Recent progress on *q*-deformations of the AdS_5 x S^5 superstring

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Refs. 1401.4855 and 1402.6147

0. Introduction

An overview of integrability behind AdS/CFT

A strategy to consider integrable deformations

A recent progress in string theory

The discovery of integrability behind AdS/CFT

The current understanding

Integrable spin chain, Bethe ansatz

[Beisert-Staudacher, ...]

$\lambda \gg 1$ — IIB string on $AdS_5 \times S^5$

$\lambda \ll 1$ $(N \to \infty)$ — 4D $\mathcal{N}=4$ SU(N) SYM

IIB string on $AdS_5 \times S^5$ ⟷ 4D $\mathcal{N}=4$ SU(N) SYM

[Frolov-Tseytlin, Bena-Polchinski-Roiban, Arutyunov-Frolov...]

[Minahan-Zarembo, BMSZ, ...]

The integrable structure → the structure of $AdS_5 \times S^5$

Next step: **Integrable deformations of AdS/CFT**

NOTE AdS/CFT belongs to the rational class (like XXX-model)

There are some kinds of integrable deformations

EX beta-deformations, gamma-deformations

These are based on TsT transformations and closed within the rational class

➡ Our interest is a trigonometric-type deformation of AdS/CFT (like XXZ-model)

to understand the wider structure behind AdS/CFT

c.f., XXX model is nothing but a degenerate limit of XXZ model.

XXZ model has a richer structure than XXX model. EX phase transition

The study of it may reveal some new aspects of AdS/CFT (Our motive)

Strategy to consider trigonometric deformations

A *q*-deformed symmetry is a characteristic of the trigonometric class.

EX *q*-deformation of su(2) [Drinfeld, Jimbo,1985]

$$[S^3, S^\pm] = \pm S^z\,, \qquad [S^+, S^-] = \frac{q^{S^z} - q^{-S^z}}{q - q^{-1}}$$

By taking the limit $q \to 1$,
the su(2) algebra is reproduced.

Three approaches:

Spin chain side

1) *q*-deformed S-matrix

[Beisert,Koroteev,Galleas,Matsumoto,
Hoare-Hollowood-Miramontes,
Arutyunov-de Leeuw-van Tongeren]

Mathematically done, but the geometry is unknown

a little understood

String theory side

2) *q*-deformed geometry

3D is easy, the very recent progress on 10D

[Cherednik, Balog-Forgacs-Palla,Kawaguchi-Matsumoto-KY,
Basso-Rej, Orlando-Reffert-Uruchurtu, Klimcik
Delduc-Magro-Vicedo, Arutyunov-Borsato-Frolov]

Gauge theory side

3) deformed N=4 SYM

? ?

Complex beta deformation?

[Berenstein-Cherkis, Correa, Lunin-Maldacena]

In this talk, we will concentrate on the string theory side.

- A summary of the recent progress on *q*-deformations of the $AdS_5 \times S^5$ superstring
- An introduction to our recent works, 1401.4855 & 1402.6147.

Io Kawaguchi will talk about them.

The contents of this talk

1. Yang-Baxter sigma models
2. *q*-deformations of the $AdS_5 \times S^5$ superstring
3. Summary and Discussion

1. Yang-Baxter sigma models

- A review of squashed S^3 and its higher-dimensional generalization
- An introduction of modified Yang-Baxter eq. & linear R-operators

C. Klimcik, hep-th/0210095, 0802.3518

Kawaguchi-KY, 1107.3662, Delduc-Magro-Vicedo, 1308.3581

A simple example of deformed sigma models - squashed S^3 sigma models

What is squashed S^3 ?

Round S^3 with the radius L

$$ds^2 = \frac{L^2}{4}[\underbrace{d\theta^2 + \cos^2\theta d\phi^2}_{S^2} + \underbrace{(d\psi + \sin\theta d\phi)^2}_{S^1\text{-fibration}}]$$

3 angles (θ, ϕ, ψ)

Isometry: $SU(2)_L \times SU(2)_R$

a deformation of round S^3

$C=0$

Squashed S^3

$$ds^2 = \frac{L^2}{4}[d\theta^2 + \cos^2\theta d\phi^2 + (1 + \underline{C})(d\psi + \sin\theta d\phi)^2]$$

squashing parameter (C)

Isometry: $SU(2)_L \times \underline{U(1)_R}$

Sigma model action on squashed S^3

[Cherednik, 1981]

$$S = \int d\tau d\sigma\, \gamma^{\alpha\beta} \left[\mathrm{Tr}(J_\alpha J_\beta) - 2C\, \mathrm{Tr}(T_3 J_\alpha) \mathrm{Tr}(T_3 J_\beta) \right]$$

