Post-Newtonian equations of motion for relativistic compact binaries

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KAGRA Data Analysis School @ RESCEU 2013

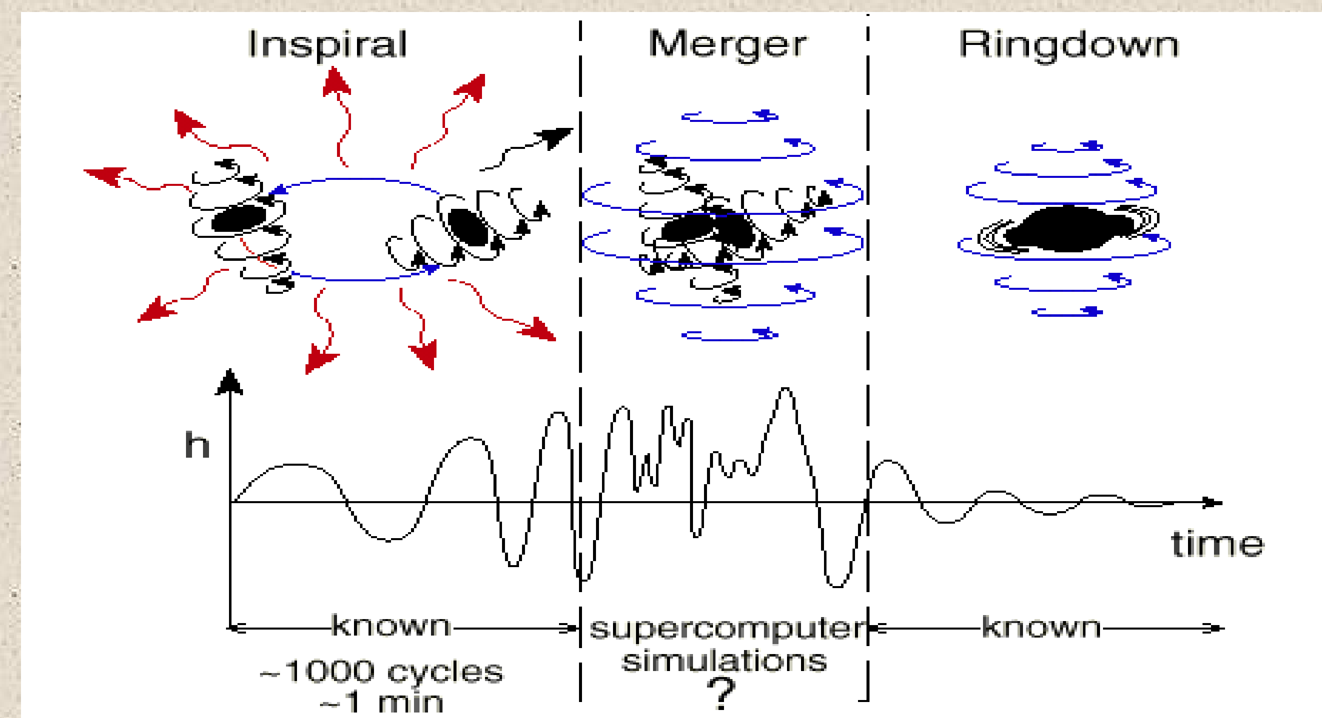
- September 27 (Fri.) 10:00 and 28 (Sat.) ~ 17:00
- RESCEU, University of Tokyo
- Continuous Gravitational Wave search
- Lecture on GW from neutron stars by Kojima san
- Lecture on pulsar timing

Where PNA is useful (current/future).

- For gravitational wave astronomy. Why?
- most promising GW sources for KAGRA/LIGO/VIRGO are inspiralling compact stars (neutron stars, black holes) binary. Because ...
  - **Existence: Such systems are known to exist.**
  - **Event rate: 40-600 events per one year for advanced LIGO**
  - **Waveform templates are available: we can use the most optimal linear filter (matched filter). How to construct?**
- PNA is useful to construct waveform templates.

PNA, Numerical relativity, Single star/BH Perturbation

Compact binary mergers



Thorne 2002

PNA **NR** **Pert.**

My talk

Waveform Templates and Equations of Motion

$$h(t) \sim \frac{G^2 Q m_1 m_2}{c^4 R r_{12}(t)} \cos\left(\int 2\pi f(t) dt\right),$$

Need to know Phase evolution

EOM → orbital evolution → GW Phase evolution

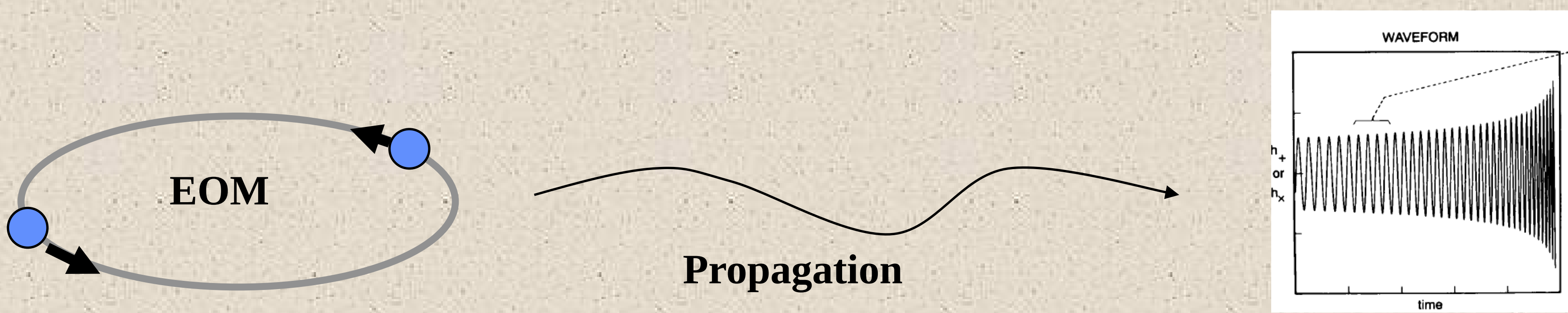
$$\mu a^i = -\frac{G(m_1+m_2)}{r_{12}^3} r_{12}^i + \cdots; \quad \frac{df}{dt} = \frac{96\pi^{8/3} G^{5/3}}{5c^5} M_{\mathrm{chirp}}^{5/3} f^{11/3} + \ldots,$$

- **More accurate the EOM is, better the quality of waveform templates becomes and we get good signal to noise ratio.**
- **For GW detection and measurements, 3.5 (~ 4 PN) EOM may be enough for stellar mass binary.**

EOM and wave propagation problem.

Both EOM and wave propagation from source to observer must be computed to construct waveform.

- Blanchet-Damour-Iyer (BDI) et al or Will-Wiseman (WW) succeeded in deriving higher order waveform.
- This talk is on EOM.



Two approaches to PNA EOM.

Two approaches to find PNA binary dynamics in insipiralling phase.

1. ADS Hamiltonian in ADMTT gauge
   - Damour, Jaranowski & Schaefer (2001) was the first who completed the 3 PN ADS Hamiltonian, partially completed 4PN.
2. Equations of motion in harmonic gauge
   - Blanchet, Faye & Esposito-Farese, Pati & Will, Itoh, Futamase & Asada.

← Hamiltonian

← EOM

Post-Newtonian equations of motion for relativistic compact binaries

Plan: Concentrate on (my contributions to) 3.5 PN EOM.
Then two slides show solutions to EOM and waveform briefly.

References:

Itoh, Futamase & Asada, Phys. Rev.D62:064002-1-12(2000).
Itoh, Futamase & Asada, Phys. Rev.D63:064038-1-21 (2001).
Itoh & Futamase, Phys. Rev. D68:121501-1-5(R)(2003).
Itoh, Phys. Rev.D69:064018-1-43 (2004).
Itoh, Class. and Quant. Grav. 21 S529-S534 (2004).
Futamase & Itoh, Living Review in Relativity 10:2 1-81 (2007).
Itoh, Phys. Rev.D80:124003-1-17 (2009).

For other approaches, see e.g.
L. Blanchet, Living Review in Relativity **9**, 4 (2006).

Key ideas in our formalism

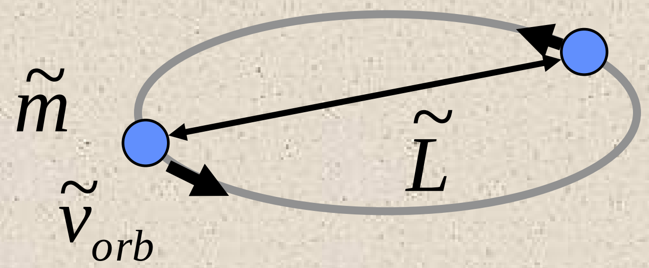
1. **Post-Newtonian approximation (PNA)**
   - Anderson & Decanio (1975)
2. **Point particle approximation**
   - Strong field point particle limit (Futamase ,1987)
3. **Surface integral approach**
   - Similar to Einstein, Infeld & Hoffmann (1938).

Key ideas 1. PNA

Post-Newtonian approximation.

Newtonian gravitational bound system : Balance between centrifugal force and gravitational force

$$\frac{G\tilde{m}}{c^2\tilde{L}} \simeq \left(\frac{\tilde{v}_{orb}}{c}\right)^2$$



Introduce scaled mass m and velocity v, PN Expansion parameter ε, Newtonian dynamical time τ.

$$\tilde{v}_{\mathrm{orb}} \equiv \frac{dx^i}{dt} \equiv \epsilon \frac{dx^i}{d\tau} \equiv \epsilon v^i, \quad \tilde{m} \equiv \epsilon^2 m$$

Newtonian dynamical time τ and near zone coord.

1. **Just a scaled time variable, but a "natural" Newtonian time.**
2. **<u>Nothing to do with proper time of anything.</u>**
3. **Newtonian equation of motion obeys "Newtonian scaling law":**

$$t = \tau/\epsilon, \tilde{\phi} = \epsilon^2 \phi$$

$$\frac{d^2 x^i}{dt^2} = -\nabla\tilde{\phi} \Rightarrow \frac{d^2 x^i}{d\tau^2} = -\nabla\phi$$

4. **Near zone coordinate (τ, x^i) and Newtonian metric η_{ij}**

$$\eta_{\mu\nu} \equiv \mathrm{diag}(-\epsilon^{-2}, \delta_{ij})$$

Key ideas 1. PNA (cont'd)

1. Expanding metric (and stress energy tensor) in ε formally.

Use lower order EOM if necessary

$$g(x;\epsilon) = \eta + \epsilon\partial_\epsilon g(x;0) + \epsilon^2\partial_\epsilon^2 g(x;0)/2! \cdots,$$

Iteration

2. Expanding Einstein Equations in ε.

Solve those for g_n's up to required order in ε as functional of m, v, …

$$G[g(x,\epsilon)] = T(x;\epsilon^2 m, \epsilon v, \cdots)$$

3. Stress energy conservation law gives EOM.

$$\nabla_{[g(x;\epsilon)]} T(x;\epsilon^2 m, \epsilon v, \cdots) = 0$$

$$m_1\frac{dv_1^i}{d\tau} = -\frac{m_1 m_2}{r_{12}^3}r_{12}^i + \epsilon^2 F_{1\mathrm{PN}} + \epsilon^4 F_{2\mathrm{PN}} + \epsilon^5 F_{2.5\mathrm{PN}} + \epsilon^6 F_{3\mathrm{PN}} + \epsilon^7 F_{3.5\mathrm{PN}} + \cdots$$

3PN: Damour, Jaranowski & Schäfer 2001, Itoh & Futamase 2003, Blanchet, Damour & Esposito-Farése 2004.

3.5PN: Blanchet 1997, Jaranowski & Schäfer 1997, Pati & Will 2002, Königsdöerffer, Faye, Schäfer 2003, Nissanke & Blanchet 2005, Itoh 2009

Key ideas 2. Point particle limit

Why point particle limit?

