Exact results on the ABJ theory

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ABJ(M) Theory on S^3

- Holographic duality

M-theory on $AdS_4 \times S^7/\mathbb{Z}_k \leftrightarrow$ ABJ(M) theory on S^3

$$Z_{\text{M-theory}} = Z_{\text{ABJ(M)}}$$

- This theory also has an interesting relation to
  - Nekrasov-Shatashvili limit of refined topological string on local $\mathbb{P}^1 \times \mathbb{P}^1$
  - quantum curve and quantum period

[Hatsuda-Mariño-Moriyama-KO]

ABJ(M) Matrix Model

- Partition function of $U(N_1)_k \times U(N_2)_{-k}$ ABJ theory

$$Z = e^{i\theta} \int \frac{d^{N_1}\mu d^{N_2}\nu \, e^{\frac{ik}{4\pi}(\mu^2-\nu^2)}}{(2\pi)^{N_1+N_2} N_1! N_2!} \left[\frac{\prod_{i<j} 2\sinh\frac{\mu_i-\mu_j}{2} \cdot 2\sinh\frac{\nu_i-\nu_j}{2}}{\prod_{i,j} 2\cosh\frac{\mu_i-\nu_j}{2}} \right]^2$$

- We consider $U(N+M)_k \times U(N)_{-k}$ theory with small M, large N limit

- Free energy exhibits the $N^{3/2}$ behavior [Drukker-Mariño-Putrov]

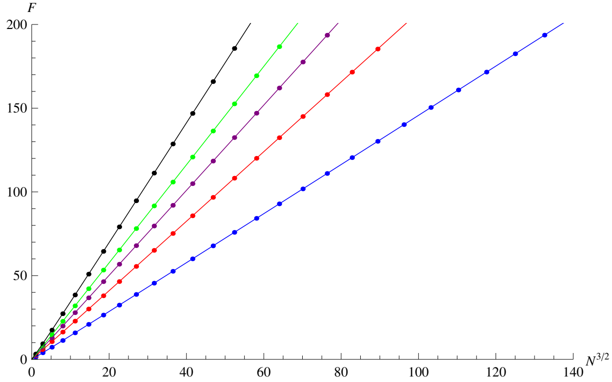
$$Z \approx \exp\left[-\frac{\pi\sqrt{2k}}{3}(N-B)^{3/2} \right]$$

Exact Values of Z

- We are interested in the instanton corrections to the free energy
- To study the instanton corrections,
  we have computed the exact partition function at finite N
- ABJM: $U(N)_k \times U(N)_{-k}$ $(M=0)$ [Hatsuda-Moriyama-KO]
  - up to $N = 44, 20$ for $k = 1, 2$, respectively (and few other k's)
- ABJ: $U(N+M)_k \times U(N)_{-k}$ $(M = 1, 2, 3)$ [Honda-KO]
  - up to $N = 62, 57$ for $(k, M) = (2, 1), (4, 1)$ (and few other (k, M)'s)

Free Energy of ABJM Theory

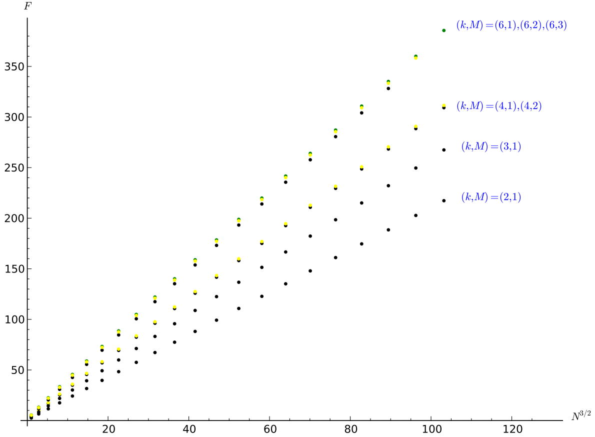
- Free energy exhibits the $N^{3/2}$ behavior even at small N



Free Energy of ABJ Theory

- Free energy of ABJ theory behaves similarly as the ABJM case

$$F \approx \frac{\pi\sqrt{2k}}{3}(N-B)^{3/2}$$



Grand Canonical Partition Function

- Partition function of ABJ theory has the following structure [Awata-Hirano-Sigemori] [Honda]

$$Z(N, M, k) = e^{i\theta'} Z_{\mathrm{CS}}(M, k)\widehat{Z}(N, M, k)$$

- A useful way to analyze ABJ theory is to consider the grand canonical ensemble [Matsumoto-Moriyama] [Honda-KO]

$$\Xi(\mu) = \sum_{N=0}^{\infty} e^{N\mu}\widehat{Z}(N, M, k)$$

- Grand partition function is written as a Fredholm determinant

$$\Xi(\mu) = \mathrm{Det}(1 + e^{\mu - H})$$

Hamiltonian of ABJ Fermi Gas

- Hamiltonian of ABJ Fermi gas (for even M)

$$e^{-H} = e^{-\frac{1}{2}U(x)} \frac{1}{2\cosh\frac{p}{2}} e^{-\frac{1}{2}U(x)}$$

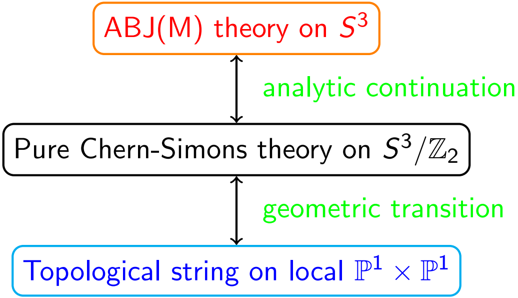
$$e^{-U(x)} = \frac{1}{2\cosh\frac{x}{2}} \prod_{m=-\frac{M-1}{2}}^{\frac{M-1}{2}} \tanh\frac{x+2\pi i m}{2k}$$

$$[x,p] = 2\pi i k$$

- Chern-Simons level k $=$ $\hbar$ of Fermi gas system
- Eigenvalue problem of H is quite interesting

work in progress [Honda-Källén-KO]