$x^\alpha = (\tau, \sigma), \quad \gamma_{\alpha\beta} = (-1, 1)$: 2D Minkowski spacetime

$$J = g^{-1} dg = J^1 T_1 + J^2 T_2 + J^3 T_3 \,, \qquad g = e^{\phi T_1} e^{\theta T_2} e^{\psi T_3} \in SU(2)$$

Symmetry

hybrid integrability

[Kawaguchi-Matsumoto-KY, 1203.3400]

$SU(2)_L$ symmetry → two $SU(2)_L$ Yangians

[Kawaguchi-KY, 1008.0776]

equivalence

$U(1)_R$ symmetry → *q*-deformed $SU(2)_R$ → quantum affine algebra

[Kawaguchi-KY, 1107.3662]

[Kawaguchi-Matsumoto-KY, 1201.3058]

This plays an important role in studying the integrability of Yang-Baxter sigma models

Yang-Baxter sigma models - the principal chiral model (PCM) case -

Let us start from a *G*-principal chiral model, whose symmetry is $G_{\rm L} \times G_{\rm R}$.

The deformed PCM action

[C. Klimcik, hep-th/0210095]

$$S = -\frac{1}{2}\int d\tau d\sigma\, \gamma^{\alpha\beta}{\rm Tr}\left(g^{-1}\partial_\alpha g\,\frac{(1+\eta^2)}{1-\eta R}(g^{-1}\partial_\beta g)\right)$$

η : deformation parameter

R : a solution of modified classical Yang-Baxter eq. (mCYBE)

A solution of mCYBE ⟷ An integrable deformation

Symmetry $G_{\rm L} \times G_{\rm R} \longrightarrow G_{\rm L} \times U(1)^r \quad r:\ \text{the rank of } G_{\rm R}$

NOTE The construction of Lax pairs [C. Klimcik, 0802.3518]

The enhancement to *q*-deformed $G_{\rm R}$ [Kawaguchi-KY, 1107.3662, Delduc-Magro-Vicedo, 1308.3581]

Possibly, quantum affine algebra (Not yet shown)

R-operator & modified classical Yang-Baxter eq.

Start from

Yang-Baxter eq. $\widehat{R}_{12}\widehat{R}_{23}\widehat{R}_{13} = \widehat{R}_{13}\widehat{R}_{23}\widehat{R}_{12}$ $\widehat{R}:$ R-matrix

Expanding the R-matrix as $\widehat{R}_{12} = 1 + \hbar\, \underline{r_{12}} + \mathcal{O}(\hbar^2)\,, \quad r_{12} \in \mathfrak{g} \otimes \mathfrak{g}$

Classical *r*-matrix

Classical Yang-Baxter eq. (CYBE) $[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0$

With an *r*-matrix, one may introduce

A linear R-operator $R \;:\; \mathfrak{g} \longrightarrow \mathfrak{g}$

$$R(X) \equiv \langle r_{12}, 1 \otimes X \rangle = \sum_i a_i \langle b_i, X \rangle \qquad \text{for} \quad X \in \mathfrak{g}$$

$$r_{12} = \sum_i a_i \otimes b_i \qquad \text{with} \qquad a_i, b_i \in \mathfrak{g}$$

Then the CYBE can be rewritten as

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = 0 \qquad \text{(CYBE)}$$

if and only if

- the bilinear form is non-degenerate
- the classical *r*-matrix is skew-symmetric

$$r_{21} = \sum_i b_i \otimes a_i = -r_{12} \quad \Longleftrightarrow \quad \langle R(X), Y \rangle = -\langle X, R(Y) \rangle$$

Then the modified classical Yang-Baxter eq. (mCYBE) is given by

$$[R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = \underline{-c^2[X, Y]} \qquad (c \in \mathbb{C})$$

We set $c = -i$ by rescaling R.

An example of R-operator for squashed S^3

Consider the su(2) algebra : $[T^3, T^{\pm}] = \pm 2T^{\pm}, \quad [T^+, T^-] = T^3$

$$T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad T^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad T^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The Drinfeld-Jimbo type r-matrix:

$$r_{\rm DJ} = -i\left[T^+ \otimes T^- - T^- \otimes T^+\right]$$

satisfies mCYBE (unique solution)

⟺ $$R(T^+) = -i\,T^+\,, \quad R(T^-) = +i\,T^-\,, \quad R(T^3) = 0$$

The resulting action:

$$S = -\frac{1}{2}\int d\tau d\sigma\,\gamma^{\alpha\beta}\left[\mathrm{Tr}(J_\alpha J_\beta) + \eta^2 \mathrm{Tr}(T^3 J_\alpha)\mathrm{Tr}(T^3 J_\beta)\right]$$

Computation:

