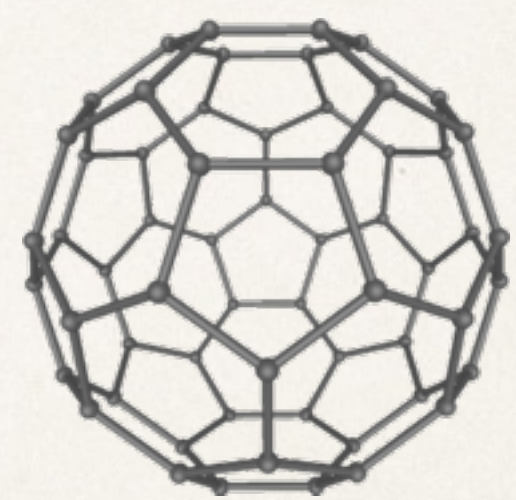
In this sense, 2d (SUSY-)YM theory (and also 3d (SUSY-)CS theory) is integrable (exactly solvable).

Introduction

Question:
Does this integrable structure still hold in lattice gauge theory (on discretized space-time)?

Answer: YES!
We can construct exactly solvable 2d gauge theories on the lattice.

On S^2, S^3, ...



On simplicial complex

Today, I explain how to apply localization method to the lattice gauge theories and give some exact results.

Harish-Chandra Itzykson-Zuber Integral

Let us first consider the so-called Harich-Chandra Itzykson-Zuber (HCIZ) integral for a lesson:

$$Z_{\rm HCIZ} = \int DU\, e^{-\beta H_{\rm HCIZ}} = \left(\frac{2\pi}{\beta}\right)^{N(N-1)/2} \frac{\det_{i,j} e^{-\beta a_i b_j}}{\Delta(a)\Delta(b)}$$

where

$$H_{\rm HCIZ} = {\rm Tr} AUBU^\dagger$$

$A,\ B$: (constant) Hermite matrices

U : $N \times N$ unitary matrix

$\Delta(a),\ \Delta(b)$: Vandermonde determinants of eigenvalues

$$\Delta(a) = \prod_{i<j} (a_i - a_j)$$

Proof by localization

We can perform HCIZ integral exactly.
Why? ⇒ This is because the *localization* works

Phase space (coadjoint action orbit) is isomorphic to a coset $M = U(N)/U(1)^N$, which possesses a symplectic 2-form ω (Kirillov-Kostant-Souriau 2-form). The Hamiltonian H_{HCIZ} generates a vector field V with ω.

$$dH_{\mathrm{HCIZ}} - \iota_V \omega = 0$$

We rewrite (identifying the right-invariant one form with a fermion)

$$Z_{\mathrm{HCIZ}} = \frac{1}{\Delta(b)} \int DU\, D\psi_R\, e^{-\beta H_{\mathrm{HCIZ}} + \omega}$$

where $\omega = \frac{1}{2} \operatorname{Tr} \psi_R [U B U^\dagger, \psi_R]$

Proof by localization

The exponent is invariant ($Q(\beta H_{\mathrm{HCIZ}}\text{-}\omega)=0$) under the following "supersymmetry" (BRST symmetry)

$$QU = i\psi_R U, \qquad Q\psi_R = i\beta A + i\psi_R\psi_R$$

We can also introduce Q-exact "action":

$$Q\Xi = \beta\,\mathrm{Tr}[A, UBU^\dagger]^2 + \cdots$$

Using these, if we deform the integral by Q-exact action

$$Z_{\mathrm{HCIZ}} = \frac{1}{\Delta(b)}\int DU\,D\psi_R\,e^{-\beta H_{\mathrm{HCIZ}}+\omega-\frac{1}{g^2}Q\Xi}$$

we can show that the integral is independent of g ($\Rightarrow$ WKB(1-loop) exact)

