Quantum Lax pairs for Painlevé systems and their solutions

Yasuhiko Yamada (Kobe Univ.)

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▶ **A correspondence**

Gauge theory $\overset{[AGT]}{\longleftrightarrow}$ CFT

$\updownarrow$

Painlevé

▶ At the classical level, it is known that the spectral curve of the Painlevé system = SW curve.

▶ **The aim of this talk** is to study the correspondence at the **quantum level**.

▸ **Plan.**

1. Lax pair for Painlevé type equations

2. Quantum Lax pairs from CFT

3. Solutions through the AGT relation

1. Lax pair for Painlevé type equations

► I will give a short overview of the Painlevé type equations and their relation to gauge theories and CFT.

► We use the **Lax pair** as a basic tool to understand the relation.

► **What is Painlevé equation?**

► In a narrow sense, the Painlevé equations mean the six nonlinear ODEs $P_{\mathrm{I}}, P_{\mathrm{II}}, \cdots, P_{\mathrm{VI}}$ discovered by P. Painlevé around 1900.

► The most important property of these equations is the **formulation as IMD (iso-monodromic deformation)**.

▸ **Example 1.** $P_{\rm II}$ **equation**

$$q'' = 2q^3 + tq + b. \quad (b \in \mathbb{C})$$

This equation for $q = q(t)$ can be written in Hamiltonian form as

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

where

$$H = \frac{p^2}{2} - \Big(q^2 + \frac{t}{2}\Big)p - aq. \quad (a = b + \frac{1}{2})$$

- $P_{\rm II}$ equation is obtained as the **compatibility** of a pair of linear differential equations $L\psi = 0$, $B\psi = 0$ (Lax pair),

$$L = \partial_x{}^2 - \Big\{2x^2 + t + \frac{1}{x-q}\Big\}\partial_x + \Big\{\frac{p}{x-q} - 2H - 2ax\Big\},$$

$$B = 2\partial_t - \frac{1}{x-q}\partial_x + \frac{p}{x-q}.$$

- $\Rightarrow P_{\rm II}$ equation is the **IMD** of the linear differential equation $L\psi = 0$ through $B\psi = 0$.

- Since $2H = p^2 - (2q^2 + t)p - 2aq$, one observe a symmety $(x, \partial_x) \leftrightarrow (q, p)$.

▶ **Example 2.** P_{VI} **equation**

Hamiltonian form:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

where

$$H = \frac{q(q-1)(q-t)}{t(t-1)}\Big\{p^2 - \Big(\frac{\alpha_0 - 1}{q-t} + \frac{\alpha_3}{q-1} + \frac{\alpha_4}{q}\Big)p\Big\}$$
$$+ \frac{(q-t)\alpha_2(\alpha_1+\alpha_2)}{t(t-1)}.$$

$$(\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1)$$

▸ P_{VI} **equation as IMD**

$$L = {\partial_x}^2 + \left\{\frac{1-\alpha_0}{x-t} + \frac{1-\alpha_3}{x-1} + \frac{1-\alpha_4}{x} - \frac{1}{x-q}\right\}\partial_x + \frac{p(q-1)q}{(x-1)x(x-q)} + \frac{\alpha_2(\alpha_1+\alpha_2)}{(x-1)x} - \frac{t(t-1)H}{(x-1)x(x-t)},$$

$$B = \frac{t(t-1)}{q-t}\partial_t + \frac{x(x-1)}{q-x}\partial_x + \frac{pq(q-1)}{x-q},$$

Compatibility of $L\psi = B\psi = 0$ gives the P_{VI} equation.

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q}.$$

▸ Classification of the 2nd order IMD

$$
\begin{array}{ccccccccc}
P_{\mathrm{VI}} & \to & P_{\mathrm{V}} & \to & P_{\mathrm{III}} & \to & P'_{\mathrm{III}} & \to & P''_{\mathrm{III}} \\
 & & & \searrow & & \searrow & & & \\
 & & & & P_{\mathrm{IV}} & \to & P_{\mathrm{II}} & \to & P_{\mathrm{I}}
\end{array}
$$

These correspond to $SU(2)$ gauge theories/Virasoro CFT

$$
\begin{array}{ccccccccc}
SW_4 & \to & SW_3 & \to & SW_2 & \to & SW_1 & \to & SW_0 \\
 & & & \searrow & & \searrow & & & \\
 & & & & AD_2 & \to & AD_1 & \to & AD_0
\end{array}
$$

- After the work of Painlevé, various generalized Painlevé type equations have been constructed as IMDs.
- They are specified by the data:

Punctured Riemann surface and singularity type at each puncture

- These IMDs are expected to correspond to higher rank gauge theory and W-CFT associated with the same geometry.