We first introduce $$J \equiv g^{-1}dg, \quad A \equiv \frac{1}{1-\eta R}J$$

and expand them as
$$\begin{cases} J = J^+T^- + J^-T^+ + J^3T^3 \\ A = A^+T^- + A^-T^+ + A^3T^3 \end{cases}$$

Then J can be rewritten as

$$\begin{aligned} J &= (1-\eta R)A \\ &= (1-i\eta)A^+T^- + (1+i\eta)A^-T^+ + A^3T^3 \end{aligned}$$

Thus we obtain $$A^+ = \frac{J^+}{1-i\eta}, \quad A^- = \frac{J^-}{1+i\eta}, \quad A^3 = J^3$$

Thus the action is computed as

$$\begin{aligned} S &= -\frac{1}{2}\int d\tau d\sigma\, \gamma^{\alpha\beta}\left[2J^+_\alpha J^-_\beta + (1+\eta^2)J^3_\alpha J^3_\beta\right] \\ &= -\frac{1}{2}\int d\tau d\sigma\, \gamma^{\alpha\beta}\left[\mathrm{Tr}(J_\alpha J_\beta) + \eta^2 \mathrm{Tr}(T^3 J_\alpha)\mathrm{Tr}(T^3 J_\beta)\right] \end{aligned}$$

The symmetric coset case

[Delduc-Magro-Vicedo, 1308.3581]

One may consider the Yang-Baxter extension for symmetric coset *M=G/H*.

The decomposition of Lie algebra: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ (direct product as vector spaces)

The grade $\mathfrak{h}: \; 0, \quad \mathfrak{m}: \; 1$ $\quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$

0, 0 0 0, 1 1 1, 1 0

The deformed coset action

$$S[g] = -\frac{1}{2}\int d\tau d\sigma\, \gamma^{\alpha\beta}\mathrm{Tr}\left(P_1(g^{-1}\partial_\alpha g)\,\frac{1+\eta^2}{1-\eta R_g \circ P_1}P_1(g^{-1}\partial_\beta g)\right)$$

$$R_g = \mathrm{Ad}g^{-1}\circ R \circ \mathrm{Ad}g \;, \qquad P_1 : \; \text{projection} \quad \mathfrak{g} \longrightarrow \mathfrak{m}$$

The symmetry $\quad G \longrightarrow U(1)^r \quad r: \;$ the rank of G

q-deformation of *G* [Delduc-Magro-Vicedo, 1308.3581]

2. *q*-deformations of the $AdS_5 \times S^5$ superstring

- A review of the $AdS_5 \times S^5$ superstring (MT)
- A *q*-deformation of the $AdS_5 \times S^5$ superstring (DMV)
- The explicit metric (ABF)
- Jordanian deformations (KMY)

The AdS_5 x S^5 background

$$\mathrm{AdS}_5 \times \mathrm{S}^5 = \frac{SO(2,4)}{SO(1,4)} \times \frac{SO(6)}{SO(5)}$$

The symmetric coset exhibits the Z_2-grading.

Including fermions

The super coset: $$\frac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$$

The Z_4-grading ensures the classical integrability

[Bene-Polchinski-Roiban, 2003]

Decomposition of **su**(2,2|4)

$$\mathfrak{su}(2,2|4) = \mathfrak{su}(2,2|4)^{(0)} \oplus \mathfrak{su}(2,2|4)^{(1)} \oplus \mathfrak{su}(2,2|4)^{(2)} \oplus \mathfrak{su}(2,2|4)^{(3)}$$

The projections $P_n : \mathfrak{su}(2,2|4) \longrightarrow \mathfrak{su}(2,2|4)^{(n)} \quad (n = 0, \ldots, 3)$

These are used to construct the Green-Schwarz (GS) string action on AdS_5 x S^5 .

The GS action of the $AdS_5 \times S^5$ superstring

[Metsaev-Tseytlin, hep-th/9805028]
[For a nice review, see 0901.4937]

$$S = -\frac{1}{2}\int d\tau \int_0^{2\pi} d\sigma \, P_-^{\alpha\beta} \, \mathrm{Str}\left(A_\alpha d \circ A_\beta\right)$$

where $\quad A_\alpha \equiv g^{-1}\partial_\alpha g \,, \quad g \in SU(2,2|4)$, $\quad d \equiv P_1 + 2P_2 - P_3$

$$P_\pm^{\alpha\beta} \equiv \frac{1}{2}(\gamma^{\alpha\beta} \pm \epsilon^{\alpha\beta}) \qquad \begin{cases} \gamma^{\alpha\beta} = \mathrm{diag}(-1,1) \\ \epsilon^{\alpha\beta} \, : \, \text{anti-symm. tensor} \end{cases}$$

NOTE

- By taking a parameterization of the group element,
  the coordinate system can be introduced.