Proof by localization

The saddle points (fixed points) are given by the equation

$$[A, UBU^\dagger] = 0 \text{ and } \psi_R = 0$$

$$\Rightarrow \; U = \Gamma_\sigma \text{ (permutation (Weyl group))}$$

Evaluating the integral around the saddle points, we finally obtain

$$Z_{\text{HCIZ}} = \left(\frac{2\pi}{\beta}\right)^{N(N-1)/2} \frac{1}{\Delta(b)} \sum_\sigma \frac{(-1)^{|\sigma|}}{\Delta(a)} e^{-\beta H_{\text{HCIZ}} + \omega}\Big|_{\text{fixed points}}$$

1-loop det

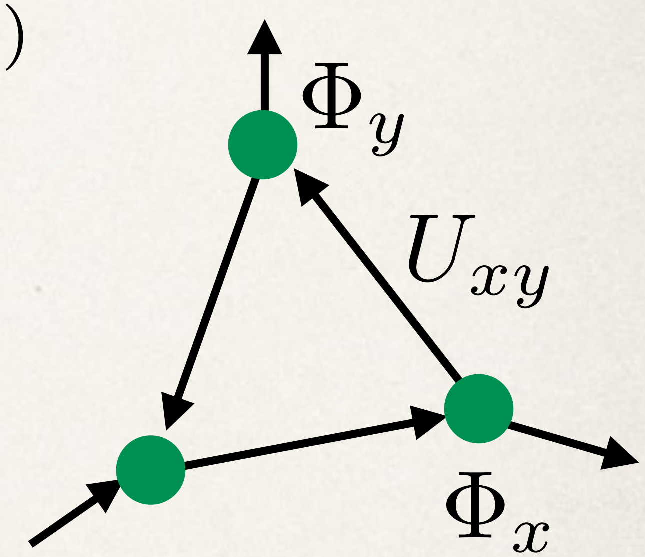
$$= \left(\frac{2\pi}{\beta}\right)^{N(N-1)/2} \frac{1}{\Delta(a)\Delta(b)} \sum_\sigma (-1)^{|\sigma|} e^{-\beta \sum_i a_i b_{\sigma(i)}}$$

Migdal-Kazakov model

Let us next extend the HCIZ integral to the multi-matrix model on the lattice (induced QCD), that is;

$$A \to \Phi_x, \quad B \to \Phi_y, \quad U \to U_{xy} \quad (x, y \in \text{sites})$$

$$Z_{\rm MK} = \int \prod \mathcal{D}U_{xy} \mathcal{D}\Phi_x \, e^{-tS_{\rm MK} - \sum_x {\rm Tr} V(\Phi_x)}$$



where

$$S_{\rm MK} = \sum_{\langle xy \rangle} {\rm Tr} \Phi_x U_{xy} \Phi_y U_{xy}^\dagger$$

This model is known as the Migdal-Kazokov model (1992).
(The relation to 2d YM is also discussed in [Caselle-D'Adda-Lorenzo-Magnea-Panzeri],[Kharchev-Marshakov-Mironov-Morozov].)

SUSY on lattice

The "action" (+symplectic 2-form) of MK model is invariant under the following supersymmetry on the 2d lattice (generalization of the Sugino model)

$$
\begin{aligned}
&QU_{xy} = \Psi_{xy}, && Q\Psi_{xy} = U_{xy}\Phi_y - \Phi_x U_{xy} \\
&Q\Phi_x = 0 && \\
&Q\bar{\Phi}_x = \eta_x, && Q\eta_x = i[\bar{\Phi}_x, \Phi_x] \\
&QY_x = i[\chi_x, \Phi_x], && Q\chi_x = Y_x
\end{aligned}
$$

where we have defined $\Psi_{xy} \equiv \psi_{R,x} U_{xy}$

We also find

$$Q^2 = \delta_{\text{gauge}}(\Phi)$$

nilpotent on gauge invariant operators
(⇒ equivariant cohomology)