1. **To make equations of motion more tractable (reduce number of degrees of freedom)**

2. **Gravitational wave data analysis may not need higher order multipoles other than spin (and quadrupole). Smaller the number of parameters (mass, spins, …) to be searched for is, easier the data analysis and lesser the computational burden become.**

Rough argument: (m, vs, R: mass, spinning velocity, and radius of a star, L: orbital separation)

1. Tidal force: m R /L^3 ~ m^2/L^3 for a compact star.

   Tidally induced quadrupole: Q ~ (tidal gravity)/(self gravity) times m R^2 ~ m^3 (R/L)^3.

   Quadrupole orbit coupling force: F ~ mQ/L^4 ~ (m/L)^7 = 5 PN

   (cf. (m/L)^2 for Newtonian Force).

2. Spin induced quadrupole: Q ~ (mRv)^2/m ~ m^3 v^2

. Quadrupole orbit coupling force: F ~ (m/L)^4 vs^2 = 2PN times (rotational velocity)^2.

See e.g. for Bildsten & Cutler (1992), Blanchet’s 2007 Liv. Rev. review.

Key ideas 2. Point particle limit (cont'd)

Strong field point particle limit:

- **"Regular" point particle limit.**
- **Can make a star have strong internal self-gravity (while keeping inter-body gravity weak).**
- **Nicely fit into post-Newtonian approximation.**

1. We would like to make a star have strong internal gravity $$\frac{\tilde{m}}{\tilde{R}} = O(1)$$

2. while keeping inter-star gravity weak and PNA valid. $$\frac{\tilde{m}}{\tilde{L}} = O(\epsilon^2)$$

3. As a consequence, we have a point particle in the ε-zero limit. $$\frac{\tilde{R}}{\tilde{L}} = O(\epsilon^2)$$

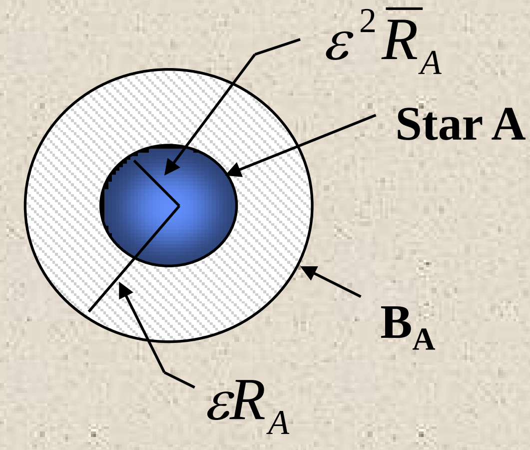
Scaling law for radius of star: (Strong field point particle limit)

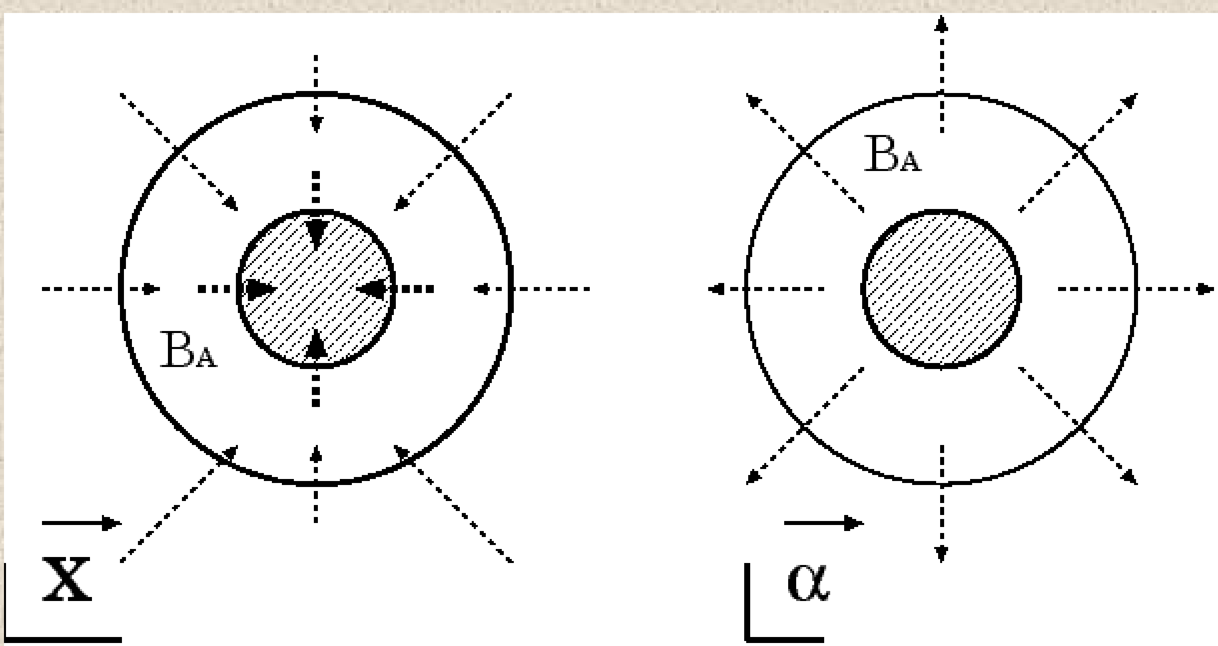
$$\tilde{R}_A \equiv \epsilon^2 \bar{R}_A$$

Body zone coordinate.

Body zone B_A: $B_A \equiv \{x^i \,|\, |x - z_A(\tau)| < \epsilon^2 R_A\}$

Body zone coordinate: $\alpha^i_A \equiv \epsilon^{-2}(x^i - z^i_A(\tau))$





- Star shrinks in (τ,x) coord.
- Star does not shrink in (τ,α) coord.
- Natural coordinate to describe the star.

Scaling law of stress energy tensor

Inside a star

$$\tilde{\rho} = \tilde{m}/\tilde{R}^3 = O(\epsilon^{-4})$$

$$\tilde{v}_{\mathsf{spin}} = O(\epsilon)$$

$$\partial_i \tilde{\phi} = O(\epsilon^{-2})$$

Source term of
Relaxed Einstein Equations.

$$\Theta^{\mu\nu} = (-g)(T^{\mu\nu} + t^{\mu\nu}_{\mathsf{LL}})$$

Scaling law of the stress energy tensor of matter + gravity.

$$\Theta^{\tau\tau} = O(\epsilon^{-2})$$

$$\Theta^{\tau\underline{i}} = O(\epsilon^{-4})$$

$$\Theta^{\underline{ij}} = O(\epsilon^{-8})$$

Natural time scale inside a star ~ $\tilde{\rho}^{-1/2} = O(\varepsilon^2)$. For such a dynamical star, $\tilde{v}_{spin} = O(1)$.

Other approaches: Dirac delta

- One can use Dirac delta to achieve point particle limit.
- Have to deal with divergent integrals.

$$\int T^{\mu\nu} d^3x = \int \frac{m v^\mu v^\nu}{\sqrt{-g g_{\rho\sigma} v^\rho v^\sigma}} \delta_D(\vec{x} - \vec{z}(t)) d^3x$$

- And Hadamard Partie Finie regularization gives non-unique answers in e.g. near zone quadrupole moment at 3 PN order.

$$\Delta I_{ij} = \frac{44}{3} \frac{G^2 m_1^3}{c^6} \left[\left(\xi + \kappa \frac{m}{m_1} \right) a_1^{\langle i} y_1^{j\rangle} + \zeta v_1^{\langle i} v_1^{j\rangle} \right] + 1 \leftrightarrow 2.$$

Eq, (10.2.5) of Blanchet, Iyer & Joguet (2002).

Key ideas 3. Surface integral approach

- (Newtonian) Force by Volume integral

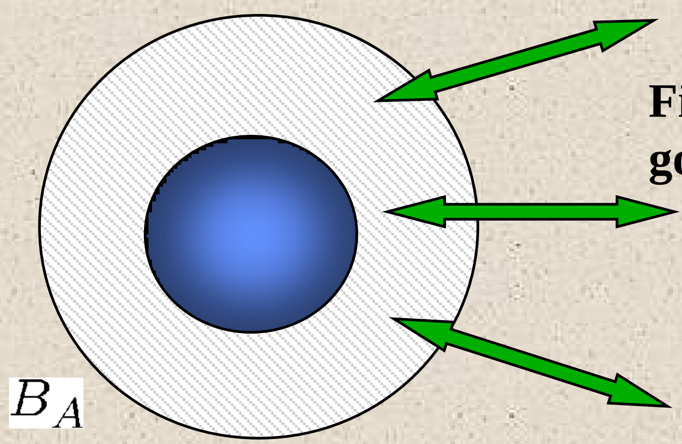
$$F_1^i = -\int_{\mathrm{B}_1} d^3x \rho \frac{\partial \phi}{\partial x^i}$$

Need ρ and φ inside the star.

- By surface integral (using Poisson eq.)

$$\begin{aligned} F_1^i &= -\oint_{\partial B_1} dS_j t^{ij}, \\ t^{ij} &\equiv \frac{1}{4\pi}\left(\frac{\partial \phi}{\partial x^i}\frac{\partial \phi}{\partial x^j} - \frac{\delta^{ij}}{2}\frac{\partial \phi}{\partial x^k}\frac{\partial \phi}{\partial x^k}\right). \end{aligned}$$

Need φ close but outside the star.



Field Momentum flux going through BA

BA and Star A shrink

Gravitational Force on the star A

Other ways to EOM

(1) Volume integral Approach: (Pati & Will)

Assume the properties of the density.

$$F_1^i = \int_{B_1} d^3x \rho(x) \frac{\partial \phi}{\partial x^i}.$$

Explicit demonstration of irrelevance of the internal structure.

(2) Regularized geodesics or, regularized action (Blanchet & Faye)

$$[u^\nu u^\mu{}_{;\nu}]^{reg} = 0$$

Physically interesting implications.

(cf.) Surface Integral Approach: (Einstein, Infeld & Hoffmann, YI, Futamase & Asada)

$$\begin{aligned} F_1^i &= -\oint_{\partial B_1} dS_j t^{ij}, \\ t^{ij} &\equiv \frac{1}{4\pi}\left(\frac{\partial \phi}{\partial x^i}\frac{\partial \phi}{\partial x^j} - \frac{\delta^{ij}}{2}\frac{\partial \phi}{\partial x^k}\frac{\partial \phi}{\partial x^k}\right). \end{aligned}$$

Avoid the internal problem up to the order where φ depends on it.

Newtonian computations.

mass $$m_A \equiv \int_{B_A} d^3\rho$$

dipole $$D^i_A \equiv \int_{B_A} d^3 x\rho(x^i - z^i_A)$$

momentum $$P^i_A \equiv \int_{B_A} d^3 x\rho v^i$$

Velocity Momentum relation

$$\frac{dD^i}{d\tau} = P^i_A - m_A v^i_A = 0$$

From Field eq. $$\phi = \sum_{A=1,2}\left[\frac{m_A}{|\vec{x}-\vec{z}_A|} + \frac{1}{2} I^{ij}_A \frac{\partial^2}{\partial x^i x^j}\left(\frac{1}{|\vec{x}-\vec{z}_A|}\right) + \cdots\right].$$

$$\begin{aligned} F^i_1 &= -\oint_{\partial B_1} dS_j t^{ij}, \\ t^{ij} &\equiv \frac{1}{4\pi}\left(\frac{\partial\phi}{\partial x^i}\frac{\partial\phi}{\partial x^j} - \frac{\delta^{ij}}{2}\frac{\partial\phi}{\partial x^k}\frac{\partial\phi}{\partial x^k}\right). \end{aligned}$$

$$m_1 \frac{d^2 z^i_1}{d\tau^2} = \sum_{p=0}\sum_{q=0} \frac{(-1)^{p+1}(2p+2q+1)!!}{p!q!} \frac{I_1^{<M_p>} I_2^{<N_q>}}{r_{12}^{p+q+2}} N^{<iM_pN_q>}$$

Newtonian computations cont'd.

