Progress in the synthesis of integrabilities arising from gauge-string duality
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Integrability of BPS equations in ABJM theory

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The worldvolume theory of M2-branes was not known until recently.

The theory should be a 3D CFT with no adjustable coupling, but people could not construct such a CFT with sufficient amount of supersymmetries.

ABJM theory (Aharony-Bergman-Jafferis-Maldacena '08)

$\mathcal{N}=6$ $\mathrm{U}(N)\times\mathrm{U}(N)$ super Chern-Simons theory with two bifundamental hypermultiplets

Chern-Simons levels are chosen to be $k_1=-k_2=k$

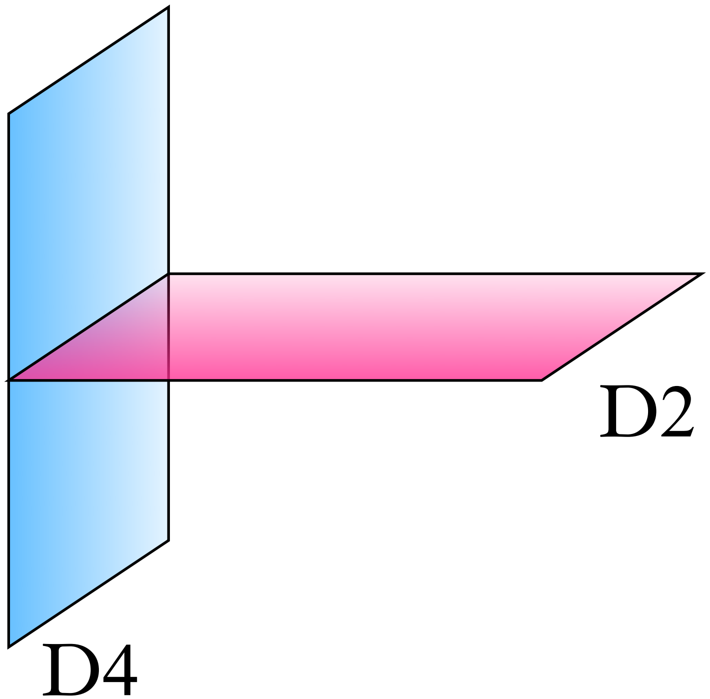
The theory describes the low energy effective theory of the worldvolume theory of N M2-branes probing a $\mathbb{C}^4/\mathbb{Z}_k$ singularity.

$k=1$: M2-branes in flat space(?)

The worldvolume theory of M5-branes is still mysterious.
(Covariant Lagrangian description might not exist.)

Two descriptions of BPS D2-D4 bound states

(Diaconescu '96)



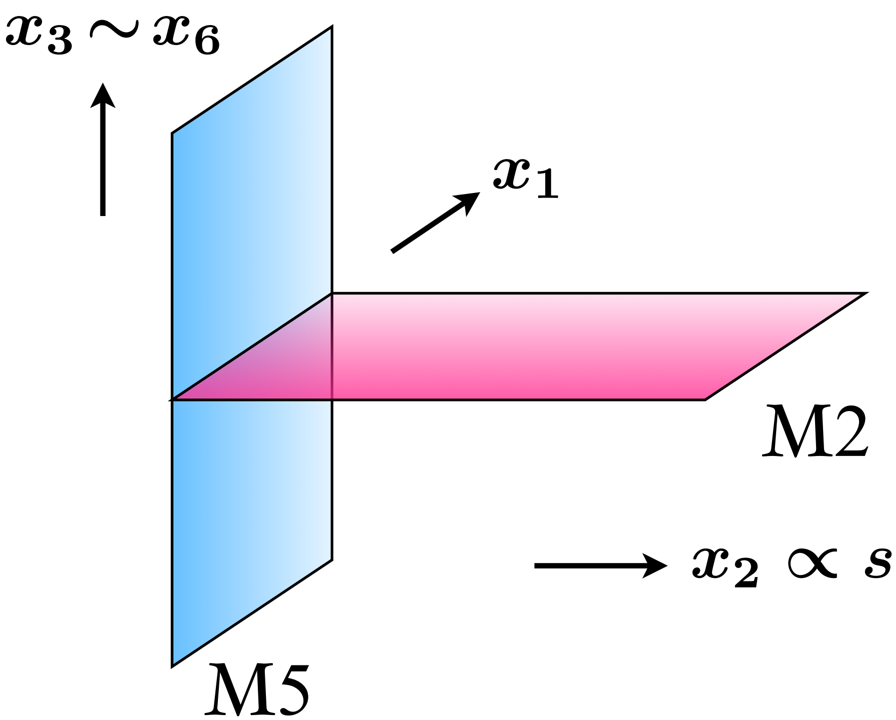
In the worldvolume theory of D2-branes, the bound states are described as solutions of the Nahm equations (Nahm data).