SUSY action

The action of 2d N=(2,2) SUSY YM on the lattice is written in Q-exact form by:

$$S_{\rm Sugino} = Q \sum_x \mathrm{Tr}\left[\Psi_{xy}(\bar{\Phi}_y U^\dagger_{xy} - U^\dagger_{xy}\bar{\Phi}_x) + \eta_x[\Phi_x, \bar{\Phi}_x]\right.$$
$$\left. + \chi_x(Y_x - 2i\mu_x)\right]$$
$$\sim \sum_x \mathrm{Tr}\left[|U_{xy}\Phi_y - \Phi_x U_{xy}|^2 + |[\Phi_x, \bar{\Phi}_x]|^2 + \mu_x^2 + \cdots\right]$$

where $\mu_x \sim W(U)\text{-}W(U)^\dagger \sim F_{\mu\nu}$ are moment maps (superpotential constraints) associated with each loops (faces).

Localization in SUSY lattice

The partition function of the supersymmetric MK model deformed by the Sugino action

$$Z_{\rm sMK} = \int \prod \mathcal{D}U_{xy}\mathcal{D}\Phi_x\mathcal{D}\Psi_{xy}\, e^{-t(S_{\rm MK}-\omega)-\sum {\rm Tr}V(\Phi)}$$

$$= \int \prod \mathcal{D}U_{xy}\mathcal{D}\Phi_x\mathcal{D}\Psi_{xy}\cdots\, e^{-t(S_{\rm MK}-\omega)-\sum {\rm Tr}V(\Phi)-\frac{1}{g^2}S_{\rm Sugino}}$$

is independent of the coupling g, since the action is Q-exact and

$$Q(S_{\rm MK}-\omega)=0$$

$$Q{\rm Tr}V(\Phi_x)=0$$

where $\omega = -\frac{1}{2}\sum_{\langle xy\rangle} {\rm Tr}\,\Psi_{xy}[\Phi_y,\Psi^\dagger_{xy}]$

So the integral becomes WKB (1-loop) exact wrt the SUSY action

1-loop determinant

To evaluate the 1-loop determinant, we fix the gauge by:

$$\Phi_x \to \mathrm{diag}(\phi_{x,1}, \phi_{x,2}, \ldots, \phi_{x,N}) \qquad (\ U(N) \to U(1)^N \ )$$

Then, we obtain the 1-loop determinant of the Sugino model:

$$(\text{1-loop det}) = \prod_{i<j} \frac{\prod_{x\in S}(\phi_{x,i}-\phi_{x,j})^2_{c,\bar{c}} \times \prod_{x\in F}(\phi_{x,i}-\phi_{x,j})_\chi}{\prod_{\langle xy\rangle\in L}(\phi_{y,i}-\phi_{x,j})_U \times \prod_{x\in S}(\phi_{x,i}-\phi_{x,j})_{\bar{\Phi}}}$$

where the subscripts mean that the determinants come from each variables, and *S*, *L* and *F* stand for sets of sites (vertices), links (edges) and loops (faces), respectively.

Exact result at fixed points

The fixed points are classified again by the permutations (Weyl group):

$$U_{xy} \to \Gamma_{\sigma_{xy}}$$

and

$$U_{xy}\Phi_y U_{xy}^\dagger - \Phi_x = 0 \quad (\Rightarrow \mathcal{D}_\mu \Phi(x) = 0)$$