Newtonian potential.

$$\phi(\vec{x}) = \sum_{A=1,2} \frac{m_A}{|\vec{x}-\vec{z}_A|}$$

Force on star 1 by surface integral over the body zone

$$F_1^i = -\left(\delta^i_k \delta^j_l - \frac{1}{2}\delta^{ij}\delta_{kl}\right) \oint_{\partial B_1} \frac{dS_j}{4\pi} \partial^k \phi \partial^l \phi$$

$\vec{x} = \vec{z}_1 + \epsilon \vec{n}_1.$ **On the body zone 1**

$$\begin{aligned}
\lim_{\epsilon\to 0} \oint_{\partial B_1} \frac{d\Omega}{4\pi} \epsilon^2 n_{1j} \partial^k \phi \partial^l \phi &= \lim_{\epsilon\to 0} \oint_{\partial B_1} \frac{d\Omega}{4\pi} \epsilon^2 n_{1j} \left(\frac{m_1^2}{\epsilon^4} n_1^k n_1^l \right. \\
&+ \frac{2m_1 m_2}{\epsilon^2 |\vec{z}_1 + \epsilon\vec{n} - \vec{z}_2|} n_1^{(k} (z_1^{l)} + \epsilon n_1^{l)} - z_2^{l)}) \\
&\left. + \frac{m_2^2}{|\vec{z}_1 + \epsilon\vec{n} - \vec{z}_2|^4} (z_1^k + \epsilon n_1^k - z_2^k)(z_1^l + \epsilon n_1^l - z_2^l) \right) \\
&= \frac{2m_1 m_2}{3|\vec{z}_1 - \vec{z}_2|} \delta_j^{(k} (z_1^{l)} - z_2^{l)})
\end{aligned}$$

Zero by surface int..

$$\int_{\partial B_1} d\Omega n_1^i n_1^j = \frac{4\pi}{3}\delta^{ij}.$$

Zero when ε→ 0.

$$\begin{aligned}
F_1^i &= -\left(\delta^i_k \delta^j_l - \frac{1}{2}\delta^{ij}\delta_{kl}\right) \oint_{\partial B_1} \frac{dS_j}{4\pi} \partial^k \phi \partial^l \phi \\
&= -\frac{m_1 m_2}{r_{12}^3} r_{12}^i
\end{aligned}$$

Field equations.

1. Gauge choice
2. Relxed Einstein Equations (REE)
3. How to solve REE.
   - Boundary conditions
   - How to deal with PNA break down
   - Field around stars: operational multipole moments.
   - Super(-duper-tuper-…)- potentials.
   - PNA iteration

Field Equation

- **Deviation field h.** $$h^{\mu\nu} \equiv \eta^{\mu\nu} - \sqrt{-g}g^{\mu\nu}$$
- **Harmonic gauge** $$h^{\mu\nu}{}_{,\nu} = 0,$$
- **Relaxed Einstein Equations (REE) Anderson & Decanio (1975).** $$\Box h^{\mu\nu} = -16\pi\Lambda^{\mu\nu}$$

flat wave operator

REE source terms

$$\Lambda^{\mu\nu} \equiv \Theta^{\mu\nu} + \chi^{\mu\nu\alpha\beta}{}_{,\alpha\beta},$$
$$\Theta^{\mu\nu} \equiv (-g)(T^{\mu\nu} + t^{\mu\nu}_{LL}),$$
$$\chi^{\mu\nu\alpha\beta} \equiv \frac{1}{16\pi}(h^{\alpha\nu}h^{\beta\mu} - h^{\alpha\beta}h^{\mu\nu}).$$

Stress energy tensor of matter plus gravity

Wave operator residual.

Conservation laws $$\Lambda^{\mu\nu}{}_{,\nu} = 0, \Theta^{\mu\nu}{}_{,\nu} = 0, \chi^{\mu\nu\alpha\beta}{}_{,\alpha\beta\nu} = 0.$$

- **Formal solution to REE.**

$$h^{\mu\nu}(\tau, x^i) = 4\int_{C(\tau,x^k)} d^3y \frac{\Lambda^{\mu\nu}(\tau - \epsilon|\vec{x} - \vec{y}|, y^k; \epsilon)}{|\vec{x} - \vec{y}|} + h^{\mu\nu}_H(\tau, x^i),$$

flat light cone **Homogeneous term**

Boundary condition:

•**Homogeneous solution:**

$$h_H^{\mu\nu}(\tau, x^i) = \oint_{\partial C(\tau, x^i)} \frac{d\Omega_y}{4\pi} \left[\frac{\partial}{\partial \rho}(\rho h^{\mu\nu}(\tau', y^i)) + \frac{\partial}{\partial \tau'}(\rho h^{\mu\nu}(\tau', y^i)) \right] \Bigg|_{\tau'=0, \rho=|\vec{x}-\vec{y}|=\tau}.$$

•**No incoming radiation condition at Minkowskian past null infinity.**

$$\lim_{\substack{\tau=r,\\ r\to\infty}} \left[\frac{\partial}{\partial r}(r h^{\mu\nu}) + \frac{\partial}{\partial \tau}(r h^{\mu\nu}) \right] = 0.$$ **or** $$h_H^{\mu\nu} = 0.$$

Other possibilities:

• Use “radiative coordinates” to incorporate system monopole effect on null characteristic (MPM of Blanchet, Damour, Iyer et al.).

-- No difference up to 3.5 PN order inclusively.

• Use initial value formalism rather than going to fictitious past null (BigBang).

-- Assume binary is immersed in (environmental/cosmological) stochastic GWs h^ij (not h^tt, h^ti). (Statistical initial condition by Schutz 1980.)

-- Not deeply investigated.

PNA break-down, Far zone field, & WWP-DIRE

Divergent integrals in formal slow motion expansion series

$$h(\tau, x^i) \sim \int d^3y \frac{f(\tau - \epsilon|\vec{x} - \vec{y}|, \vec{y})}{|\vec{x} - \vec{y}|} \sim \sum_n^{\infty} \frac{(-1)^n}{n!} \epsilon^n \int d^3y |\vec{x} - \vec{y}|^{n-1} \frac{d^n}{dt^n} f(t, \vec{y}).$$

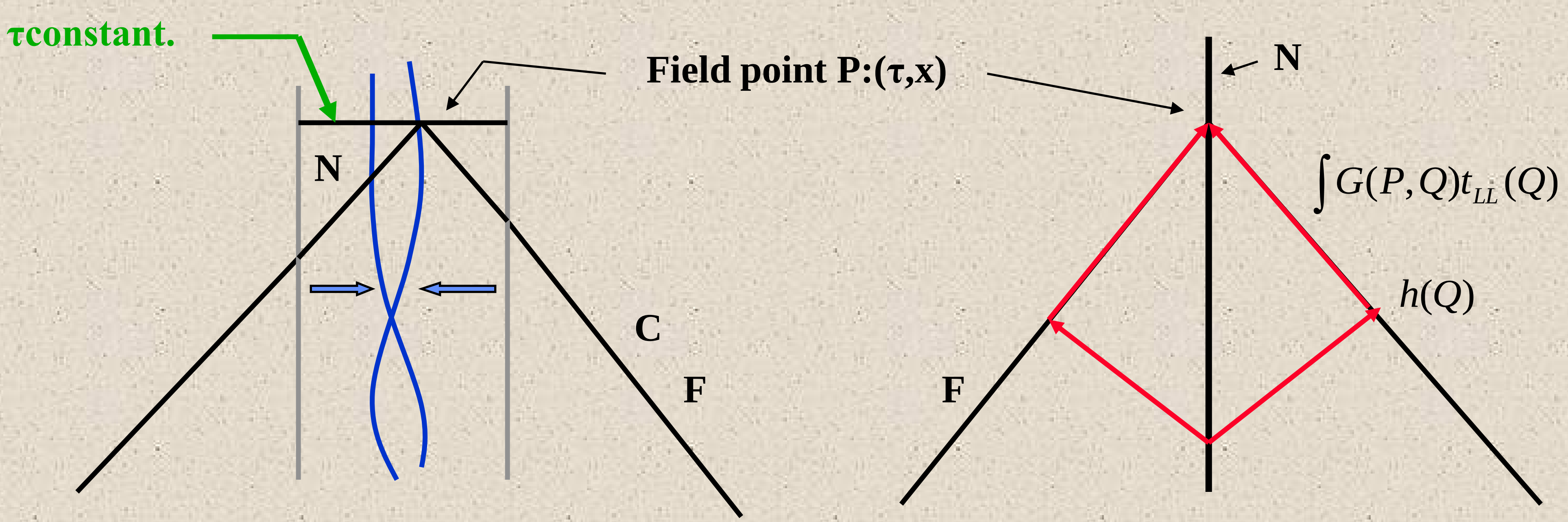
- Multipolar-Post-Minkowskian formalism (MPM)
  - Blanchet, Damour, Iyer et al. (e.g. Blanchet 2007 review)
    - PMA in radiative coordinates for far zone
    - PNA for near zone
    - Matching between two.
- Direct Integration of Relaxed Einstein Equations (DIRE)
  - Will & Wiseman (1996)
    - same coordinates in far and near zone (harmonic).

Will-Wiseman-Pati's DIRE

$$h^{\alpha\beta}_{N(C)}(P) = h^{\alpha\beta}_{N(N)}(P) + h^{\alpha\beta}_{N(F)}(P);$$

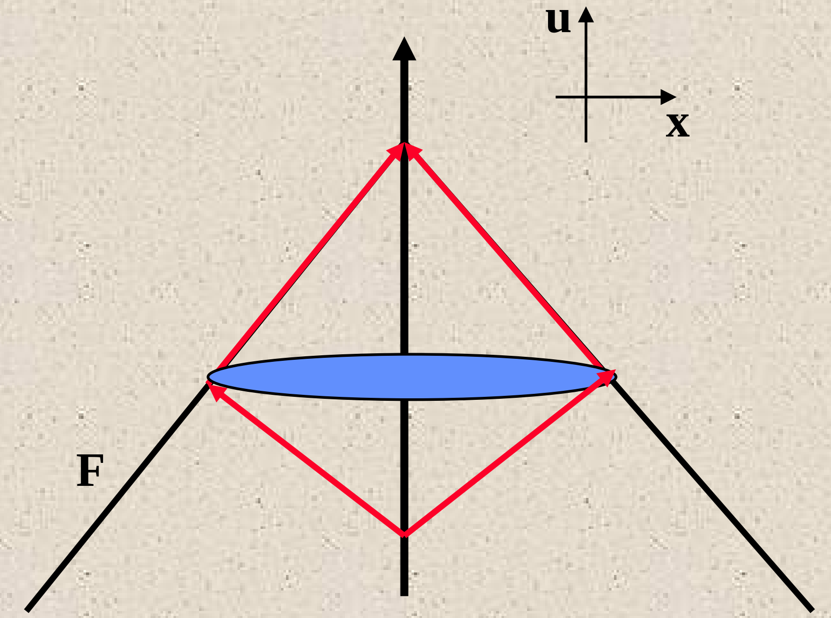
$$h^{\alpha\beta}_{N(N)}(P) = 4\int_{N=\{y:|y|\le R/\epsilon\}} d^3y\,\frac{\tau^{\alpha\beta}(\tau-\epsilon|x-y|, y^k; \epsilon)}{|x-y|};$$

$$h^{\alpha\beta}_{N(F)}(P) = 4\int_{F=\{y:|y|>R/\epsilon\}} d^3y\,\frac{\tau^{\alpha\beta}(\tau-\epsilon|x-y|, y^k; \epsilon)}{|x-y|};$$



Will-Wiseman-Pati's DIRE cont'd

$$\begin{aligned} h^{\mu\nu}_{F(F)}(t,\vec{x}) &= 4\int_F d^3x' \frac{\Lambda^{\mu\nu}(t-|\vec{x}-\vec{x}'|,\vec{x}')}{|\vec{x}-\vec{x}'|} \\ &= 4\int_{-\infty}^{u} du' \oint_F \frac{\Lambda^{\mu\nu}(u'+r',\vec{x}')}{t-u'-\vec{n}'\cdot\vec{x}}[r'(u',\Omega')]^2 d\Omega, \end{aligned}$$



STF expansion

$$\Lambda^{\mu\nu} \sim f_{B,L} r^{-B} n^{<L>}$$

General formula for far zone contribution to near zone field

$$\begin{aligned} h^{\mu\nu}_{N(F)}(t,x^i) &= \sum_{\substack{B\neq 2\\ l}} \left(\frac{2}{r}\right)^{B-2} n^{<L>} \sum_{q=0} \mathcal{E}_{B,L}(z)^q r^q \frac{d^q f_{B,L}(t)}{dt^q} \\ &+ \frac{n^{<L>}}{r}\int_0^\infty f_{2,L}(u-s)\mathrm{Q}_l\left(1+\frac{s}{r}\right) + n^{<L>}\sum_{q=0}\mathcal{E}^q_{2,L}(z) r^q \frac{d^q f_{2,L}(t)}{dt^q}, \end{aligned}$$

PNA

appear at 4 PN EOM as PN tail.