Nahm transformation

In the worldvolume theory of D4-branes, the bound states are described as monopole solutions.

It would be very interesting if one could promote this picture to M-theory.

Two descriptions of BPS M2-M5 bound states



In the worldvolume theory of M2-branes (ABJM theory), the bound states are described as solutions of the BPS equations.

analog of Nahm transformation?

In the worldvolume theory of M5-branes, the bound states should be given as some solutions.

BPS equations in the ABJM theory

(Terashima '08)

(Gomis, Rodriguez-Gomez, Van Ramsdonk and Verlinde '08)

$$\dot{Y}^a = Y^b Y^{b\dagger} Y^a - Y^a Y^{b\dagger} Y^b$$

$Y^a(s)$ $(a = 1, 2)$: $N \times N$ complex matrices

s : a real coordinate $\qquad \dot{Y}^a := \frac{d}{ds} Y^a$

Automorphism

$$Y^a \to Y'^a = e^{i\varphi} \Lambda^a{}_b U Y^b V^\dagger$$

$$U, V \in \mathrm{SU}(N), \quad (\Lambda^a{}_b) \in \mathrm{SU}(2), \quad e^{i\varphi} \in \mathrm{U}(1)$$

Y'^a again satisfy the above equations

We argue that the BPS equations are classically integrable.

The BPS equations admit a Lax representation

$$\dot{A} = [A, B]$$

$$A(s;\lambda) = \begin{pmatrix} O & Y^1 + \lambda Y^2 \\ Y^{1\dagger} - \lambda^{-1} Y^{2\dagger} & O \end{pmatrix}$$

$$B(s;\lambda) = \begin{pmatrix} \lambda^{-1} Y^1 Y^{2\dagger} + \lambda Y^2 Y^{1\dagger} & O \\ O & \lambda Y^{1\dagger} Y^2 + \lambda^{-1} Y^{2\dagger} Y^1 \end{pmatrix}$$

$\lambda \in \mathbb{C}$: the spectral parameter

Making use of this structure we formulate an efficient way of constructing solutions of the BPS equations.

Nahm equations

$$\dot{T}^I = i\epsilon_{IJK} T^J T^K$$

$T^I \ (I = 1, 2, 3)$: $N \times N$ hermitian matrices

- Relation between the BPS equations and the Nahm equations

$$T_1^I := (\sigma^I)_{ab} Y^a Y^{b\dagger}, \qquad T_2^I := (\sigma^I)_{ab} Y^{b\dagger} Y^a$$

$\sigma^I \ (I = 1, 2, 3)$: Pauli matrices

If Y^a are solutions to the BPS equations, both T_1^I and T_2^I satisfy the Nahm equations.

(Nosaka-Terashima '12)

Lax representation for the Nahm equations

(Hitchin '83)

$$\dot{A}_\alpha = [A_\alpha, B_\alpha]$$

$$A_\alpha := T^3_\alpha + \frac{\lambda}{2}\left(T^1_\alpha - iT^2_\alpha\right) - \frac{1}{2\lambda}\left(T^1_\alpha + iT^2_\alpha\right)$$

$$B_\alpha := \frac{\lambda}{2}\left(T^1_\alpha - iT^2_\alpha\right) + \frac{1}{2\lambda}\left(T^1_\alpha + iT^2_\alpha\right)$$

The above Lax forms are related to those of the BPS equations in a remarkably simple way:

$$A^2 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

Idea of our construction of solutions to the BPS equations

- The Lax equation is regarded as the compatibility condition of the following auxiliary linear problem:

$$\boldsymbol{A}(\boldsymbol{s};\boldsymbol{\lambda})\psi(\boldsymbol{s};\boldsymbol{\lambda}) = \boldsymbol{\eta}(\boldsymbol{\lambda})\psi(\boldsymbol{s};\boldsymbol{\lambda})$$

$$\boldsymbol{B}(\boldsymbol{s};\boldsymbol{\lambda})\psi(\boldsymbol{s};\boldsymbol{\lambda}) = -\dot{\psi}(\boldsymbol{s};\boldsymbol{\lambda})$$

Two sets of solutions of the Nahm equations ⇒ Solutions of the aux. linear problem for the Nahm eqs.

