Remainder function of N = 4 SYM and massless TBA

Yuji Satoh

Univ. of Tsukuba

based on work with

K. Ito (TIT), Y. Hatsuda (DESY) and J. Suzuki (Shizuoka U)

to appear

Introduction

- Stimulated by duality N=4 SYM $\longleftrightarrow$ string on AdS5 x S5

  one may hope planar N=4 SYM can be solved

  - spectrum (scaling dim.) at $\forall$ $\lambda = g_{YM}^2 N_c$ ['t Hooft coupling]
  - scattering amplitudes/Wilson loops

$\vdots$

In this talk,

- we derive analytic form of (AdS5) 6-pt amplitudes at strong coupling around certain kinematic pt.

- take new approach
  - massless (not massive) thermodynamic Bethe ansatz (TBA)
  - quantum Wronskian relation

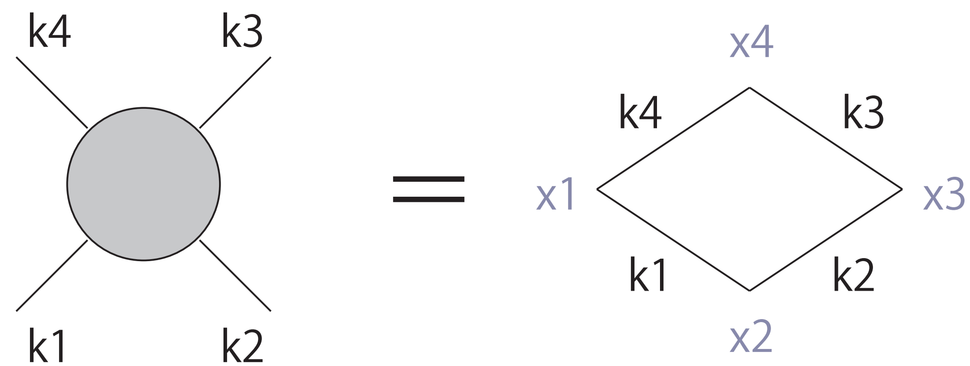
Plan of talk

1. Introduction
2. Amplitudes/Wilson loops in N=4 SYM
3. 6pt remainder fn. at strong coupling
4. Analytic expansion from massless TBA
5. Comparison with perturbative results
6. Summary

Amplitudes/Wilson loops in N=4 SYM

- Amplitudes are equivalent to null polygonal Wilson loops

e.g.)



- Dual conformal symmetry implies for n-pt fn.

$$\log \mathcal{M}_n^{MHV} = A_n^{BDS} + \sum_k \lambda^k R_n^{(k)}$$

[max. helicity violating amp.]

A_n^{BDS} : Bern-Dixon-Smirnov formula; all order in λ

$R_n^{(k)}$: remainder fn.; fn. of cross-ratios of cusp coord.

- 1 loop: BDS formula is exact

- Analytic result of 6–pt remainder fn. at 2,3,4 loops $R_6^{(2,3,4)}$

  [Del Duca, Duhr, Smirnov '09, '10; Goncharov et al '10, Dixon et al '13, '14]

- Strong coupling: area of null polygonal minimal surface

  [Alday, Maldacena '07;
  Itoyama, Mironov, Morozov '07; Astefanesei, Dobashi, Ito, Nastase '07, ...]

  $\Uparrow$ thermodynamic Bethe ansatz (TBA) of 2D integrable model

  [Alday, Gaiotto, Maldacena, Sever, Vieira, '09,'10; Ito, Hatsuda, Sakai, YS '10]

- Finite (any) coupling: OPE method based on integrability

  [Alday, Gaiotto, Maldacena, Sever, Vieira '10; Basso, Sever, Vieira '13]

In the following,

we focus on 6-pt remainder fn. at strong coupling

- Detailed analysis: yet to be done

- More analytic data at strong coupling may be of some use as well as perturbative ones

6pt remainder fn. at strong coupling

TBA equation

- To get 6–pt remainder fn. at strong coupling, first solve

$$\log \tilde{Y}_1(\theta) = m \cosh\theta + K_2 * L_2 + K_1 * L_{13}$$

$$\log \tilde{Y}_2(\theta) = \sqrt{2}\, m \cosh\theta + 2K_1 * L_2 + K_2 * L_{13}$$

$$L_2 = \log(1 + \tilde{Y}_2^{-1}),\ \ L_{13} = \log(1 + \mu \tilde{Y}_1^{-1})(1 + \mu^{-1}\tilde{Y}_1^{-1})$$

$$K_1 = 1/\cosh\theta,\ \ K_2 = 2\sqrt{2}\cosh\theta/\cosh 2\theta$$

$$\tilde{Y}_a(\theta) := Y_a(\theta + i\varphi)$$

TBA of twisted (μ) Z4 integrable model   [m : mass parameter]