Near zone field and slow motion expansion

- **Slow motion expansion**

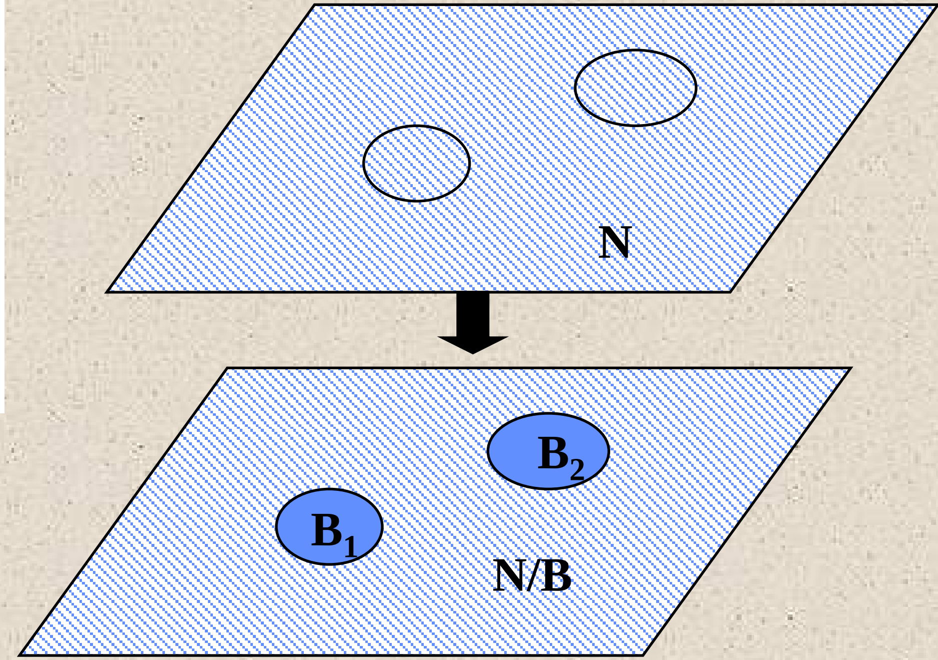
$$h^{\mu\nu} = 4\sum_{n=0} \frac{(-\epsilon)^n}{n!}\left(\frac{\partial}{\partial\tau}\right)^n \int_N d^3y|\vec{x}-\vec{y}|^{n-1}\Lambda_N^{\mu\nu}(\tau, y^k;\epsilon).$$

- **Then split it into Body zone contribution + N/B contribution**

$$h = h_B + h_{N/B},$$

$$h_B = \epsilon^6 \sum_{A=1,2}\int_{B_A} d^3\alpha_A \frac{f(\tau, \vec{z}_A + \epsilon^2\vec{\alpha}_A)}{|\vec{r}_A - \epsilon^2\vec{\alpha}_A|^{1-n}},$$

$$h_{N/B} = \int_{N/B} d^3y \frac{f(\tau,\vec{y})}{|\vec{x}-\vec{y}|^{1-n}},$$



Body zone field to field and multipole expansion

Body zone contribution : Multipole expansion

$$h^{\tau\tau}_{Bn=0} = 4\epsilon^4 \sum_{A=1,2} \left(\frac{P^\tau_A}{r_A} + \epsilon^2 \frac{D^k_A r^k_A}{r^3_A} + \epsilon^4 \frac{3I^{<kl>}_A r^k_A r^l_A}{2r^5_A} + \epsilon^6 \frac{5I^{<klm>}_A r^k_A r^l_A r^m_A}{2r^7_A} \right)$$

$$h^{\tau i}_{Bn=0} = 4\epsilon^4 \sum_{A=1,2} \left(\frac{P^i_A}{r_A} + \epsilon^2 \frac{J^{ki}_A r^k_A}{r^3_A} + \epsilon^4 \frac{3J^{kli}_A r^k_A r^l_A}{2r^5_A} \right)$$

$$h^{ij}_{Bn=0} = 4\epsilon^2 \sum_{A=1,2} \left(\frac{Z^{ij}_A}{r_A} + \epsilon^2 \frac{Z^{kij}_A r^k_A}{r^3_A} + \epsilon^4 \frac{3Z^{<kl>ij}_A r^k_A r^l_A}{2r^5_A} + \epsilon^6 \frac{5Z^{<klm>ij}_A r^k_A r^l_A r^m_A}{2r^7_A} \right)$$

Operational multipoles

$$I^{K_l}_A \equiv \epsilon^2 \int_{B_A} d^3\alpha_A \Lambda^{\tau\tau}_N \alpha^{K_l}_A ,$$

$$J^{K_l i}_A \equiv \epsilon^4 \int_{B_A} d^3\alpha_A \Lambda^{\tau i}_N \alpha^{K_l}_A ,$$

$$Z^{K_l ij}_A \equiv \epsilon^8 \int_{B_A} d^3\alpha_A \Lambda^{ij}_N \alpha^{K_l}_A ,$$

Integrands include gravitational stress energy tensor →

Self-gravitating star.

$$P^\tau_A \equiv I^{K_0}_A$$

$$D^{k_1}_A \equiv I^{K_1}_A$$

$$P^{k_1}_A \equiv J^{K_1}_A$$

Reduction of multipoles via conservation law.

Reduction of operational multipoles.

$$P^i_A = P^\tau_A v^i_A + Q^i_A + \epsilon^2 \frac{dD^i_A}{d\tau},$$

$$J^{ij}_A = \frac{1}{2}\left(M^{ij}_A + \epsilon^2 \frac{dI^{ij}_A}{d\tau}\right) + v^{(i}_A D^{j)}_A + \frac{1}{2}\epsilon^{-2} Q^{ij}_A,$$

$$\begin{aligned} Z^{ij}_A &= \epsilon^2 P^\tau_A v^i_A v^j_A + \frac{1}{2}\epsilon^6 \frac{d^2 I^{ij}_A}{d\tau^2} + 2\epsilon^4 v^{(i}_A \frac{dD^{j)}_A}{d\tau} + \epsilon^4 \frac{dv^{(i}_A}{d\tau} D^{j)}_A \\ &+ \epsilon^2 Q^{(i}_A v^{j)}_A + \epsilon^2 R^{(ij)}_A + \frac{1}{2}\epsilon^2 \frac{dQ^{ij}_A}{d\tau}, \end{aligned}$$

$$Z^{kij}_A = \frac{3}{2} A^{kij}_A - A^{(ij)k}_A,$$

Velocity-Momentum relation

$$M^{ij}_A \equiv 2\epsilon^4 \int_{B_A} d^3\alpha_A \alpha^{[i}_A \Lambda^{j]\tau}_N,$$

Operationally defined spin

$$Q^{K_l i}_A \equiv \epsilon^{-4} \oint_{\partial B_A} dS_k \left(\Lambda^{\tau k}_N - v^k_A \Lambda^{\tau\tau}_N\right) y^{K_l}_A y^i_A,$$

$$R^{K_l ij}_A \equiv \epsilon^{-4} \oint_{\partial B_A} dS_k \left(\Lambda^{kj}_N - v^k_A \Lambda^{\tau j}_N\right) y^{K_l}_A y^i_A,$$

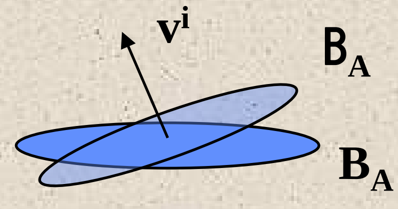
Residual due to non-compactness of integrands.

Operational multipole and "true" multipole (n=2)

Volume integral of the total stress energy tensor $\Lambda^{\nu\mu}$ in B_A on τ constant 3-space

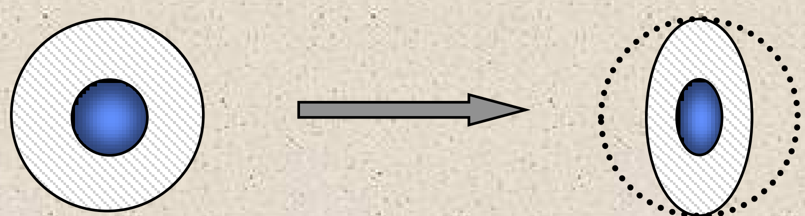
$$I_A^L \equiv \epsilon^2 \int_{B_A} d^3\alpha_A \Lambda_N^{\tau\tau} \alpha_A^{\underline{L}},$$

$$\Rightarrow \bar{I}_A^L \equiv \epsilon^2 \int_{\mathcal{B}_A} d^3\alpha_A \Lambda_N^{\tau\tau} \alpha_A^{\underline{L}} = I_A^L + \delta I_A^L$$



Even "Spherical-in-rest-frame compact stars" has non-zero operationally defined quadrupole due to Lorentz contraction..

$$\vec{v}_1 = 0 \quad \downarrow \quad [I_1^{ij}]^{STF} = 0$$



$$\vec{v}_1 \neq 0 \quad \downarrow \quad [I_1^{ij}]^{STF} \neq 0$$

$$[\delta I_A^{<ij>}]^{STF} = -\frac{4}{5} m_1^3 [v_1^i v_1^j]^{STF}$$

3PN contribution

Operational multipole and “true” multipole (n=1)

•**Spin defined in the Fermi-normal coordinate**

$$\mathcal{M}_A^{\mu\nu} = \frac{2\epsilon^{-6}}{\sqrt{-g}} \int_{\mathcal{B}_A} d^3\Sigma_\rho \sigma_A^{[\mu} \Lambda_N^{\nu]\rho},$$
$$\mathcal{S}_{A\mu} = \frac{1}{2}\epsilon_{\alpha\rho\sigma\mu} \mathcal{M}_A^{\rho\sigma} u_A^\alpha,$$

•**Relation between “ope.” spin and “true” spin**

$$\mathcal{M}_1^{\hat{i}\hat{\tau}} = \left(\left(1 + \frac{1}{2}\epsilon^2 v_1^2 - \epsilon^2 \frac{2m_2}{r_{12}}\right)\delta^{ij} - \frac{1}{2}\epsilon^2 v_1^i v_1^j\right) D_1^j - \epsilon^2 M_1^{ik} v_1^k + O(\epsilon^3),$$
$$\mathcal{M}_1^{\hat{i}\hat{j}} = M_1^{ij} + \epsilon^2 v_1^k v_1^{[i} M_1^{j]k} + O(\epsilon^3).$$

•**Spin supplementary condition or definition of center of mass**

$$\mathcal{S}_{A\mu} u_A^\mu = 0 \quad \text{, or equivalently,} \quad \mathcal{D}_A^\mu = -\mathcal{M}_A^{\mu\nu} u_{A\mu} = 0,$$ **--(●)**

•**Eq. (●) means** $D_A^i = \epsilon^2 M_1^{ik} v_1^k.$

Chi part of the operational multipole moments.

$$I_A^L \equiv \epsilon^2 \int_{B_A} d^3\alpha_A \Lambda_N^{\tau\tau} \alpha_A^L, = \epsilon^2 \int_{B_A} d^3\alpha_A (\Theta_N^{\tau\tau} + \chi^{\mu\nu\alpha\beta}_{,\alpha\beta}) \alpha_A^L,$$

- **Chi is just the difference between the curved space time wave operator and the flat-space one.**
- **It is not natural to include chi-part as a integrand of multipole moments.**
- **Up to the 3.5 PN order, we can subtract the chi-part of the multipoles from the operational multipoles.**

$$D^i_{A\chi} = \epsilon^2 \int_{B_A} d^3\alpha_A\, \alpha^i_A \chi_N^{\tau\tau\alpha\beta}{}_{,\alpha\beta} = \epsilon^4 \frac{175\, m_1^3\, m_2\, r^i_{12}}{18\, r^3_{12}} + \mathcal{O}(\epsilon^5).$$

$$M^{ij}_{A\chi} = \mathcal{O}(\epsilon^3).$$

QR integrals: due to non-compactness.