⇓

Solutions of the BPS equations ⇐ Solutions of the aux. linear problem for the BPS eqs.

The most general semi-infinite solutions with $N = 2$

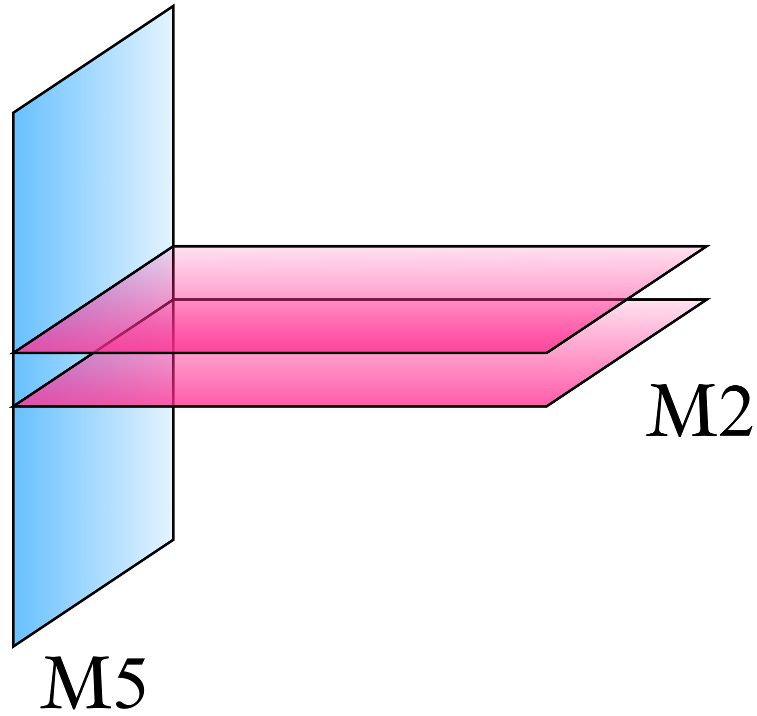
$$T^1_\alpha = \frac{c}{\sinh(x - x_\alpha)} \frac{\sigma^1}{2} + t^1 1_2$$

$$T^2_\alpha = \frac{c}{\sinh(x - x_\alpha)} \frac{\sigma^2}{2} + t^2 1_2$$

$$T^3_\alpha = \frac{c}{\tanh(x - x_\alpha)} \frac{\sigma^3}{2} + t^3 1_2$$

$x = cs, \quad c \geq 0$ $\quad \sigma^I$: Pauli matrices

$x_1 = 0, \quad x_2 = -l, \quad l \geq 0, \quad t^I \in \mathbb{R}$



$\Rightarrow$

$$Y^1 = \sqrt{\frac{c}{2 \sinh l \sinh x \sinh(x+l)}} \begin{pmatrix} \sinh(x+l) \cos\frac{\theta}{2}\, e^{i\phi} & \sinh l \sin\frac{\theta}{2} \\ 0 & \sinh x \cos\frac{\theta}{2}\, e^{i\phi} \end{pmatrix}$$

$$Y^2 = \sqrt{\frac{c}{2 \sinh l \sinh x \sinh(x+l)}} \begin{pmatrix} \sinh x \sin\frac{\theta}{2} & 0 \\ \sinh l \cos\frac{\theta}{2}\, e^{i\phi} & \sinh(x+l) \sin\frac{\theta}{2} \end{pmatrix}$$

$(x = cs, \quad c, l, \theta, \phi \in \mathbb{R})$

The most general solution with $N = 2$

- Matrices of the form

$$Y^1 = \frac{1}{2}\left(f_1 \sin\frac{\theta}{2}\,\sigma^1 + f_2 \sin\frac{\theta}{2}\, i\sigma^2 + f_3 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^3 - f_0 e^{i\phi}\cos\frac{\theta}{2}\,1_2 \right)$$

$$Y^2 = \frac{1}{2}\left(f_1 e^{i\phi}\cos\frac{\theta}{2}\,\sigma^1 - f_2 e^{i\phi}\cos\frac{\theta}{2}\, i\sigma^2 - f_3 \sin\frac{\theta}{2}\,\sigma^3 - f_0 \sin\frac{\theta}{2}\,1_2 \right)$$

with any real functions $f_i(s)$ satisfying

$$\dot{f}_i = f_j f_k f_l$$

are solutions of the BPS equations.