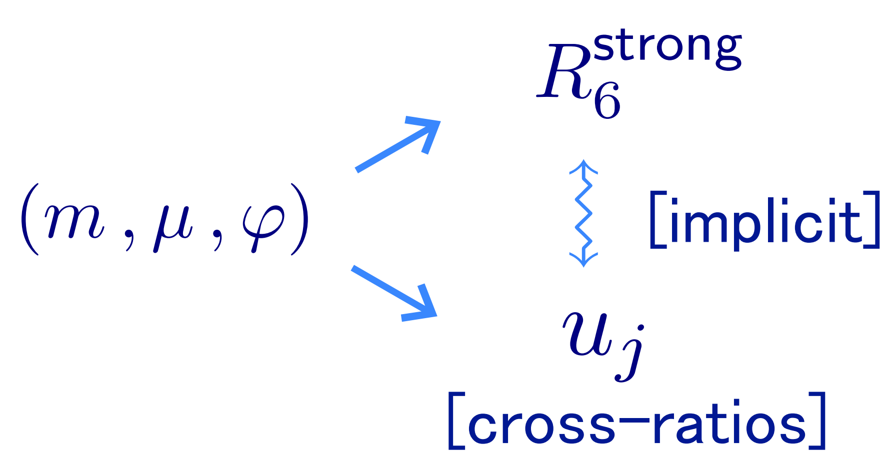
- 3 param. $(m\,, \mu\,, \varphi)$ give Y–fn. (Y_a)   [cf. μ = 0 : AdS4]

- Y−fn. give basis of indep. cross−ratios

$$1+Y_2\Big(\frac{(2j+1)}{4}i\Big)=u_j^{-1}:=\frac{x_{j+2,j-1}^2x_{j,j+3}^2}{x_{j,j+2}^2x_{j+3,j-1}^2}\qquad (j=1,2,3)$$

$$k_i=x_{i+1}-x_i\quad (i=1,\ldots,6\mod 6)$$

$$x_{ij}=x_i-x_j$$



- Remainder fn. at strong coupling is

$$R_6^{\text{strong}}=A_L+A_{\text{period}}+A_{\text{free}}$$

$$A_L=\frac{1}{4}\sum_{j=1}^{3}\mathrm{Li}_2(1-u_j^{-1})\,,\qquad A_{\text{period}}=\frac{1}{4}m^2$$

$$A_{\text{free}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\theta\cdot m\cosh\theta\Big[L_{13}(\theta)+\sqrt{2}L_2(\theta)\Big]$$

[free energy of Z4 int. model]

Y/T-system

- Y-fn. satisfy Y-system

$$Y_1^+ Y_1^- = 1 + Y_2$$

$$Y_2^+ Y_2^- = (1 + \mu Y_1)(1 + \mu^{-1} Y_1)$$

$$Y_a^{\pm} = Y_a^{[\pm 1]}\,,\ Y_a^{[k]} = Y_a(\theta + \frac{\pi i}{4} k)$$

- Y-system is rewritten as T-system

$$T_a^+ T_a^- = 1 + T_{a-1} T_{a+1} \qquad (a = 1, 2)$$

$$T_a = Y_a \quad (a = 1, 2)$$

$$T_3 = T_1 + \mu + \mu^{-1}\,,\ T_0 = 1$$

- From Y/T-system, one finds

$$Y_a^{[6]} = Y_a(\theta + \frac{3}{2}\pi i) = Y_a(\theta)$$

- This periodicity, small m behavior, conformal perturbation theory (CPT) [for AdS4 case] imply

$$\tilde{Y}_a(\theta) = \sum_{p,q=0} y_a^{(p,q)} m^{\frac{4}{3}(p+q)} \cosh \frac{4p}{3}\theta$$

[similarly for T-fn.]

- m → 0 : UV/regular polygonal limit

UV expansion from massless TBA

- AdS3, AdS4 case : remainder fn. are expanded around m=0

  by
  - massive TBA of homogeneous sine-Gordon model
  - conformal perturbation theory (CPT) for A_{free}
  - boundary CPT for Y/T-fn.