$$Q_A^{Li} = \epsilon^{-4} \sum_{n=4}^{8} \epsilon^n \oint_{\partial B_A} dS_k \left[{}_n\Lambda_N^{\tau k} - v_A^k {}_n\Lambda_N^{\tau\tau} \right] y_A^{Li} = O(\epsilon^5) \text{ or } \epsilon R_A - \text{dependent},$$

for $^\forall$ l and,

$$R_A^{Lij} = \epsilon^{-4} \sum_{n=4}^{8} \epsilon^n \oint_{\partial B_A} dS_k \left[{}_n\Lambda_N^{jk} - v_A^k {}_n\Lambda_N^{j\tau} \right] y_A^{Li}$$

$$= \begin{cases} -\epsilon^4 \frac{3m_1^3 m_2}{5r_{12}^5} r_{12}^{<ij>} + O(\epsilon^5) \text{ or } \epsilon R_A - \text{dependent}, & \text{for } l = 0, \\ \epsilon^4 \frac{3m_1^3 m_2}{5r_{12}^3} \left(4\delta^{ki} r_{12}^j - \delta^{jk} r_{12}^i - \delta^{ij} r_{12}^k \right) & \\ +O(\epsilon^5) \text{ or } \epsilon R_A - \text{dependent}, & \text{for } l = 1, \\ O(\epsilon^5) \text{ or } \epsilon R_A - \text{dependent} & \text{otherwise.} \end{cases}$$

And $${}_{\leq 6}Q_{1\Theta}^i = -\epsilon^6 \frac{m_1^3 m_2 n_{12}^{<ij>} v_{12}^j}{2r_{12}^3} = -\epsilon^6 \frac{d}{d\tau} \left(\frac{m_1^3 m_2}{6r_{12}^3} r_{12}^i \right) = \epsilon^6 \frac{d}{d\tau} \left(\frac{1}{6} m_1^3 a_1^i \right),$$

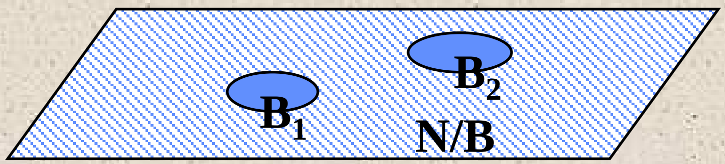
for monopoles.

For a star with n = 1 multipoles.

$$Q_1^i = \epsilon^4 \left(\frac{2m_2 M_1^{ik} n_{12}^k}{3r_{12}^2} + \frac{2m_2 D_1^i (\vec{n}_{12} \cdot \vec{v}_{12})}{r_{12}^2} + \frac{m_2 (\vec{D}_1 \cdot \vec{v}_1) n_{12}^i}{r_{12}^2} - \frac{m_2 (\vec{D}_1 \cdot \vec{r}_{12}) v_1^i}{3r_{12}^2} - \frac{4m_2 (\vec{D}_1 \cdot \vec{v}_2) n_{12}^i}{3r_{12}^2} + \frac{4m_2 (\vec{D}_1 \cdot \vec{r}_{12}) v_2^i}{3r_{12}^2} \right) + O(\epsilon^5),$$

It's too lengthy to show expressions for other QR's.

N/B field and super-potentials



$$\int_{N/B} d^3y \frac{f(\vec{y})}{|\vec{x}-\vec{y}|} = -4\pi g(\vec{x}) + \oint_{\partial(N/B)} dS_k \left[\frac{1}{|\vec{x}-\vec{y}|}\frac{\partial g(\vec{y})}{\partial y^k} - g(\vec{y})\frac{\partial}{\partial y^k}\left(\frac{1}{|\vec{x}-\vec{y}|}\right)\right].$$

$$\Delta g(\vec{x}) = f(\vec{x})$$

g: (Super-)potential of (non-compact) source f.

- **There's no need to worry about homogeneous solutions.**
- **There's no need to use Dirac delta to prove above equation (Appendix B of Itoh 2004).**
- **Analytic closed form expressions of all the necessary super-potentials are available up to 2.5 PN order inclusively and 3.5 PN order.**
- **At 3 PN order, we could not find all. We instead find the potentials in the neighborhood of the body zone, which are what we need to evaluate surface integrals to derive EOM, or change the order of integrations: compute surface integral first and then compute remaining Poisson integral.**

Poisson integral without super-potentials.

Decompose the integrand using symmetric trace-free (STF) tensors

$$f(\vec{y}_1) = \sum_l g_l(\cos\theta, y_1) n_1^{<L>} \qquad \textbf{with} \quad \cos\theta = -\vec{n}_{12}\cdot\vec{n}_1$$

Formulae for changing the order of integrations:

$$\frac{1}{4\pi}\oint_{\partial B_1} d\Omega \operatorname*{disc}_{\epsilon R_A} \int_{N/B} d^3y_1 \frac{f(\vec{y}_1)}{|\vec{r}_1 - \vec{y}_1|} = \frac{1}{2}\sum_l n_{12}^{<L>} \lim_{r'_1\to\epsilon R_1} \operatorname*{disc}_{\epsilon R_A}\Bigg[\int_{N/B'} dy_1 y_1 \int dt \mathrm{P}_l(-t) g_l(t, y_1) + \int_{\epsilon R_1}^{r'_1} \frac{dy_1 y_1^2}{r'_1} \oint_{-1}^{1} dt \mathrm{P}_l(-t) g_l(t, y_1)\Bigg],$$

$$\oint_{\partial B_1} dS_k \frac{r_2^l}{r_2^3}\int_{N/B} d^3y \frac{f(\vec{y}_1)}{|\vec{r}_1 - \vec{y}_1|}$$

$$= \lim_{r'_1\to\epsilon R_A} \operatorname*{disc}_{\epsilon R_A} 4\pi \sum_{m=0} (-1)^m \frac{(2m-1)!!}{(2m+3)m!}$$

$$\times\Bigg[\int_{N/B'} d^3y_1 f(\vec{y}_1)\Bigg[\frac{r_1'^{2m+3}}{y_1^{m+2}} N_1^{<kM>} \partial_{z_2^l}\left(\frac{n_{12}^{<M>}}{r_{12}^{m+1}}\right) - \frac{r_1'^{2m+3}}{y_1^{m+1}} N_1^{<M>} \partial_{z_2^l}\left(\frac{n_{12}^{<kM>}}{r_{12}^{m+2}}\right)\Bigg]$$

$$+ \int_{B'_1/B_1} d^3y_1 f(\vec{y}_1)\Bigg[y_1^{m+1} N_1^{<kM>} \partial_{z_2^l}\left(\frac{n_{12}^{<M>}}{r_{12}^{m+1}}\right) - r_1'^2 y_1^m N_1^{<M>} \partial_{z_2^l}\left(\frac{n_{12}^{<kM>}}{r_{12}^{m+2}}\right)\Bigg]\Bigg]$$

Equations of motion

1. Conservation law
2. Surface integral approach
3. Mass-Energy relation
4. Momentum-velocity relation
5. General form of equations of motion

Conservation law and surface integral approach

Separate Theta part and chi part

$$
\begin{aligned}
P_A^{\mu}(\tau) &= \epsilon^2 \int_{B_A} d^3\alpha_A \Lambda_N^{\tau\mu} \\
&= \epsilon^2 \int_{B_A} d^3\alpha_A \Theta_N^{\tau\mu} \\
&+ \epsilon^{-4} \int_{\partial B_A} \frac{dS_k}{16\pi} (h^{\mu k} h^{\tau\alpha} - h^{\tau\tau} h^{k\alpha})_{,\alpha} \\
&\equiv P_{\Theta A}^{\mu}(\tau) + P_{\chi A}^{\mu}(\tau)
\end{aligned}
$$

Conservation law:

$$\Lambda^{\mu\nu}{}_{,\nu} = 0, \Theta^{\mu\nu}{}_{,\nu} = 0, \chi^{\mu\nu\alpha\beta}{}_{,\alpha\beta\nu} = 0.$$

Surface integral form for evolution equation of 4-momentum as a result of energy-momentum conservation:

$$\frac{dP_{\Theta A}^{\mu}}{d\tau} = -\epsilon^{-4} \oint_{\partial B_A} dS_k \Theta_N^{k\mu} + \epsilon^{-4} v_A^k \oint_{\partial B_A} dS_k \Theta_N^{\tau\mu}.$$

Mass Energy relation

$$\frac{dP^{\mu}_{\Theta A}}{d\tau} = -\epsilon^{-4}\oint_{\partial B_A} dS_k \Theta_N^{k\mu} + \epsilon^{-4} v_A^k \oint_{\partial B_A} dS_k \Theta_N^{\tau\mu}.$$

$$h^{\tau\tau} = \sum_A \frac{4P^{\tau}_A}{r_A} + \cdots.$$

$$\begin{aligned}\frac{dP^{\tau}_{\Theta 1}}{d\tau} &= -\epsilon^{-4}\oint_{\partial B_1} dS_k \Theta_N^{k\tau} + \epsilon^{-4} v_1^k \oint_{\partial B_1} dS_k \Theta_N^{\tau\tau} \\ &= -\epsilon^2 \frac{m_1 m_2}{r_{12}^2}\left[4(\vec{n}_{12}\cdot\vec{v}_1) - 3(\vec{n}_{12}\cdot\vec{v}_2)\right] + \cdots\end{aligned}$$

Integrate this equation functionally as

$$P^{\tau}_{\Theta A=1} = m_1\left[1 + \epsilon^2\left(\frac{1}{2}v_1^2 + \frac{3m_2}{r_{12}}\right)\right] + \cdots$$

Mass is defined as a integration constant, and independent of epsilon and time.

$$m_A \equiv \lim_{\epsilon\to 0} P^{\tau}_{\Theta A} \longleftarrow P^{\mu}_{\Theta A} = \int d^3\alpha_A \epsilon^2 \Theta^{\tau\mu}.$$

NB: 1) when epsilon is zero, there's no motion, no companion star. So This mass is defined on the rest frame of the star.

2) if body zone were extended to spatial infinity, this mass would become ADM mass of the star A (since epsilon → zero, there's no companion star).

Momentum Velocity relation

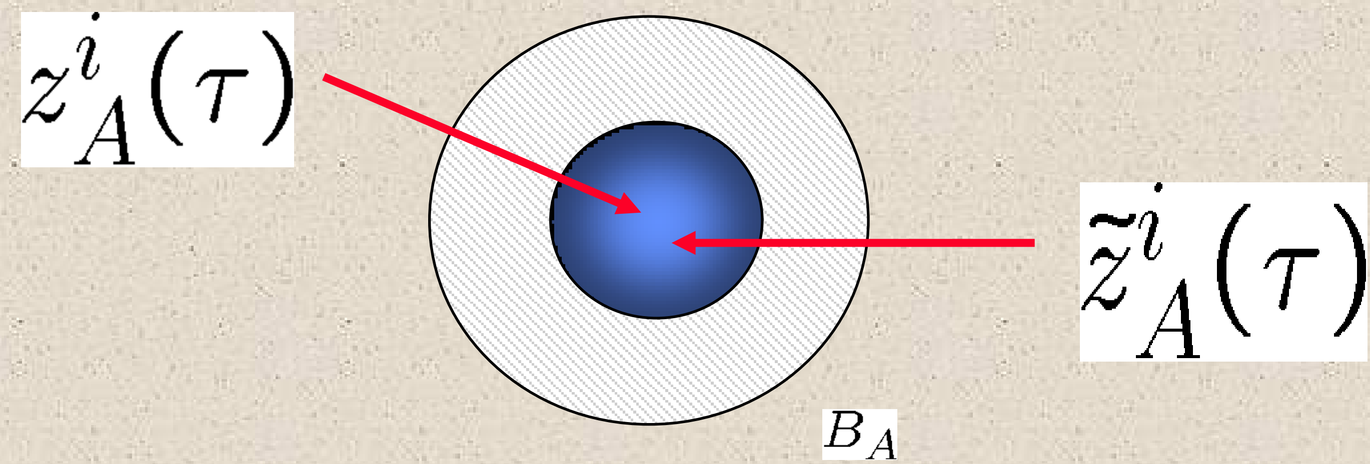
$$P^i_{\Theta A} = P^\tau_{\Theta A} v^i_A + Q^i_{\Theta A} + \epsilon^2 \frac{dD^i_{\Theta A}}{d\tau},$$

$$\frac{dP^\mu_{\Theta A}}{d\tau} = -\epsilon^{-4} \oint_{\partial B_A} dS_k \Theta^{k\mu}_N + \epsilon^{-4} v^k_A \oint_{\partial B_A} dS_k \Theta^{\tau\mu}_N .$$

P, not v.

