@ Ritsumeikan University

NO. 1

DATE 2014 3 7
Friday
(пятница)

# On Cherednik algebra and its root of unity limit

Takeshi Oota.

Joint work with H. Itoyama and R. Yoshioka.

(work in progress)

## 1. Introduction

[Itoyama – T.O. – Yoshioka, 1308.2068]

(→ Talk by Yoshioka)

$q$-lift of 4d/2d correspondence:
/deformation

[Shiraishi-Kubo-Awata-Odake, Frenkel-Reshetikhin]

5d gauge theories on $\mathbb{R}^4 \times S^1$ ⟷ $q$-Virasoro/W-algebra parameters $q, t = q^\beta$

$( q = e^{R\varepsilon_2},\ t = e^{-R\varepsilon_1} )$

→ root of unity limit $\begin{cases} q \to \omega = e^{2\pi i/r} \\ t \to \omega (= e^{2\pi i/r}) \end{cases}$

$\mathbb{R}^4/\mathbb{Z}_r$ (ALE) ⟷ super Virasoro/parafermions

[Ref] Cherednik's book.

→ The Cherednik algebra (double affine Hecke algebra, DAHA) of type $G = GL_n$ also plays an important role.

① well-known degenerate limits.

DAHA (Cherednik)

↓

degenerate DAHA (DDAHA) (→ Matsuo san's talk)
(trigonometric Cherednik algebra)

↓

doubly degenerate DAHA (DDDAHA)
(rational Cherednik algebra).

[our expectation]

② root of unity limit of DAHA
will play important roles in the gauge theories
on ALE space (still, work in progress).

## 2. Cherednik algebra of type $G = GL_n$

$$\mathcal{C}_n = \mathcal{C}_n(q,t)$$
$$\simeq \mathbb{C}[T] \otimes \mathbb{C}[T'] \otimes \mathcal{H}_n(t)$$

↑ as a vector space

$T$ : the maximal torus of $G$

$\mathbb{C}[T] = \mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}]$ : $\boxed{X_i X_j = X_j X_i}$

the group algebra of $T$. (Laurent polynomial ring)

$\mathbb{C}[T'] = \mathbb{C}[Y_1^{\pm 1}, \cdots, Y_n^{\pm 1}]$, $\boxed{Y_i Y_j = Y_j Y_i}$

→ $Y_i$ ; the Dunkl-Cherednik (difference) operators.

- $\mathcal{H}_n(t)$ : the finite Hecke algebra of type $GL_n$

  generators $T_1, \cdots, T_{n-1}$

  with relations

  - $\boxed{(T_i - t^{1/2})(T_i + t^{-1/2}) = 0}$ $(i = 1, \cdots, n-1)$

    → eigenvalues of $T_i = \pm t^{\pm 1/2}$.

  - Coxeter relations

$$\boxed{\begin{cases} T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \\ T_i T_j = T_j T_i & |i-j| > 1. \end{cases}}$$

$$\boxed{\begin{array}{ll} T_i X_i T_i = X_{i+1} & T_i^{-1} Y_i T_i^{-1} = Y_{i+1} \\ T_i X_j = X_j T_i \quad (j \neq i, i+1). & T_i Y_j = Y_j T_i \quad (j \neq i, i+1) \end{array}}$$

• $\boxed{\pi := Y_1^{-1} T_1 \cdots T_{n-1}.}$

or $Y_1 = T_1 \cdots T_{n-1} \pi^{-1}$

$\rightarrow$ $Y_2 = T_1^{-1} Y_1 T_1^{-1}$

$\quad = T_2 \cdots T_{n-1} \pi^{-1} T_1^{-1}$

$Y_3 = T_2^{-1} Y_2 T_2^{-1}$

$\quad = T_3 \cdots T_{n-1} \pi^{-1} T_1^{-1} T_2^{-1}$

$\vdots$

$$\boxed{Y_i = T_i T_{i+1} \cdots T_{n-1} \pi^{-1} T_1^{-1} \cdots T_{i-1}^{-1}}$$

$$\rightarrow \boxed{\begin{cases} \pi X_i = X_{i+1} \pi & (i = 1, \cdots, n-1) \\ \pi X_n = q^{-1} X_1 \pi \end{cases}}$$

## 3. Polynomial rep. $\Phi$ of DAHA

construct on $\mathbb{K}[X_1, \cdots, X_n]$. $\mathbb{K} = \mathbb{C}(q, t^{1/4})$.

with.