[Ito, Hatsuda, Sakai, YS, '11; Ito, Hatsuda, YS, '11, '12]

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- AdS_5 $n = 6 \Leftarrow Z_4$ integrable model

[cf. A_{free} part, numerics: Ito, Hatsuda, Sakai, YS, '10]

but, how to incorporate μ into boundary CPT ?

- AdS_5 $n \geqslant 7$: corresponding integrable model ?

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- AdS5 n $\geqslant$ 7: corresponding integrable model ?

$\Rightarrow$ We derive UV expansion of Y-fn. and 6-pt remainder by

- "massless" TBA
- quantum Wronskian relation

CFT/chiral limit and quantum Wronskian

- T-system for 6pt case: obtained from XXZ (6-vertex) model
  - anisotropy param. $\Delta = \cos\gamma = -\frac{1}{2} \quad (\gamma = \frac{2}{3}\pi)$
  - twist by $D_\phi = \mathrm{diag}(e^{-\phi/2}, e^{-\phi/2})\,, \quad \mu =: e^{3i\phi/2}$
  - T-fn = transfer matrices

$T_1(\theta) =$ [Λ: inhomogeneity]

$$\mathrm{Tr}_0 D_\phi R_{0,2N}(\tfrac{2i}{3}\theta + i\Lambda) R_{0,2N-1}(\tfrac{2i}{3}\theta - i\Lambda) \cdots R_{0,2}(\tfrac{2i}{3}\theta + i\Lambda) R_{0,1}(\tfrac{2i}{3}\theta - i\Lambda)$$

- First take scaling limit : $N, \Lambda \to \infty\,;\ 4Ne^{-\frac{3}{2}\Lambda} = m$: fixed

[2N: # of sites]

- Further take CFT/chiral limit : $\theta \to \theta - \log(m/2)$, $m \to 0$

$$T_a^{\mathsf{CFT}}(\lambda^2) := \lim_{m\to 0} T_a\big(\theta - \log(m/2)\big) = \sum_{p=0} t_a^{(p,0)} 2^{\frac{4p}{3}} \lambda^{2p} \quad [\, \lambda = e^{\frac{2}{3}\theta} \,]$$

  - driving term : $\lim_{m\to 0} m \cosh\big(\theta - \log(m/2)\big) \to e^{\theta}$

"massless TBA"

- One then has quantum Wronskian relations

[Bazhanov, Lukyanov, Zamolodchikov; '94, '96]

$$2i \sin\frac{\phi}{2} \cdot T_a^{\mathsf{CFT}}\big(e^{i(a-1)\pi}\lambda^2\big)$$

$$= e^{\frac{i}{2}(a+1)\phi} A_+(\lambda q^{-\frac{(a+1)}{2}}) A_-(\lambda q^{\frac{(a+1)}{2}}) - e^{-\frac{i}{2}(a+1)\phi} A_+(\lambda q^{\frac{(a+1)}{2}}) A_-(\lambda q^{-\frac{(a+1)}{2}})$$

$A_\pm \propto$ limits of Q-fn. $Q_\pm(\lambda) = e^{\frac{\pm\theta}{3\gamma}\phi} \prod_j 2\sinh\big(\lambda - \lambda_j(\pm\phi)\big)$

λ_j: Bethe roots , $q = e^{i\gamma}$

- A± are expanded as $\quad (\, 0 \leq \phi \leq 2(\pi - \gamma)\,)$

$$\log A_\pm = - \sum_{n=1} a_n^\pm(\phi)\,\lambda^{2n}\,, \qquad a_n^-(\phi) = a_n^+(-\phi)$$

- Substituting expansions into quant. Wronskian, e.g. , for a = 2

$$t_2^{(0,0)} = 2\cos\phi + 1\,, \quad 2^{\frac{4}{3}} t_2^{(1,0)} = (2\cos\phi + 1)\left(a_1^- + a_1^+\right)$$

$$2^{\frac{8}{3}} t_2^{(2,0)} = \frac{1}{2}(2\cos\phi + 1)\left(2a_1^+ a_1^- + (a_1^-)^2 - 2a_2^- + (a_1^+)^2 - 2a_2^+\right)$$

$$\vdots$$

- Relation among $a_n^\pm$ are given by Wronskian for a=0 $[\, T_0^{\mathsf{CFT}} = 1\,]$

$$\sin(n\gamma - \frac{\phi}{2})a_n^+ - \sin(n\gamma + \frac{\phi}{2})a_n^- = R_n(\phi)$$

$$R_1(\phi) = 0\,, \quad R_2(\phi) = \frac{\sin\frac{\phi}{2}}{2}\Big((a_1^+)^2 + (a_1^-)^2 + 2\cos 2\gamma\, a_1^+ a_1^-\Big)$$

$$\vdots$$

- Assuming
  - analyticity: $a_n^+ \ (a_n^-)$ analytic for $\phi > -\gamma \ (\phi < \gamma)$