▸ **Example 1.** 2×2 L-operator on $\mathbb{P}^1$ with $n+3$ regular singularities

$$L = \frac{\partial}{\partial x} - A, \quad A = \frac{A_0}{x} + \frac{A_1}{x-1} + \sum_{i=1}^{n} \frac{A_{t_i}}{x - t_i}$$

$\to$ **n-Garnier system** (non-autonomous deformation of Gaudin model)

$\leftrightarrow$ $SU(2)^{\otimes n}$ quiver theory .

▸ **Example 2.** $N \times N$ L-operator $\mathbb{P}^1$ with 4 regular singularities

$$L = \frac{\partial}{\partial x} - A, \quad A = \frac{A_0}{x} + \frac{A_1}{x-1} + \frac{A_t}{x-t}$$

where the **spectral type** (=multiplicity of eigenvalues) of A_0, A_1, A_t, A_∞ are $(1^N), (1, N-1), (1, N-1), (1^N)$

$\to$ IMD [Fuji-Suzuki][Tsuda] $\leftrightarrow$ $SU(N)$, $N_f = 2N$.

▸ Example 3. Difference analogs

$SU(2)$ case

$$
\begin{array}{llllllllllllllll}
\text{Ell} & E_8 & & & & & & & & & & & & & & \\
 & & & & & & & & & & & & & & \nearrow & A_1 \\
q & E_8 & \to & E_7 & \to & E_6 & \to & D_5 & \to & A_4 & \to & A_{2+1} & \to & A_{1+1} & \to & A_1 \\
 & & & & & & & & & & & & & & & \\
d & E_8 & \to & E_7 & \to & E_6 & & \to & & D_4 & \to & A_3 & \to & A_{1+1} & \to & A_1 \\
 & & & & & & & & & & & & \searrow & & \searrow & \\
 & & & & & & & & & & & & & A_2 & \to & A_1
\end{array}
$$

The **difference/q-difference/elliptic difference** cases are expected to correspond to **4d/5d/6d** gauge theories.

▸ **Duality**

There may be **many** Lax pairs for **one** Painlevé type equation.

▸ **Example 1.** $SU(2)$ with $N_f = 4$.

(1) Scalar form with five regular singular points (one is apparent)

(2) 2×2 form with four regular singular points

(3) 8×8 form with one regular singularity $x = 0$ and one irregular singularity $x = \infty$ ($\leftrightarrow D_4^{(1)}$ Drinfeld-Sokolov hierarchy)

(4) $r \times r$ form ($3 \leq r \leq 7$) are also known

These Lax pairs are related by some integral transformations.

▸ **Example 2.** $SU(2)$ with E_6-flavor symmetry.

(1) 2×2 difference operator [Arinkin-Borodin]

$$\Psi(x+1) = A(x)\Psi(x), \quad A(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}$$

where $\mathrm{ord}(\mathrm{det} A) = 6$

(2) 3×3 differential operator with 3 regular singular points [Boalch]

$$\frac{\partial}{\partial x}\Psi(x) = \Big(\frac{A_0}{x} + \frac{A_1}{x-1}\Big)\Psi(x)$$

There is no continuous deformation.

2. Quantum Lax pairs

- The Lax operators L, B in Section.1 are differential operators in (x, ∂_x) depending on (q, p) variables as parameter.

- There is a **natural quantization of (q, p) variables**

$\rightarrow$ Quantum Lax operators are symmetric in (x, ∂_x) and (q, ∂_q), and they are obtained as the BPZ equations in $2d$ CFT.