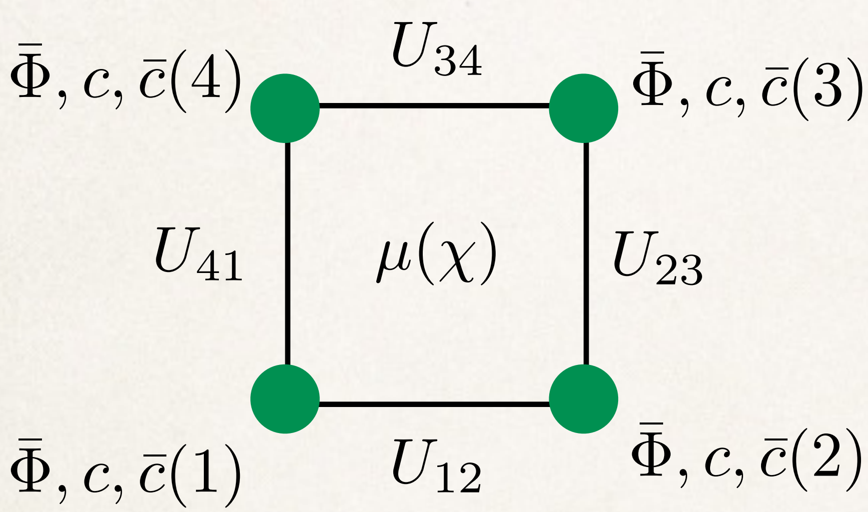
Then we get at the fixed points

$$Z_{\mathrm{sMK}} = \sum_{\{\sigma_{xy}\}}{}' \prod_{\langle xy \rangle} (-1)^{|\sigma_{xy}|} \int \prod_i d\phi_i \prod_{i<j} (\phi_i - \phi_j)^\chi e^{-t\ell \sum_i \phi_i^2 - s \sum_i V(\phi_i)}$$

where χ=*s*-*l*+*f*, and *s*, *l* and *f* are respectively the number of vertices (sites), edges (links) and faces, which come from the number of each variables (matrices).

Examples

On one plaquette;



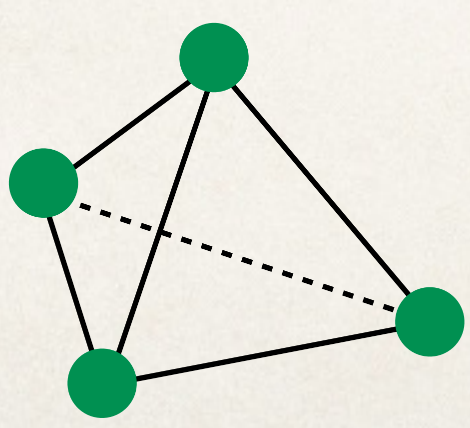
$\sharp$ of $\bar\Phi$ and $c, \bar c = 4$

$\sharp$ of $U_{xy} = 4$

$\sharp$ of χ(or μ) $= 1$

$$\prod_{i<j}(\phi_i - \phi_j)^1$$ ~ disk

On a tetrahedron;



$\sharp$ of $\bar\Phi$ and $c, \bar c = 4$

$\sharp$ of $U_{xy} = 6$

$\sharp$ of χ(or μ) $= 4$

$$\prod_{i<j}(\phi_i - \phi_j)^2$$ ~ sphere

Comparison with 2d YM

2d YM partition function on a general Riemann surface (Migdal)

$$Z_{\text{2d YM}} = \sum_{\{n_i\}\in\mathbb{Z}^N} \prod_{i<j} (n_i - n_j)^\chi \, e^{-\frac{g^2 A}{2}\sum_i n_i^2}$$

One branch (trivial permutation) of sMK model (with $V = \frac{m}{2}\Phi_x^2$)

$$Z_{\text{sMK}} = \int \prod_i d\phi_i \prod_{i<j} (\phi_i - \phi_j)^\chi \, e^{-(t\ell + \frac{m}{2}s)\sum_i \phi_i^2}$$

Remarks:

- Multiple integrals still remain because of flat directions of SUSY theory (not fixed points but fixed lines)
- The partition function is independent of the simplicial decomposition but depends only on the topology (and area) (⇒ 2d YM is almost topological)

Conclusion and Discussion

Results:

- We exactly evaluated the partition function of the MK model under the restricted symmetry (constraints)
- Reversing the logic, we exactly calculated a vev of physical observable in 2d SUSY YM theory on the lattice
- We also found other observables and useful Ward-Takahashi identities

Problems:

- Application to other integrable systems (spin chain, etc.)
- Relation to (or realization in) string/M theory or gravity (topological invariants, etc. in mathematics)

Our model is closely related to quiver gauge theory, deconstruction, etc...