A sufficiently general solution is given by

$$f_i = \frac{\vartheta_{i+1}(u)}{\vartheta_{i+1}(u_*)}\sqrt{\frac{\pi}{2\omega_1}\frac{\vartheta_1(u_*)\vartheta_2(u_*)\vartheta_3(u_*)\vartheta_4(u_*)}{\vartheta_1(u_*+u)\vartheta_1(u_*-u)}}$$

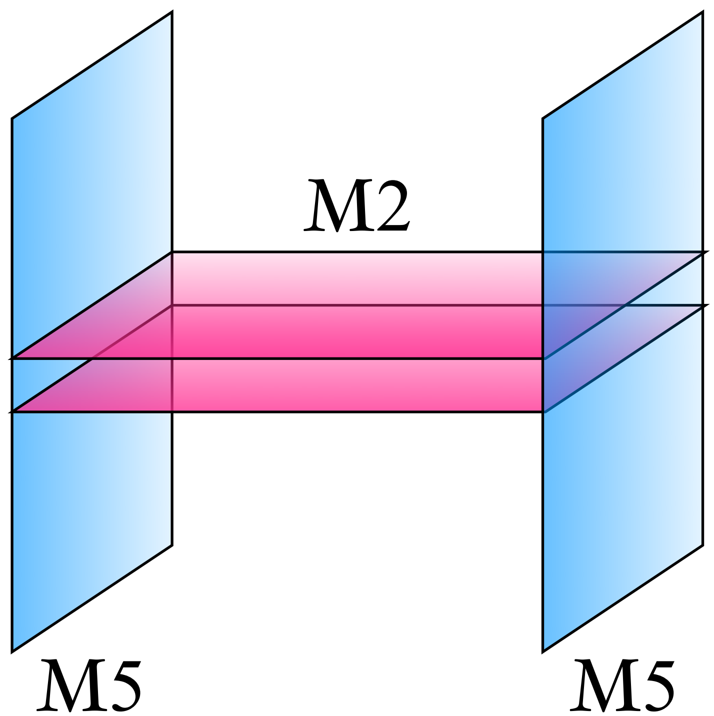
$$\vartheta_{i+1}(u) := \vartheta_{i+1}(u,\tau) \qquad (i = 0, 1, 2, 3)$$

$$u = \frac{s - s_0}{2\omega_1}, \qquad s_0 \in \mathbb{R}, \qquad 0 < u_* < \frac{1}{2}, \qquad \omega_1 \in \mathbb{R}_{>0}, \qquad \tau \in i\mathbb{R}_{>0}$$

The solution is defined over the region

$$-u_* < u < u_*$$

and f_i diverge at each boundary of the region.



Reduction in connection with the periodic Toda chain

- Let us make an ansatz of $Y^a(s)$ as follows:

$$(Y^1)_{mn} = g_m(s)\delta_{m,n}, \qquad (Y^2)_{mn} = h_n(s)\delta_{m,n+1}$$

$$(m, n = 1, \ldots, N)$$

The BPS equations become

$$\dot{g}_m = \left(h_{m-1}^2 - h_m^2\right) g_m, \qquad \dot{h}_m = \left(g_{m+1}^2 - g_m^2\right) h_m.$$

If we introduce

$$a_m := g_{m+1} h_m, \qquad \tilde{a}_m := g_m h_m,$$
$$b_m := g_m^2 - h_m^2, \qquad \tilde{b}_m := g_m^2 - h_{m-1}^2,$$

a_m, b_m satisfy (and the same is true for $\tilde{a}_m, \tilde{b}_m$)

$$\dot{a}_m = a_m(b_{m+1} - b_m), \qquad \dot{b}_m = -2(a_m^2 - a_{m-1}^2).$$

These are the equations for the periodic Toda chain!

Summary

- We have shown that the BPS equations in the ABJM theory is classically integrable.
- The integrable structure of the BPS equations is closely related to that of the Nahm equations.
- By making use of this fact, we have formulated an efficient way of constructing solutions of the BPS equations.
- By way of illustration we have constructed the most general solution describing two M2-branes.

Outlook

- What is the structure of the moduli space of the solutions?

- Are there any other integrable BPS equations?

- What is the analog of the Nahm construction?

- What is the role of integrability in the theory of M5-branes and in the whole M-theory?