▸ **Quantum Lax pair for $P_{\rm VI}$**: $\widehat{L}\Psi = \widehat{B}\Psi = 0$:

$$\widehat{L} = x(x-1)(x-t)\Big\{\frac{\alpha_0^{(2)}}{x}+\frac{\alpha_1^{(2)}}{x-1}+\frac{\alpha_t^{(2)}}{x-t}-\frac{\epsilon_1-\epsilon_2}{x-q}\Big\}\epsilon_1\partial_x$$
$$-q(q-1)(q-t)\Big\{\frac{\alpha_0^{(1)}}{q}+\frac{\alpha_1^{(1)}}{q-1}+\frac{\alpha_t^{(1)}}{q-t}-\frac{\epsilon_2-\epsilon_1}{q-x}\Big\}\epsilon_2\partial_q$$
$$+x(x-1)(x-t){\epsilon_1}^2{\partial_x}^2-q(q-1)(q-t){\epsilon_2}^2{\partial_q}^2+C(x-q),$$

$$\widehat{B} = q(q-1)\Big\{\frac{\alpha_0^{(1)}}{q}+\frac{\alpha_1^{(1)}}{q-1}+\frac{\alpha_t}{q-t}-\frac{\epsilon_2}{q-x}\Big\}\epsilon_2\partial_q$$
$$+\frac{t(t-1)}{q-t}\epsilon_1\epsilon_2\partial_t+\frac{x(x-1)}{q-x}\epsilon_1\epsilon_2\partial_x+q(q-1){\epsilon_2}^2{\partial_q}^2+C.$$

$\alpha_i^{(j)} = \alpha_i-\epsilon_j$, $C = (3\epsilon-\alpha_0-\alpha_1-\alpha_t-\alpha_\infty)(\epsilon-\alpha_0-\alpha_1-\alpha_t+\alpha_\infty)/4$,
$\epsilon = \epsilon_1+\epsilon_2$.

▸ The operators $\widehat{L}, \widehat{B}$ were obtained by using affine Weyl group symmetry ([Nagoya-Y]1206.5963)

▸ The operators $\widehat{L}, \widehat{B}$ give the classical Lax pair L, B for $P_{\rm VI}$ under the **Nekrasov-Shatashvili limit**:

$$\epsilon_2 \partial_q \to p. \quad (\epsilon_2 \to 0)$$

▸ Relation to CFT

▸ $\widehat{L}\psi = \widehat{B}\psi = 0$ are the **BPZ equations for 6-points block ψ** on $\mathbb{P}^1$ with two degenerate fields

$$\psi(x,q,t) = \left\langle V_{-\epsilon_2}(x) V_{-\epsilon_1}(q) \mathcal{O}_{\mathrm{VI}} \right\rangle,$$

$$\mathcal{O}_{\mathrm{VI}} = V_{\alpha_0}(0) V_{\alpha_1}(1) V_{\alpha_t}(t) V_{\alpha_\infty}(\infty),$$

where $V_\alpha(z)$s are the Virasoro primary operators:

$$c = 1 + 6\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}, \quad h(a) = \frac{a}{2\epsilon_1\epsilon_2}(\epsilon_1+\epsilon_2-\frac{a}{2}),$$

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial_w T(w) + \cdots$$

▸ Quantum Lax pair for P_{II}:

$$\begin{aligned}
\widehat{L} &= {\epsilon_1}^2 \partial_x^2 - \left(t + 2x^2 + \frac{\epsilon_1 - \epsilon_2}{x - q} \right) \epsilon_1 \partial_x - 2ax \\
&\quad - {\epsilon_2}^2 \partial_q^2 + \left(t + 2q^2 + \frac{\epsilon_2 - \epsilon_1}{q - x} \right) \epsilon_2 \partial_q + 2aq, \\
\widehat{B} &= \frac{\epsilon_1 \epsilon_2}{x - q} \partial_x - 2\epsilon_1 \epsilon_2 \partial_t \\
&\quad - {\epsilon_2}^2 \partial_q^2 + \left(t + 2q^2 + \frac{\epsilon_2}{q - x} \right) \epsilon_2 \partial_q + 2aq.
\end{aligned}$$