Need to care for which point in the star is representative.

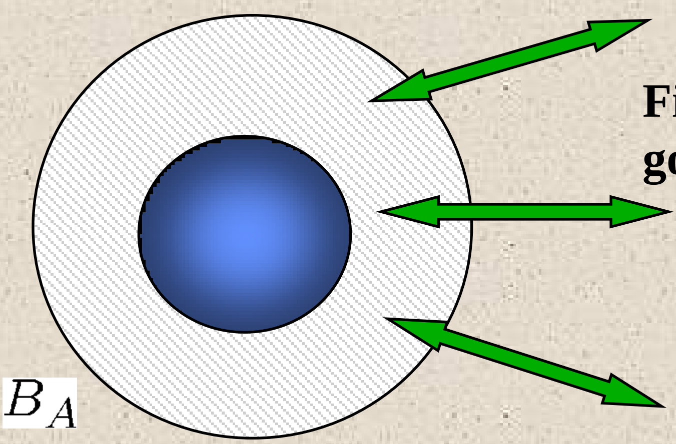
→Specify the dipole moment freely and determine which point inside the star represents the star in the point particle limit.



General form of equations of motion

$$
\begin{aligned}
m_A \frac{dv^i_A}{d\tau} &= -\epsilon^{-4} \oint_{\partial B_A} dS_k \Theta^{ki}_N + \epsilon^{-4} v^k_A \oint_{\partial B_A} dS_k \Theta^{\tau i}_N \\
&\quad + \epsilon^{-4} v^i_A \left(\oint_{\partial B_A} dS_k \Theta^{k\tau}_N - v^k_A \oint_{\partial B_A} dS_k \Theta^{\tau\tau}_N \right) \\
&\quad - \frac{dQ^i_{\ominus A}}{d\tau} - \epsilon^2 \frac{d^2 D^i_{\ominus A}}{d\tau^2} + (m_A - P^{\tau}_{\ominus A}) \frac{dv^i_A}{d\tau}
\end{aligned}
$$

The general form of the equation of motion (Itoh, Futamase & Asada (2000))



Field Momentum flux going through B_A

B_A and Star A shrink

Gravitational Force on the star A

PNA iteration flow

n PN field: ${}_{2n+4}h^{\tau\tau}$, ${}_{2n+2}h^{\mu i}$, ${}_{4}h^{\tau\tau} = 4\sum_{A=1,2} m_A/r_A$,
n PN mass-energy relation: $P^\tau_A = m_A + m_A \sum_{k=1}^{2n} \epsilon^k{}_k\Gamma_A$,
n PN momentum-velocity relation: $P^i_A = P^\tau_A v^i_A + \sum_{k=1}^{2n} \epsilon^k{}_k Q^i_A$,
n PN equations of motion: $m^i_A dv^i_A/d\tau = -m_1 m_2 r^i_{12}/r^3_{12} + \sum_{k=1}^{2n} \epsilon^k{}_k F^i_A$.

$n+1$ PN mass-energy relation:
$(dP^\tau_A/d\tau)_{n+1\ \mathrm{PN}} = \epsilon^{-4} \oint_{\partial B_A} dS_k[-{}_{2n+4}\Lambda^{\tau k}_N + v^k_A {}_{2n+4}\Lambda^{\tau\tau}_N]$,
$P^\tau_A = m_A + m_A \sum_{k=1}^{2n+2} \epsilon^k{}_k\Gamma_A$.

$n+1$ PN Q^i_A integral: $(Q^i_A)_{n+1\ \mathrm{PN}} = \oint_{\partial B_A} dS_k[{}_{2n+6}\Lambda^{\tau k}_N - v^k_A {}_{2n+6}\Lambda^{\tau\tau}_N]$,
Define z^i_A up to $n+1$ PN order by, for instance, setting $D^i_A = 0$.
$n+1$ PN momentum-velocity relation: $P^i_A = P^\tau_A v^i_A + \sum_{k=1}^{2n+2} \epsilon^k{}_k Q^i_A$.

$n+1$ PN $Q^{K_l i}_A$ and $R^{K_l ij}_A$ integrals,
$n+1$ PN N/B contribution:
${}_{2n+6}h^{\tau\tau}_{N/B} = \sum_{k=0}^{2n}(-\epsilon)^k/k! \int_{N/B} d^3y|x-y|^{k-1}{}_{2n-k+6}{}^{(k)}\Lambda^{\tau\tau}(\tau, y^i)$,
${}_{2n+4}h^{\mu i}_{N/B} = \sum_{k=0}^{2n}(-\epsilon)^k/k! \int_{N/B} d^3y|x-y|^{k-1}{}_{2n-k+4}{}^{(k)}\Lambda^{\mu i}(\tau, y^i)$
($n+1$ PN far zone contribution, if necessary).

$n+1$ PN field: ${}_{2n+6}h^{\tau\tau}, {}_{2n+4}h^{\mu i}$.

Evolution equation for P^i_A: $(dP^i_A/d\tau)_{n+1\ \mathrm{PN}} = -\oint_{\partial B_A} dS_k\, {}_{2n+6}\Lambda^{ki} + v^k_A \oint_{\partial B_A} dS_k\, {}_{2n+6}\Lambda^{\tau i}$,
$n+1$ PN equations of motion: $m^i_A dv^i_A/d\tau = -m_1 m_2 r^i_{12}/r^3_{12} + \sum_{k=1}^{2n+2} \epsilon^k{}_k F^i_A$.

3.5 PN evolution equation for energy.

1PN

$$\left(\frac{dP^{\tau}_{1\Theta}}{d\tau}\right) = \boxed{-\epsilon^2\frac{m_1m_2}{r_{12}^2}\left[4(\vec{n}_{12}\cdot\vec{v}_1)-3(\vec{n}_{12}\cdot\vec{v}_2)\right]}$$

2PN

$$+\epsilon^4\frac{m_1m_2}{r_{12}^2}\Bigg[-\frac{9}{2}(\vec{n}_{12}\cdot\vec{v}_2)^3+\frac{1}{2}v_1^2(\vec{n}_{12}\cdot\vec{v}_2)+6(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2$$
$$-2v_1^2(\vec{n}_{12}\cdot\vec{v}_1)+4(\vec{v}_1\cdot\vec{v}_2)(\vec{n}_{12}\cdot\vec{V})+5v_2^2(\vec{n}_{12}\cdot\vec{v}_2)-4v_2^2(\vec{n}_{12}\cdot\vec{v}_1)$$
$$+\frac{m_1}{r_{12}}(-4(\vec{n}_{12}\cdot\vec{v}_2)+6(\vec{n}_{12}\cdot\vec{v}_1))+\frac{m_2}{r_{12}}(-10(\vec{n}_{12}\cdot\vec{v}_1)+11(\vec{n}_{12}\cdot\vec{v}_2))\Bigg]$$

3PN

$$+\epsilon^6\frac{m_1m_2}{r_{12}}\Bigg[-\left(\frac{3}{2}v_1^4+2v_1^2v_2^2+4v_2^4\right)(\vec{n}_{12}\cdot\vec{v}_1)+\left(\frac{5}{8}v_1^4+\frac{3}{2}v_1^2v_2^2+7v_2^4\right)(\vec{n}_{12}\cdot\vec{v}_2)$$
$$+\left(2v_1^2+4v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)-\left(2v_1^2+8v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)$$
$$+\left(3v_1^2+12v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-\left(\frac{3}{4}v_1^2+12v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2)^3$$
$$+2(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)^2-6(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)+6(\vec{n}_{12}\cdot\vec{v}_2)^3(\vec{v}_1\cdot\vec{v}_2)$$
$$-\frac{15}{2}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^4-\frac{45}{8}(\vec{n}_{12}\cdot\vec{v}_2)^5$$
$$+\frac{m_1}{r_{12}}\Bigg(\left(-42v_1^2-\frac{117}{4}v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)+60(\vec{n}_{12}\cdot\vec{v}_1)^3$$
$$+\left(\frac{137}{4}v_1^2+\frac{37}{2}v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2)$$
$$+\frac{297}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}.\vec{v}_2)-\frac{219}{4}(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}.\vec{v}_2)-151(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)$$
$$+109(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-23(\vec{n}_{12}\cdot\vec{v}_2)^3\Bigg)$$
$$+\frac{m_2}{r_{12}}\Bigg(-\left(13v_1^2+18v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_1)+\left(17v_1^2+25v_2^2\right)(\vec{n}_{12}\cdot\vec{v}_2)$$
$$+26(\vec{n}_{12}\cdot\vec{v}_1)(\vec{v}_1\cdot\vec{v}_2)-28(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)+2(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{n}_{12}\cdot\vec{v}_2)$$
$$+16(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)^2-20(\vec{n}_{12}\cdot\vec{v}_2)^3\Bigg)$$
$$+\frac{m_1^2}{r_{12}^2}\left(\frac{33}{4}(\vec{n}_{12}\cdot\vec{v}_1)-\frac{13}{2}(\vec{n}_{12}\cdot\vec{v}_2)\right)-\frac{m_1m_2}{r_{12}^2}\left(\frac{35}{4}(\vec{n}_{12}\cdot\vec{v}_1)+\frac{17}{4}(\vec{n}_{12}\cdot\vec{v}_2)\right)$$
$$+\frac{m_2^2}{r_{12}^2}\left(-12(\vec{n}_{12}\cdot\vec{v}_1)+\frac{23}{2}(\vec{n}_{12}\cdot\vec{v}_2)\right)\Bigg].$$

$$+\frac{8m_1{}^4m_2}{15r_{12}{}^5}+\frac{56m_1{}^3m_2{}^2}{15r_{12}{}^5}+\frac{16m_1{}^2m_2{}^3}{5r_{12}{}^5}-\frac{m_1^2m_2^2}{r_{12}^4}\Bigg[\frac{148}{15}v_1{}^2+\frac{392}{15}(\vec{n}_{12}\cdot\vec{v}_1)^2+\frac{184}{15}(\vec{n}_{12}\cdot\vec{v}_2)^2$$
$$-\frac{52}{15}v_2{}^2-\frac{192}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)+\frac{40}{3}(\vec{v}_1\cdot\vec{v}_2)\Bigg]+\frac{m_1^3m_2}{r_{12}^4}\Bigg[-\frac{16}{5}(\vec{n}_{12}\cdot\vec{v}_1)^2+\frac{8}{5}(\vec{n}_{12}\cdot\vec{v}_2)^2+\frac{4}{5}v_1{}^2$$
$$-\frac{4}{5}v_2{}^2+\frac{8}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)\Bigg]+\frac{m_1^2m_2}{r_{12}^3}\Bigg[-\frac{8}{15}v_1{}^4+\frac{4}{15}v_2{}^4+\frac{8}{5}(\vec{n}_{12}\cdot\vec{v}_1)^2v_1{}^2-\frac{4}{5}(\vec{n}_{12}\cdot\vec{v}_2)^2v_1{}^2$$
$$-\frac{4}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_1{}^2-\frac{8}{15}(\vec{v}_1\cdot\vec{v}_2)^2+\frac{8}{5}(\vec{n}_{12}\cdot\vec{v}_1)^2v_2{}^2-\frac{4}{5}(\vec{n}_{12}\cdot\vec{v}_2)^2v_2{}^2-\frac{4}{15}v_1{}^2v_2{}^2$$
$$-\frac{4}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)v_2{}^2-\frac{4}{15}(\vec{v}_1\cdot\vec{v}_2)v_2{}^2-\frac{16}{5}(\vec{n}_{12}\cdot\vec{v}_1)^2(\vec{v}_1\cdot\vec{v}_2)+\frac{8}{5}(\vec{n}_{12}\cdot\vec{v}_2)^2(\vec{v}_1\cdot\vec{v}_2)$$
$$+\frac{4}{3}v_1{}^2(\vec{v}_1\cdot\vec{v}_2)+\frac{8}{5}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)(\vec{v}_1\cdot\vec{v}_2)\Bigg].$$

3.5PN

We can integrate this!