[consistent w/ BAE]

  - asymptotics for $\phi \to \infty$:

$$a_n^+ \to \alpha_n \Big(\frac{\phi}{2\pi}\Big)^{1-\frac{4n}{3}}, \qquad \alpha_n = \frac{\Gamma(\frac{n}{3})\Gamma(\frac{2n}{3}-\frac{1}{2})}{4\pi^{\frac{1}{2}} n!}\Big(\pi^{\frac{1}{2}}\frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}\Big)^{-\frac{4n}{3}}$$

[deduces from BAE/NLIE, data for AdS4 case]

$\Rightarrow$ determine $a_n^\pm$ $\Rightarrow$ $t_a^{(p,0)}$ $(T_a^{\mathrm{CFT}}, Y_a^{\mathrm{CFT}})$

e.g.)

$$a_1^+(\phi) = \alpha_1 \frac{\Gamma\big(\frac{1}{3}+\frac{\phi}{2\pi}\big)}{\Gamma\big(\frac{2}{3}+\frac{\phi}{2\pi}\big)}$$

$$a_2^+(\phi) = \frac{3\alpha_1^2}{8\pi^3}\frac{\Gamma(\frac{2}{3}+\frac{\phi}{2\pi})}{\Gamma(\frac{1}{3}+\frac{\phi}{2\pi})}\int_{-\infty}^{\infty}\frac{dx}{2\pi}\frac{\sinh\frac{x}{2}}{x+i\phi}\Big(\Gamma(\frac{1}{3}+\frac{ix}{2\pi})\Gamma(\frac{1}{3}-\frac{ix}{2\pi})\Big)^3$$

UV expansion of 6pt remainder function

- Substituting expansion of Y/T–fn. into R_6^{strong}

A_{free} by (bulk) CPT

[Ito, Hatsuda, Sakai, YS, '10]

$\Rightarrow$ expansion around m = 0 [UV/regular polygonal limit]

$$R_6^{\text{strong}} = \sum_{k=0}^{\infty} r_6^{(k)}(\phi, \varphi)\, m^{\frac{4}{3}k} \qquad [\, \mu =: e^{3i\phi/2}\,]$$

$$r_6^{(0)} = \frac{\pi}{6} - \frac{3}{4\pi}\phi^2 + \frac{3}{4}\,\mathrm{Li}_2\big(1 - u_0^{-1}\big)\,, \qquad r_6^{(1)} = 0$$

$$r_6^{(2)} = \frac{3\kappa_4^2}{32(2\pi)^{\frac{2}{3}}}\left[\Big(1 - \frac{8\sqrt{3}}{9}\Big)(1 - u_0^{-1}) + \log u_0\right] \cdot B^2\Big(\frac{1}{3} + \frac{\phi}{2\pi}, \frac{1}{3} - \frac{\phi}{2\pi}\Big)$$

$$\kappa_n = \frac{1}{2\pi}\left[\gamma\Big(\frac{1}{n+2}\Big)\gamma\Big(\frac{3}{n+2}\Big)\right]^{\frac{1}{2}}\left[\frac{\Gamma(2/n)}{\gamma(1/n)}\right]^{\frac{2n}{n+2}}$$

$$u_0^{-1} = 4\cos^2(\phi/2)\,, \qquad \gamma(z) = \Gamma(z)/\Gamma(1-z)$$

- φ dependence : allowed only for k = 3n

[Z6−symm.]