These are also obtained from CFT (with Gaiotto states), and reproduce the classical Lax pair under the NS limit

$$\epsilon_2 \partial_q \to p. \quad (\epsilon_2 \to 0)$$

▸ The integral formula for special solutions $\psi(x, q, t)$:

$$\psi = \int_C U(x, q, t, \{u_i\}, \{v_j\}) \prod_{i=1}^{N_u} du_i \prod_{j=1}^{N_u} dv_j,$$

where

$$U = \prod_{1 \le i < j \le N_u} (u_i - u_j)^{-\frac{2\epsilon_1}{\epsilon_2}} \prod_{1 \le i < j \le N_v} (v_i - v_j)^{-\frac{2\epsilon_2}{\epsilon_1}}$$

$$\times \prod_{i=1}^{N_u} \prod_{j=1}^{N_v} (u_i - v_j)^{-2} \prod_{i=1}^{N_u} F(u_i)^{\frac{1}{\epsilon_1}} \prod_{i=1}^{N_v} F(v_i)^{\frac{1}{\epsilon_2}},$$

and the function $F(z)$ is given by

$F(z) = (z-x)^{\epsilon_2}(z-q)^{\epsilon_1}F_J(z)$ for P_{J} with

$$F_{\mathrm{II}}(z) = e^{\frac{2}{3}z^3+zt},$$
$$F_{\mathrm{III}}(z) = z^a e^{\frac{t}{z}+z},$$
$$F_{\mathrm{IV}}(z) = z^a e^{\frac{1}{4}z^2+tz},$$
$$F_{\mathrm{V}}(z) = z^a(z-1)^b e^{-tz},$$
$$F_{\mathrm{VI}}(z) = z^{\alpha_0}(z-1)^{\alpha_1}(z-t)^{\alpha_t}.$$

- The integral ψ solves the linear problem $\widehat{L}\psi = \widehat{B}\psi = 0$ if parameters take special values related to $N_u, N_v \in \mathbb{Z}_{\geq 0}$: e.g. $\alpha_0+\alpha_1+\alpha_t+\alpha_\infty = (3-2N_u)\epsilon_1+(3-2N_v)\epsilon_2$ for P_{VI}.

▸ Proofs

The free fields representation or a direct computation using the identities:

$$\frac{1}{\epsilon_1\epsilon_2}\widehat{L}(U) = \sum_{i=1}^{N_u} \frac{\partial}{\partial u_i}\Big(U\frac{(x-q)G_{\mathsf{J}}(u_i)}{(u_i-x)(u_i-q)}\Big) + (u\to v),$$

$$\frac{1}{\epsilon_1\epsilon_2}\widehat{B}(U) = \sum_{i=1}^{N_u} \frac{\partial}{\partial u_i}\Big(U\frac{G_{\mathsf{J}}(u_i)}{(u_i-q)}\Big) + (u\to v),$$

where

$$G_{\mathrm{II}}(z) = 1, \quad G_{\mathrm{III}}(z) = z^2, \quad G_{\mathrm{IV}}(z) = z,$$

$$G_{\mathrm{V}}(z) = z(z-1), \quad G_{\mathrm{VI}}(z) = z(z-1)(z-t).$$

▶ The construction of quantum Lax pairs and the inegral formulae can be extended to N- Garnier system :

$$\psi(x, q_1, \cdots, q_N, t) = \left\langle V_{-\epsilon_2}(x) \prod_{i=1}^{N} V_{-\epsilon_1}(q_i)\, \mathcal{O}_{\mathrm{VI}} \right\rangle,$$

$$\mathcal{O}_{\mathrm{VI}} = V_{\alpha_0}(0) V_{\alpha_1}(1) V_{\alpha_t}(t) V_{\alpha_\infty}(\infty).$$

3. Solutions through the AGT relation

- In view of the **AGT** relation, the Nekrasov functions should give a solution for the quantum Lax linear problems $\widehat{L}\psi = \widehat{B}\psi = 0$.

- We will check $\psi = Z_{\text{cft}} = Z_{\text{Nek}}$ (up to $U(1)$ factors).