- The Lax pair is constructed.
  There are many works on the classical integrability.

A q-deformation of the $AdS_5 \times S^5$ superstring

[Delduc-Magro-Vicedo, PRL122(2014)051601, 1309.5850]

$$S = -\frac{(1+\eta^2)^2}{2(1-\eta^2)} \int d\tau \int_0^{2\pi} d\sigma \, P_-^{\alpha\beta} \, \mathrm{Str} \left[A_\alpha d \circ \frac{1}{1-\eta R_g \circ d} (A_\beta) \right]$$

where $A_\alpha \equiv g^{-1}\partial_\alpha g \,, \quad g \in SU(2,2|4)$, $\quad d \equiv P_1 + \frac{2}{1-\eta^2} P_2 - P_3$

$$P_\pm^{\alpha\beta} \equiv \frac{1}{2}(\gamma^{\alpha\beta} \pm \epsilon^{\alpha\beta}) \qquad \left[\begin{array}{l} \gamma^{\alpha\beta} = \mathrm{diag}(-1,1) \\ \epsilon^{\alpha\beta} : \text{ anti-symm. tensor} \end{array} \right.$$

NOTE

- R-operation: $$R(M) = \begin{cases} -iM & (\text{if } M \text{ is a positive root}) \\ +iM & (\text{if } M \text{ is a negative root}) \end{cases}$$

- kappa-invariant
- the Lax pair has been constructed

The deformed metric (in the string frame) [Arutyunov-Borsato-Frolov, 1312.3542]

$$ds^2 = ds^2_{\rm AdS} + ds^2_{\rm S}$$

$$\begin{aligned} ds^2_{\rm AdS} &= \frac{1}{1-\kappa^2\sinh^2\rho}\left[-\cosh^2\rho\, dt^2 + d\rho^2\right] \\ &+ \sinh^2\rho\left[\frac{1}{1+\kappa^2\sinh^4\rho\,\sin^2\zeta}\left(d\zeta^2+\cos^2\zeta\, d\psi_1^2\right)+\sin^2\zeta\, d\psi_2^2\right] \\ ds^2_{\rm S} &= \frac{1}{1+\kappa^2\sin^2\theta}\left[d\theta^2+\cos^2\theta\, d\phi^2\right] \\ &+ \sin^2\theta\left[\frac{1}{1+\kappa^2\sin^4\theta\,\sin^2\xi}\left(d\xi^2+\cos^2\xi\, d\phi_1^2\right)+\sin^2\xi\, d\phi_2^2\right] \end{aligned}$$

where $\kappa \equiv \dfrac{2\eta}{1-\eta^2}$ is a deformation parameter.

NOTE this metric is singular.

The NS-NS 2-form

$$B_2 = \frac{\kappa}{2}\left[\frac{\sinh^4\rho\,\sin 2\zeta}{1+\kappa^2\sinh^4\rho\,\sin^2\zeta}d\psi_1\wedge d\zeta - \frac{\sin^4\theta\,\sin 2\xi}{1+\kappa^2\sin^4\theta\,\sin^2\xi}d\phi_1\wedge d\xi\right]$$

NS-NS 2-form is turned on both the AdS and sphere parts.

NOTE: the other components of type IIB SUGRA have NOT been fixed yet.

The world-sheet S-matrix

The perturbative world-sheet S-matrix of the bosonic particles have been computed.

➡ It exactly agrees the q-deformed S-matrix (in the large string tension limit)

Another kind of integrable deformations

[Kawaguchi-Matsumoto-KY, 1401.4855]

There are two types of q-deformations

1) Standard q-deformation (Drinfeld-Jimbo type) $\longrightarrow \mathrm{mCYBE}$

2) Non-standard q-deformation (Jordanian deformations) $\longrightarrow \mathrm{CYBE}$

Delduc-Magro-Vicedo, Arutyunov-Borsato-Frolov	➡	1) standard
Kawaguchi-Matsumoto-KY	➡	2) Jordanian

Advantages of Jordanian deformations

i) Partial deformations are possible (for example, only the AdS_5 part)

ii) A type IIB SUGRA solution has been found

[Kawaguchi-Matsumoto-KY, 1402.6147]

➡ In the next talk

3. Summary and Discussion

Summary

An overview of the recent progress on

q-deformations of the $AdS_5 \times S^5$ superstring

↑ Yang-Baxter sigma model description

Future problems

(for the standard *q*-deformation)

SUGRA solutions → find out dilaton and R-R fluxes

singularity is resolved?

One-loop beta function?

For Jordanian deformations of the string action & type IIB SUGRA solution,

→ Io Kawaguchi's talk

Thank you!