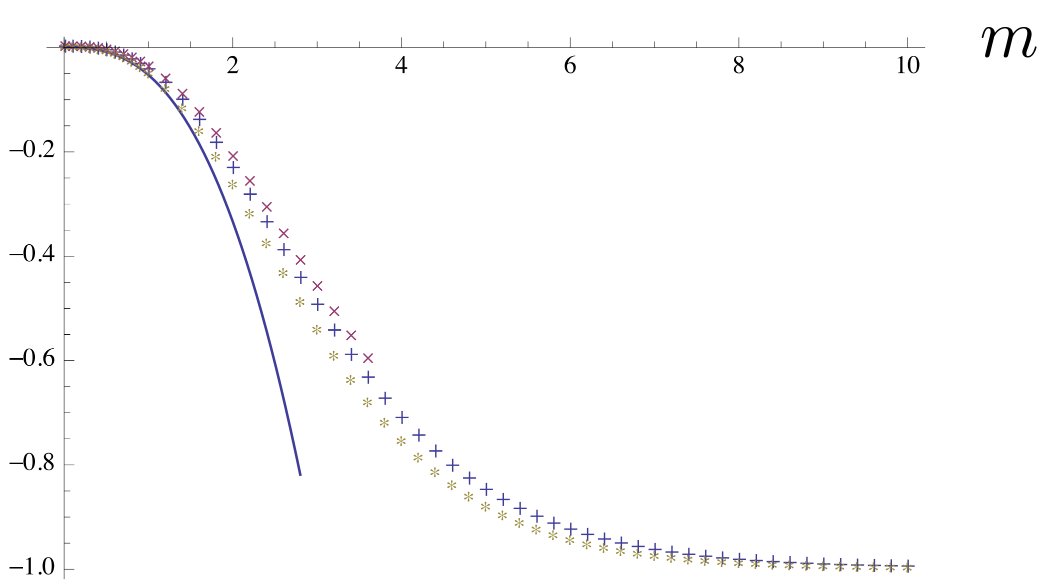
- for k=2, $t_a^{(1,0)}$ is enough

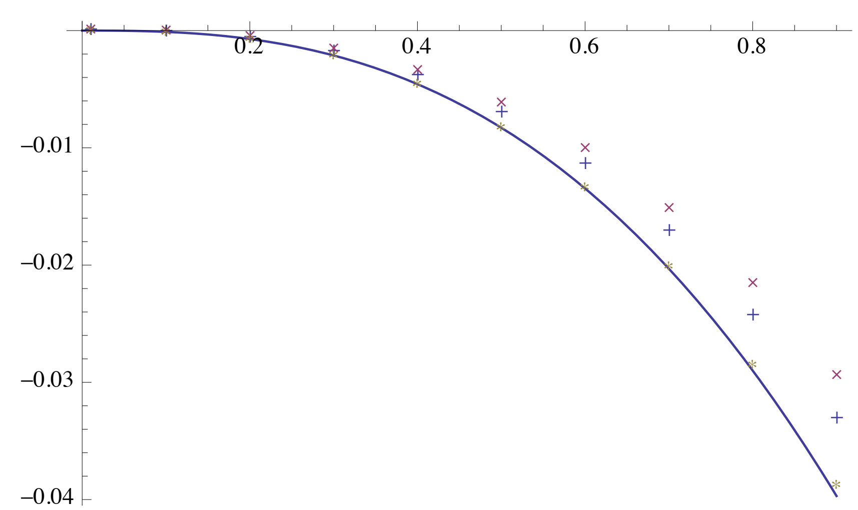
- for k ⩾ 3, need $t_a^{(p,q\neq 0)}$ (massive corrections)

Comparison w/ perturbative results

- 6pt remainder fn. : evaluated analytically at L=2,3,4 loops
- for large range of param. $R_6^{(L)}/R_6^{(L-1)}$: relatively constant
- useful to introduce rescaled remainder fn.

$$\bar{R}_6 = \frac{R_6 - R_6(m=0)}{R_6(m=0) - R_6(m=\infty)}$$





$\bar{R}_6^{(2)}(+)\,,\ \bar{R}_6^{(3)}(\times)\,,\ \bar{R}_6^{\text{strong}}(*)\,,\ \bar{R}_6^{\text{strong;Exp}}(-)$

$[\ \mu = e^{1/5}\,,\ \varphi = -\pi/20\ ]$

Summary

- "Massless" TBA, quantum Wronskian give

  UV expansion of chiral part of Y/T–fn.

  $\Rightarrow$ UV expansion of 6–pt remainder fn.

- New formalism to compute

  strong–coupling remainder fn. around UV (regular–polygonal) lim.

  - allowed us to deal w/ AdS5 ($\mu \neq 0$) case
  - part of higher order terms ($t_a^{(p,0)}$) : derived systematically

- As in other known cases,

  (rescaled) remainder fn. are similar to each other