▸ For the quantum Lax linear problems $\widehat{L}\psi = \widehat{B}\psi = 0$ for P_{VI}, there exist a power series solution of the form

$$\psi(x,q,t) = x^{\frac{v}{\epsilon_1}} q^{\frac{v}{\epsilon_2}} t^{-\frac{v(v+\alpha_0+\alpha_t-\epsilon)}{\epsilon_1\epsilon_2}} Z_{\mathrm{cft}}(x,q,t),$$

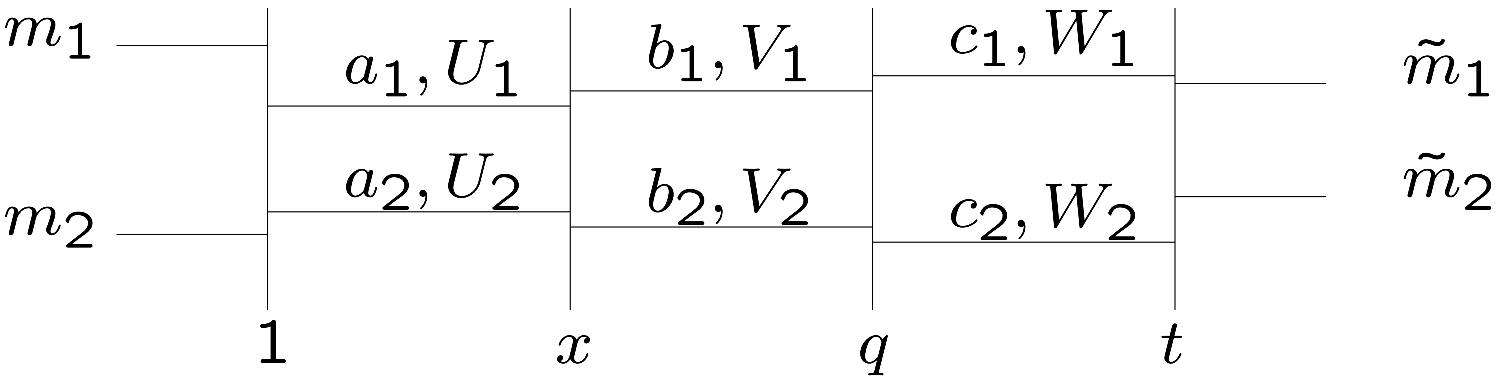
$$Z_{\mathrm{cft}}(x,q,t) = 1 + \cdots \in \mathbb{C}[[x, \frac{q}{x}, \frac{t}{q}]].$$

▸ Z_{cft} will be related to the 6 points block on $\mathbb{P}^1$ with following intermediate states:

$$\begin{array}{ccccccccccc} & & \alpha_1 & & -\epsilon_2 & & -\epsilon_1 & & \alpha_t & & \\ & & | & & | & & | & & | & & \\ & & | & & | & & | & & | & & \\ \alpha_\infty & -- & 1 & -- & x & -- & q & -- & t & -- & \alpha_0 \\ & & & v+\epsilon & & v+\epsilon_1 & & v & & & \end{array}$$

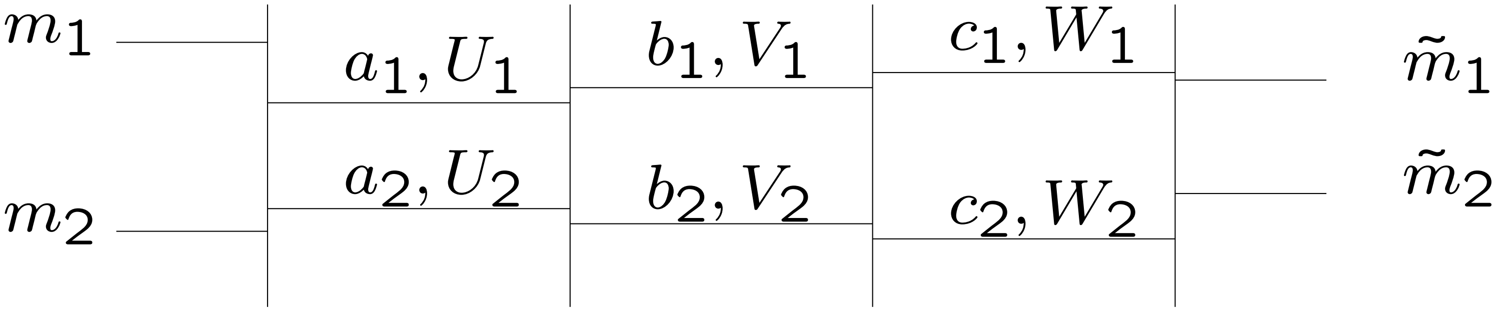
▸ Z_{cft} is represented in terms of Nekrasov function ([AGT], [Alba-Morozov](0912.2535))

▸ $Z_{\text{Nek}} = Z_{\text{Nek}}(x, q, t; \{m_1, m_2, \tilde{m}_1, \tilde{m}_2\}, \{a, b, c\})$
$a_{1,2} = \pm a,\ b_{1,2} = \pm b,\ c_{1,2} = \pm c$.