Relation to Topological String



- Grand potential $J(\mu) = \log \Xi(\mu)$ is essentially identified with the free energy of toplogical string on local $\mathbb{P}^1 \times \mathbb{P}^1$

Structure of Grand Potential

- $J(\mu) = \log \Xi(\mu)$ has parturbative and non-perturbative parts

$$J(\mu) = J_{\rm pert}(\mu) + J_{\rm np}(\mu)$$

- Perturbative part is given by an integral of Kähler form

$$J_{\rm pert}(\mu) = \frac{1}{6g_s^2}\int \mathcal{J}^3 + \frac{1}{24}\left(\frac{1}{g_s^2} - 1\right)\int \mathcal{J} \wedge c_2$$

$$\mathcal{J} = T_1 E_1 + T_2 E_2, \quad T_{1,2} = T \pm ib$$
$$T = 2g_s\mu, \quad b = \frac{M}{k} - \frac{1}{2}, \quad g_s = \frac{2}{k}$$

- Seiberg duality $M \leftrightarrow k - M$ is realized as the exchange of two $\mathbb{P}^1$'s

Instanton Corrections

There are three types of instanton corrections in $J_{\rm np}(\mu)$

1. Worldsheet instanton $\quad e^{-T} = e^{-2g_s\mu}$
   $\Leftrightarrow$ Un-refined (ordinary) topological string $F_{\rm top}(T_1, T_2, g_s)$

2. Membrane instanton $\quad e^{-T/g_s} = e^{-2\mu}$
   $\Leftrightarrow$ NS limit of refined topological string $F_{\rm NS}(\frac{T_1}{g_s}, \frac{T_2}{g_s}, \frac{1}{g_s})$
   $\Leftrightarrow$ quantum B-period

3. (n, m)-bound state $\quad e^{-(n+m/g_s)T}$
   $\Leftrightarrow$ absorbed into worldsheet instanton via quantum mirror map

$$T_{1,2}^{\rm eff} = T_{1,2} + g_s \Pi_A$$

Quantum Periods

- Quantum periods are determined by the wavefunction of D-brane [Mironov-Morozov, Aganagic-Cheng-Dijkgraaf-Krefl-Vafa]

$$H(x,\hbar\partial_x)\Psi(x)=\mu\Psi(x),\quad \Psi(x)=e^{\frac{1}{\hbar}S(x;\hbar,\mu)}$$
$$\Pi_A=\oint_A dS,\quad \Pi_B=\oint_B dS$$

- Quantum periods are the $\hbar$-deformed version of the classical periods of the mirror curve of local $\mathbb{P}^1\times\mathbb{P}^1$
  - quantum A-period Π_A gives the quantum mirror map
  - quantum B-period Π_B is related to the derivative of F_{NS}

Non-Perturbative Grand Potential

- Combining various analysis (exact Z, small k expansion, numerical fitting, etc.), we arrived at an elegant formula

$$J_{\rm np}(\mu_{\rm eff}) = F_{\rm top}(T_{\rm eff}, g_s) + \frac{1}{2\pi i}\frac{\partial}{\partial g_s}\left[g_s F_{\rm NS}\left(\frac{T_{\rm eff}}{g_s}, \frac{1}{g_s}\right)\right]$$

$$F_{\rm top}(T_{\rm eff}, g_s) = -\sum_{m=1}^{\infty}\sum_{j_L, j_R, d} N^d_{j_L, j_R}\frac{(2j_R+1)\chi_{j_L}(q_s^m)}{(q_s^{m/2} - q_s^{-m/2})^2}\frac{e^{-md\cdot T_{\rm eff}}}{m}$$

$$F_{\rm NS}\left(\frac{T_{\rm eff}}{g_s}, \frac{1}{g_s}\right) = \sum_{n=1}^{\infty}\sum_{j_L, j_R, d} N^d_{j_L, j_R}\frac{\chi_{j_L}(q^{n/2})\chi_{j_R}(q^{n/2})}{q^{n/2} - q^{-n/2}}\frac{e^{-\frac{nd\cdot T_{\rm eff}}{g_s}}}{n^2}$$

$$q_s = e^{2\pi i g_s}, \qquad q = e^{2\pi i/g_s}$$

- Instanton coefficients are determined by the BPS invariants $N^d_{j_L, j_R}$

HMO Cancellation Mechanism

- Worldsheet instanton coefficient diverges at $g_s = n/m$ (in particular, for a physical $k \in \mathbb{Z}$)

- This divergence is precisely canceled by the membrane instanton [Hatsuda-Moriyama-KO]

$$\lim_{g_s \to n/m} \Big[(\textit{worldsheet inst}) + (\textit{membrane inst}) \Big] = \mathrm{finite}$$

- Finite part is completely fixed by the structure $F_{\mathrm{top}} + \partial F_{\mathrm{NS}}$

- $F_{\mathrm{top}} + \partial F_{\mathrm{NS}}$ works for both ABJM and ABJ [Honda-KO]

Conclusions

- Membrane instanton are determined by the quantum periods
- Effect of bound states is absorbed into the worldsheet instantons by the quantum mirror map
- Membrane instanton corrections to the ABJ(M) grand potential is given by the NS limit of refined topological string

- Quantum Picard-Fuchs equation for quantum periods?
- Higher spin limit? $(M, k \gg N)$
  work in progress [Hirano-Honda-KO-Shigemori]