- permutation op. $K_{ij} = K_{ji}$, $(K_{ij})^2 = 1$,

$$K_{ij} X_j = X_i K_{ij}$$

- the $q$-shift operators. $\mathcal{T}_{q, X_i} : X_i \to q X_i$.

$$\begin{cases} (\mathcal{T}_{q, X_i}) X_i = q X_i (\mathcal{T}_{q, X_i}) \\ (\mathcal{T}_{q, X_i}) X_j = X_j (\mathcal{T}_{q, X_i}) \quad (i \neq j) \end{cases}$$

( relation
- $Y_i = T_i T_{i+1} \cdots T_{n-1} \pi T_1^{-1} \cdots T_{i-1}^{-1}$ ).

$$\begin{cases} \Phi(X_i) = X_i \\ \Phi(T_i) = t^{1/2} + \left( \dfrac{t^{1/2} X_i - t^{-1/2} X_{i+1}}{X_i - X_{i+1}} \right) (K_{i, i+1} - 1), \\ \Phi(\pi) = (\mathcal{T}_{q, 1})^{-1} K_{1,2} K_{2,3} \cdots K_{n-1, n}. \end{cases}$$

- omit $\Phi$ in the following part

DATE

## 4. Polynomial representation and Macdonald polynomial

$$(1+uY_1)(1+uY_2)\cdots(1+uY_n)\Big|_{\text{symmetric polynomial of } X_i}$$

restricted to

$$= \sum_{m=0}^{n} M_m u^m. \qquad (u: \text{a variable})$$

the Macdonald (difference) operators

$$M_m = \sum_{\substack{I\subset\{1,\cdots,n\}\\ |I|=m}} \left( \prod_{\substack{i\in I\\ j\notin I}} \frac{t^{1/2}x_i - t^{-1/2}x_j}{x_i - x_j} \right) \cdot \prod_{i\in I} (T_{q,x_i}).$$

On the Macdonald polynomial $\qquad (\lambda:\text{a partition})$

$$P_\lambda(X) = P_\lambda(X;q,t),$$

$$(1+uY_1)(1+uY_2)\cdots(1+uY_n)\,P_\lambda(X)$$
$$= \prod_{i=1}^{n}\left(1+u\,t^{\frac{1}{2}(n+1)-i}q^{\lambda_i}\right)\cdot P_\lambda(X).$$

## 5. Ordinary degeneration limits.

Cherednik alg. $\mathcal{C}_n \simeq \mathcal{H}_n(t) \otimes \mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}] \otimes \mathbb{C}[\underbrace{Y_1^{\pm 1}, \cdots, Y_n^{\pm 1}}_{\text{Dunkl-Cherednik (difference) op.s.}}]$

Parameters $q = e^h$, $t = q^\beta = e^{\beta h}$

$\beta$: fixed, $h \to 0$. $q \to 1$, $t \to 1$

the finite Hecke $\mathcal{H}_n(t) \longrightarrow \mathbb{C}[S_n] \simeq \mathbb{C}[W]$ ← Weyl group. of $GL_n$.

$T_i \longmapsto K_{i,i+1} \in S_n$. $((K_{i,i+1})^2 = 1)$.

with

- Limit (A): $X_i, Y_i$ fixed. $\mathcal{C}_n \longrightarrow$ the elliptic Weyl group of $GL_n$. [Saito] (or 2-extended Weyl group)

- Limit (B): $\begin{cases} X_i : \text{fixed} \\ Y_i = 1 + h y_i + O(h^2) \end{cases}$

$$\mathcal{C}_n \longrightarrow \mathcal{C}_n^{(trig.)} \simeq \mathbb{C}[S_n] \otimes \mathbb{C}[X_1^{\pm 1}, \cdots, X_n^{\pm 1}] \otimes \mathbb{C}[\underbrace{y_1, \cdots, y_n}]$$

↑ trigonometric Cherednik alg. (degenerate DAHA) → [Matsuo san's talk]

trigonometric Dunkl-Cherednik (differential) op.s.

- Limit (C): $\begin{cases} X_i = 1 + h x_i + O(h^2) \\ Y_i = 1 + h y_i + O(h^2) \end{cases}$

rational Dunkl (differential) op.s.

$$\mathcal{C}_n \longrightarrow \mathcal{C}_n^{(rat)} \simeq \mathbb{C}[S_n] \otimes \mathbb{C}[x_1, \cdots, x_n] \otimes \mathbb{C}[\underbrace{y_1, \cdots, y_n}]$$

↑ rational Cherednik alg. (doubly degenerate DAHA).

Comment

rational Dunkl op.s.

- ○ W-equivariant
- ○ mutually commuting.

trigonometric Dunkl-Cherednik op.s.

- × W-equivariant
- ○ mutually commuting.

trigonometric Dunkl-Heckman op.s

- ○ W-equivariant
- × mutually commuting.

Dunkl-Cherednik (difference) op.s.

- ○ mutually commuting.