Z_{Nek} is defined as a sum over the 6 partitions U_i, V_i, W_i ($i = 1, 2$):

$$Z_{\text{Nek}} = \sum_{U,V,W} \left(\prod_{i,j=1}^{2} w_{ij} \right) x^{|U_1|+|U_2|} (\frac{q}{x})^{|V_1|+|V_2|} (\frac{t}{q})^{|W_1|+|W_2|}.$$



$$w_{ij} = \frac{z_f(a_i, U_i, m_j) z_b(a_i - b_j + \frac{\epsilon_1}{2}, U_i, V_j) z_b(b_i - c_j + \frac{\epsilon_2}{2}, V_i, W_j) z_f(c_i, W_i, \tilde{m}_j)}{z_b(a_i - a_j, U_i, U_j) z_b(b_i - b_j, V_i, V_j) z_b(c_i - c_j, W_i, W_j)},$$

$$\begin{aligned} z_f(a, Y, m) &= \prod_{(i,j)\in Y} (a + \epsilon_1(i-1) + \epsilon_2(j-1) + m) \\ z_b(a, Y, W) &= \prod_{(i,j)\in Y} (a + \epsilon_1(-W'_j + i) + \epsilon_2(Y_i - j + 1)) \\ &\times \prod_{(i,j)\in W} (a + \epsilon_1(Y'_j - i + 1) + \epsilon_2(-W_i + j)). \end{aligned}$$

Y' is a dual Young diagram of Y.

- We have $Z_{\text{Nek}} = Z_{U(1)} Z_{\text{cft}}$, where

$$Z_{U(1)} = (1-x)^{k_1}(1-q)^{k_2}(1-t)^{k_3}(1-\frac{q}{x})^{k_4},$$

$$k_1 = \frac{\epsilon(\epsilon_1 + 2\epsilon - \alpha_t)}{\epsilon_1\epsilon_2}, \quad k_2 = \frac{\epsilon(\epsilon_2 + 2\epsilon - \alpha_t)}{\epsilon_1\epsilon_2},$$
$$k_3 = \frac{\alpha_1(\epsilon - \alpha_t)}{\epsilon_1\epsilon_2}, \qquad k_4 = -\frac{\epsilon_1}{\epsilon_2} - 2,$$

$$(\alpha_0, \alpha_t, \alpha_1, \alpha_\infty) = (m_1 + m_2 + \epsilon, m_1 - m_2 + 2\epsilon,$$
$$\tilde{m}_1 + \tilde{m}_2, \tilde{m}_1 - \tilde{m}_2 + \epsilon),$$
$$v - (0, \frac{\epsilon_1}{2}, \frac{\epsilon}{2}) = m_2 - \epsilon - (a, b, c), \quad \epsilon = \epsilon_1 + \epsilon_2.$$

▶ The example given above is for $N_f = 4$ case (quantum $P_{\rm VI}$) $\leftrightarrow$ usual CFT with regular singular points.

▶ $N_f = 0$ case was also checked. Hence, the gauge/CFT/Painlevé correspondence is consistent with degenerations.

Conclusion:

- Quantum gauge/CFT/Painlevé correspondence is studied.

- From this correspondence, it is expected that **Nekrasov partition functions solve the quantum Lax linear problem for IMDs**.

- We have checked this in $SU(2)$ case.

- Classical Lax is recovoerd by **NS limit:** $c \to \infty$. Relation to the work [Iorgov, Lisovyy, Teschner, $\cdots$] (conformal bock at $c = 1$ gives the τ-function of classical Painlevé equation) will be an interesting problem.

Thank you!