P¿£ A = ²2 ⁻ BA d3®A(¡ g)(£ ¿¿ + t¿¿ LL)

Tensor density of weitht -2, not -1.

P¿£ A ??= p ¡ g mA u¿A

3.5 PN mass-energy relation.

$$P^{\tau}_{1\Theta} = m_1 \sum_{k=0}^{7} \epsilon^k {}_k\Gamma_1 + O(\epsilon^8).$$

$${}_2\Gamma_1 = \frac{1}{2}v_1^2 + \frac{3m_2}{r_{12}},$$

1PN

$${}_4\Gamma_1 = -\frac{3m_2}{2r_{12}}(\vec{n}_{12}\cdot\vec{v}_2)^2 + \frac{2m_2}{r_{12}}v_2^2 + \frac{7m_2}{2r_{12}}v_1^2 - \frac{4m_2}{r_{12}}(\vec{v}_1\cdot\vec{v}_2) + \frac{3}{8}v_1^4 + \frac{7m_2^2}{2r_{12}^2} - \frac{5m_1m_2}{2r_{12}^2}.$$

2PN

$$\begin{aligned}
{}_6\Gamma_1 = {} & \frac{m_1^2m_2}{2r_{12}^3} + \frac{21m_1m_2^2}{4r_{12}^3} + \frac{5m_2^3}{2r_{12}^3} + \frac{5}{16}v_1^6 \\
& + \frac{m_2^2}{r_{12}^2}\left(\frac{45}{4}v_1^2 + \frac{19}{2}v_2^2 + \frac{1}{2}(\vec{n}_{12}\cdot\vec{v}_1)^2 - 19(\vec{v}_1\cdot\vec{v}_2) - (\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2) - 3(\vec{n}_{12}\cdot\vec{v}_2)^2\right) \\
& + \frac{m_1m_2}{r_{12}^2}\left(\frac{43}{8}v_1^2 + \frac{53}{8}v_2^2 - \frac{69}{8}(\vec{n}_{12}\cdot\vec{v}_1)^2 - \frac{53}{4}(\vec{v}_1\cdot\vec{v}_2) + \frac{85}{4}(\vec{n}_{12}\cdot\vec{v}_1)(\vec{n}_{12}\cdot\vec{v}_2)\right. \\
& \left. - \frac{69}{8}(\vec{n}_{12}\cdot\vec{v}_2)^2\right) + \frac{m_2}{r_{12}}\left(\frac{33}{8}v_1^4 + \frac{3}{2}v_1^2v_2^2 + v_2^4 - 6v_1^2(\vec{v}_1\cdot\vec{v}_2) - 4v_2^2(\vec{v}_1\cdot\vec{v}_2)\right. \\
& - \frac{7}{4}v_1^2(\vec{n}_{12}\cdot\vec{v}_2)^2 - \frac{5}{2}v_2^2(\vec{n}_{12}\cdot\vec{v}_2)^2 + 2(\vec{v}_1\cdot\vec{v}_2)^2
\end{aligned}$$

3PN

$$\begin{aligned}
{}_7\Gamma_1 = {} & -\frac{8{m_1}^2m_2(\vec{n}_{12}\cdot\vec{V})}{15{r_{12}}^3} + \frac{4m_1m_2(\vec{n}_{12}\cdot\vec{V})V^2}{15{r_{12}}^2} \\
& - \frac{16m_1{m_2}^2(\vec{n}_{12}\cdot\vec{V})}{5{r_{12}}^3},
\end{aligned}$$

3.5PN

$$P^{\tau}_{\Theta A} = m_A[\sqrt{-g}u^{\tau}_A]^{HPF}$$

HPF: Hadamard Partie Finie.

- **We need 2.5 PN field to derive 3.5 PN mass-energy relation.**
- **This "natural" relation supports use of HPF up to 2.5 PN order.**

(Blanchet et al needs dimensional reg. at 3 PN order.)

3.5 PN velocity-momentum relation.

3.5 PN velocity-momentum relation for monopole.

$$P^i_{\ast 1} = P^{\tau}_{\ast 1} v^i_1 + Q^i_{\ast 1} + \epsilon^2 \frac{dD_{\ast 1}}{d\tau}$$

$$Q^i_{\ast 1} = -\epsilon^6 \frac{d}{d\tau}\left(\frac{1}{6} m_1^3 a_1^i\right) + \epsilon^4 \frac{2 m_2 M^{ik}_{\ast 1}}{3 r_{12}^3} r^i_{12}$$

We choose the center of the mass of the star 1 to be

$$D_{\ast 1} = \frac{1}{6} m_1^3 a_1^i - \epsilon^6 \frac{22 m_1^3 a_1^i}{3} \ln\left(\frac{r_{12}}{\epsilon R_1}\right) + \epsilon^2 M^{ij}_{\ast 1} v^j_1$$

Use CMF def. freedom to erase gauge term as in Blanchet & Faye (2000).

3.5 PN EOM (monopole terms)

$$m_1\frac{dv_1^i}{d\tau} = -\frac{m_1m_2}{r_{12}^2}n_{12}^i$$

1PN

$$+\epsilon^2\frac{m_1m_2}{r_{12}^2}n_{12}^i\left[-v_1^2-2v_2^2+4(\vec v_1\cdot\vec v_2)+\frac{3}{2}(\vec n_{12}\cdot\vec v_2)^2+\frac{5m_1}{r_{12}}+\frac{4m_2}{r_{12}}\right]$$
$$+\epsilon^2\frac{m_1m_2}{r_{12}^2}V^i\left[4(\vec n_{12}\cdot\vec v_1)-3(\vec n_{12}\cdot\vec v_2)\right]$$

2PN

$$+\epsilon^4\frac{m_1m_2}{r_{12}^2}n_{12}^i\Big[-2v_2^4+4v_2^2(\vec v_1\cdot\vec v_2)-2(\vec v_1\cdot\vec v_2)^2+\frac{3}{2}v_1^2(\vec n_{12}\cdot\vec v_2)^2+\frac{9}{2}v_2^2(\vec n_{12}\cdot\vec v_2)^2$$
$$-6(\vec v_1\cdot\vec v_2)(\vec n_{12}\cdot\vec v_2)^2-\frac{15}{8}(\vec n_{12}\cdot\vec v_2)^4-\frac{57}{4}\frac{m_1^2}{r_{12}^2}-9\frac{m_2^2}{r_{12}^2}-\frac{69}{2}\frac{m_1m_2}{r_{12}^2}$$
$$+\frac{m_1}{r_{12}}\left(-\frac{15}{4}v_1^2+\frac{5}{4}v_2^2-\frac{5}{2}(\vec v_1\cdot\vec v_2)+\frac{39}{2}(\vec n_{12}\cdot\vec v_1)^2-39(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)+\frac{17}{2}(\vec n_{12}\cdot\vec v_2)^2\right)$$
$$+\frac{m_2}{r_{12}}\left(4v_2^2-8(\vec v_1\cdot\vec v_2)+2(\vec n_{12}\cdot\vec v_1)^2-4(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)-6(\vec n_{12}\cdot\vec v_2)^2\right)\Big]$$
$$+\epsilon^4\frac{m_1m_2}{r_{12}^2}V^i\Big[\frac{m_1}{r_{12}}\left(-\frac{63}{4}(\vec n_{12}\cdot\vec v_1)+\frac{55}{4}(\vec n_{12}\cdot\vec v_2)\right)+\frac{m_2}{r_{12}}\left(-2(\vec n_{12}\cdot\vec v_1)-2(\vec n_{12}\cdot\vec v_2)\right)$$
$$+v_1^2(\vec n_{12}\cdot\vec v_2)+4v_2^2(\vec n_{12}\cdot\vec v_1)-5v_2^2(\vec n_{12}\cdot\vec v_2)-4(\vec v_1\cdot\vec v_2)(\vec n_{12}\cdot\vec V)$$
$$-6(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2+\frac{9}{2}(\vec n_{12}\cdot\vec v_2)^3\Big]$$

2.5PN

$$+\epsilon^5\frac{4m_1^2m_2}{5r_{12}^3}\left[n_{12}^i(\vec n_{12}\cdot\vec V)\left(-6\frac{m_1}{r_{12}}+\frac{52}{3}\frac{m_2}{r_{12}}+3V^2\right)+V^i\left(2\frac{m_1}{r_{12}}-8\frac{m_2}{r_{12}}-V^2\right)\right]$$

3PN

$$+\epsilon^6\frac{m_1m_2}{r_{12}^2}n_{12}^i\Big[\frac{35}{16}(\vec n_{12}\cdot\vec v_2)^6-\frac{15}{8}(\vec n_{12}\cdot\vec v_2)^4v_1^2+\frac{15}{2}(\vec n_{12}\cdot\vec v_2)^4(\vec v_1\cdot\vec v_2)$$
$$+3(\vec n_{12}\cdot\vec v_2)^2(\vec v_1\cdot\vec v_2)^2-\frac{15}{2}(\vec n_{12}\cdot\vec v_2)^4v_2^2+\frac{3}{2}(\vec n_{12}\cdot\vec v_2)^2v_1^2v_2^2$$
$$-12(\vec n_{12}\cdot\vec v_2)^2(\vec v_1\cdot\vec v_2)v_2^2-2(\vec v_1\cdot\vec v_2)^2v_2^2+\frac{15}{2}(\vec n_{12}\cdot\vec v_2)^2v_2^4$$
$$+4(\vec v_1\cdot\vec v_2)v_2^4-2v_2^6$$
$$+\frac{m_1}{r_{12}}\Big(-\frac{171}{8}(\vec n_{12}\cdot\vec v_1)^4+\frac{171}{2}(\vec n_{12}\cdot\vec v_1)^3(\vec n_{12}\cdot\vec v_2)$$
$$-\frac{723}{4}(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)^2+\frac{383}{2}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^3$$
$$-\frac{455}{8}(\vec n_{12}\cdot\vec v_2)^4+\frac{229}{4}(\vec n_{12}\cdot\vec v_1)^2v_1^2-\frac{205}{2}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)v_1^2$$
$$+\frac{191}{4}(\vec n_{12}\cdot\vec v_2)^2v_1^2-\frac{91}{8}v_1^4-\frac{229}{2}(\vec n_{12}\cdot\vec v_1)^2(\vec v_1\cdot\vec v_2)$$
$$+244(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)-\frac{225}{2}(\vec n_{12}\cdot\vec v_2)^2(\vec v_1\cdot\vec v_2)$$
$$+\frac{91}{2}v_1^2(\vec v_1\cdot\vec v_2)-\frac{177}{4}(\vec v_1\cdot\vec v_2)^2+\frac{229}{4}(\vec n_{12}\cdot\vec v_1)^2v_2^2$$
$$-\frac{283}{2}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)v_2^2+\frac{259}{4}(\vec n_{12}\cdot\vec v_2)^2v_2^2-\frac{91}{4}v_1^2v_2^2$$
$$+43(\vec v_1\cdot\vec v_2)v_2^2-\frac{81}{8}v_2^4\Big)$$
$$+\frac{m_2}{r_{12}}\Big(-6(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)^2+12(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^3$$
$$+6(\vec n_{12}\cdot\vec v_2)^4+4(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)$$
$$+12(\vec n_{12}\cdot\vec v_2)^2(\vec v_1\cdot\vec v_2)+4(\vec v_1\cdot\vec v_2)^2$$
$$-4(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)v_2^2-12(\vec n_{12}\cdot\vec v_2)^2v_2^2-8(\vec v_1\cdot\vec v_2)v_2^2+4v_2^4\Big)$$
$$+\frac{m_2^2}{r_{12}^2}\Big(-(\vec n_{12}\cdot\vec v_1)^2+2(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)+\frac{43}{2}(\vec n_{12}\cdot\vec v_2)^2$$
$$+18(\vec v_1\cdot\vec v_2)-9v_2^2\Big)$$
$$+\frac{m_1m_2}{r_{12}^2}\Big(\frac{415}{8}(\vec n_{12}\cdot\vec v_1)^2-\frac{375}{4}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)+\frac{1113}{8}(\vec n_{12}\cdot\vec v_2)^2$$
$$-\frac{615\pi^2}{64}(\vec n_{12}\cdot\vec V)^2+18v_1^2+\frac{123\pi^2}{64}V^2+33(\vec v_1\cdot\vec v_2)-\frac{33}{2}v_2^2\Big)$$
$$+\frac{m_1^2}{r_{12}^2}\Big(-\frac{2069}{8}(\vec n_{12}\cdot\vec v_1)^2+543(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)-\frac{939}{4}(\vec n_{12}\cdot\vec v_2)^2$$
$$+\frac{471}{8}v_1^2-\frac{357}{4}(\vec v_1\cdot\vec v_2)+\frac{357}{8}v_2^2\Big)$$
$$+\frac{16m_2^3}{r_{12}^3}+\frac{m_1^2m_2}{r_{12}^3}\left(\frac{547}{3}-\frac{41\pi^2}{16}\right)-\frac{13m_1^3}{12r_{12}^3}+\frac{m_1m_2^2}{r_{12}^3}\left(\frac{545}{3}-\frac{41\pi^2}{16}\right)\Big]$$
$$+\epsilon^6\frac{m_1m_2}{r_{12}^2}V^i\Big[\frac{15}{2}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^4-\frac{45}{8}(\vec n_{12}\cdot\vec v_2)^5-\frac{3}{2}(\vec n_{12}\cdot\vec v_2)^3v_1^2$$
$$+6(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2(\vec v_1\cdot\vec v_2)-6(\vec n_{12}\cdot\vec v_2)^3(\vec v_1\cdot\vec v_2)$$
$$-2(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)^2-12(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2v_2^2+12(\vec n_{12}\cdot\vec v_2)^3v_2^2$$
$$+(\vec n_{12}\cdot\vec v_2)v_1^2v_2^2-4(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)v_2^2+8(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)v_2^2$$
$$+4(\vec n_{12}\cdot\vec v_1)v_2^4-7(\vec n_{12}\cdot\vec v_2)v_2^4$$
$$+\frac{m_2}{r_{12}}\Big(-2(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)+8(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2+2(\vec n_{12}\cdot\vec v_2)^3$$
$$+2(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)+4(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)$$
$$-2(\vec n_{12}\cdot\vec v_1)v_2^2-4(\vec n_{12}\cdot\vec v_2)v_2^2\Big)$$
$$+\frac{m_1}{r_{12}}\Big(-\frac{243}{4}(\vec n_{12}\cdot\vec v_1)^3+\frac{565}{4}(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)$$
$$-\frac{269}{4}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2-\frac{95}{12}(\vec n_{12}\cdot\vec v_2)^3+\frac{207}{8}(\vec n_{12}\cdot\vec v_1)v_1^2$$
$$-\frac{137}{8}(\vec n_{12}\cdot\vec v_2)v_1^2-36(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)+\frac{27}{4}(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)$$
$$+\frac{81}{8}(\vec n_{12}\cdot\vec v_1)v_2^2+\frac{83}{8}(\vec n_{12}\cdot\vec v_2)v_2^2\Big)$$
$$+\frac{m_2^2}{r_{12}^2}\left(4(\vec n_{12}\cdot\vec v_1)+5(\vec n_{12}\cdot\vec v_2)\right)$$
$$+\frac{m_1m_2}{r_{12}^2}\left(-\frac{307}{8}(\vec n_{12}\cdot\vec v_1)+\frac{479}{8}(\vec n_{12}\cdot\vec v_2)+\frac{123\pi^2}{32}(\vec n_{12}\cdot\vec V)\right)$$
$$+\frac{m_1^2}{r_{12}^2}\left(\frac{311}{4}(\vec n_{12}\cdot\vec v_1)-\frac{357}{4}(\vec n_{12}\cdot\vec v_2)\right)\Big]$$

3.5PN

$$+\frac{m_1^4m_2}{r_{12}^5}\left[n_{12}^i\left\{\frac{3992}{105}(\vec n_{12}\cdot\vec v_1)-\frac{4328}{105}(\vec n_{12}\cdot\vec v_2)\right\}-\frac{184}{21}V^i\right]+\frac{m_1^3m_2^2}{r_{12}^5}\Big[\frac{6224}{105}V^i$$
$$+n_{12}^i\left\{\frac{2872}{21}(\vec n_{12}\cdot\vec v_2)-\frac{13576}{105}(\vec n_{12}\cdot\vec v_1)\right\}\Big]+\frac{m_1^3m_2}{r_{12}^4}V^i\Big[-\frac{132}{35}{v_1}^2-\frac{48}{35}{v_2}^2+\frac{52}{15}(\vec n_{12}\cdot\vec v_1)^2$$
$$+\frac{152}{35}(\vec v_1\cdot\vec v_2)-\frac{56}{15}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)-\frac{44}{15}(\vec n_{12}\cdot\vec v_2)^2\Big]+\frac{m_1^3m_2}{r_{12}^4}n_{12}^i\Big[-\frac{4888}{105}(\vec n_{12}\cdot\vec v_1){v_1}^2$$
$$+\frac{5056}{105}(\vec n_{12}\cdot\vec v_2){v_1}^2-\frac{1028}{21}{v_2}^2(\vec n_{12}\cdot\vec v_1)+48(\vec n_{12}\cdot\vec v_1)^3+\frac{5812}{105}{v_2}^2(\vec n_{12}\cdot\vec v_2)+\frac{2056}{21}(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)$$
$$-\frac{2224}{21}(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)-\frac{696}{5}(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)+\frac{744}{5}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2-\frac{288}{5}(\vec n_{12}\cdot\vec v_2)^3\Big]$$
$$+\frac{m_1^2m_2^3}{r_{12}^5}\Big[\frac{6388}{105}V^i-\frac{3172}{21}(\vec n_{12}\cdot\vec V)n_{12}^i\Big]+\frac{m_1^2m_2}{r_{12}^3}V^i\Big[\frac{334}{35}{v_1}^4+\frac{654}{35}{v_2}^2{v_1}^2-\frac{1336}{35}(\vec v_1\cdot\vec v_2){v_1}^2+\frac{292}{35}{v_2}^4$$
$$-\frac{348}{5}(\vec n_{12}\cdot\vec v_1)^2V^2+60(\vec n_{12}\cdot\vec V)^4-\frac{1252}{35}{v_2}^2(\vec v_1\cdot\vec v_2)+\frac{684}{5}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)V^2+\frac{1308}{35}(\vec v_1\cdot\vec v_2)^2$$
$$-66(\vec n_{12}\cdot\vec v_2)^2V^2\Big]+\frac{m_1^2m_2}{r_{12}^3}n_{12}^i\Big[-\frac{246}{35}(\vec n_{12}\cdot\vec V){v_1}^4-\frac{534}{35}{v_2}^2(\vec n_{12}\cdot\vec v_1){v_1}^2+\frac{90}{7}{v_2}^2(\vec n_{12}\cdot\vec v_2){v_1}^2$$
$$+\frac{1068}{35}(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2){v_1}^2-\frac{984}{35}(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2){v_1}^2-\frac{204}{35}{v_2}^4(\vec n_{12}\cdot\vec v_1)+60(\vec n_{12}\cdot\vec v_1)^3V^2$$
$$-56(\vec n_{12}\cdot\vec V)^5+\frac{24}{7}{v_2}^4(\vec n_{12}\cdot\vec v_2)+\frac{984}{35}{v_2}^2(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)-\frac{732}{35}{v_2}^2(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)$$
$$-180(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)V^2-\frac{1068}{35}(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)^2+174(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2V^2$$
$$+\frac{180}{7}(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)^2-54V^2(\vec n_{12}\cdot\vec v_2)^3\Big]+\frac{m_1^2{m_2}^2}{r_{12}^4}V^i\Big[-\frac{152}{21}{v_1}^2-\frac{1768}{105}{v_2}^2+\frac{454}{15}(\vec n_{12}\cdot\vec v_1)^2$$
$$+\frac{2864}{105}(\vec v_1\cdot\vec v_2)-\frac{372}{5}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)+\frac{854}{15}(\vec n_{12}\cdot\vec v_2)^2\Big]+\frac{m_1^2{m_2}^2}{r_{12}^4}n_{12}^i\Big[\frac{1432}{35}(\vec n_{12}\cdot\vec v_1){v_2}^2$$
$$-\frac{5752}{105}(\vec n_{12}\cdot\vec v_2){v_2}^2-\frac{582}{5}(\vec n_{12}\cdot\vec v_1)^3+\frac{3568}{105}(\vec n_{12}\cdot\vec V)v_1^2-\frac{2864}{35}(\vec n_{12}\cdot\vec v_1)(\vec v_1\cdot\vec v_2)$$
$$+\frac{10048}{105}(\vec n_{12}\cdot\vec v_2)(\vec v_1\cdot\vec v_2)+\frac{1746}{5}(\vec n_{12}\cdot\vec v_1)^2(\vec n_{12}\cdot\vec v_2)-\frac{1954}{5}(\vec n_{12}\cdot\vec v_1)(\vec n_{12}\cdot\vec v_2)^2+158(\vec n_{12}\cdot\vec v_2)^3$$

This EOM is Lorentz-invariant (perturbation sense), admits conserved energy (when excluding rad. reac.), and has no undetermined coeff. We also checked 3.5 PN harmonic condition.

Leading order SO, SS, QO coupling forces and spin precessions.

If we de¯ne the center of the mass by $S_{A^1} u^1_A = 0$ or equivalently $D^i_A = {}^2 2 M^{ik}_A v^k_A$

$$F^i_{1\mathrm{SO}} = \epsilon^4 \frac{m_1}{r^3_{12}} \left[6(\vec{s}_2 \times \vec{n}_{12}) \cdot \vec{V} n^i_{12} + 4\vec{s}_2 \times \vec{V} - 6\vec{s}_2 \times \vec{n}_{12}(\vec{n}_{12} \cdot \vec{V})\right]$$
$$+\epsilon^4 \frac{m_2}{r^3_{12}} \left[6(\vec{s}_1 \times \vec{n}_{12}) \cdot \vec{V} n^i_{12} + 3\vec{s}_1 \times \vec{V} - 3\vec{s}_1 \times \vec{n}_{12}(\vec{n}_{12} \cdot \vec{V})\right],$$

$$F^i_{1\mathrm{SS}} = \epsilon^6 \left[-\frac{15 M_1^{jk} M_2^{jl} r^k_{12} r^l_{12} r^i_{12}}{r^7_{12}} + \frac{3 M_1^{jk} M_2^{jk} r^i_{12}}{r^5_{12}} - \frac{3 M_1^{ij} M_2^{jk} r^k_{12}}{r^5_{12}} - \frac{3 M_1^{jk} M_2^{ki} r^j_{12}}{r^5_{12}}\right]$$
$$= \epsilon^6 \frac{1}{r^4_{12}} \left[15(\vec{n}_{12} \cdot \vec{s}_1)(\vec{n}_{12} \cdot \vec{s}_2) n^i_{12} - 3 s^i_1 (\vec{n}_{12} \cdot \vec{s}_2) - 3 s^i_2 (\vec{n}_{12} \cdot \vec{s}_1) - 3 n^i_{12} (\vec{s}_1 \cdot \vec{s}_2)\right].$$

$$\frac{dM^{ij}_A}{d\tau} = -2\epsilon^{-2} v^{[i}_A P^{j]}_A - 2\epsilon^{-2} R^{[ij]}_A. \quad \Longleftrightarrow \quad \frac{d\vec{\mathcal{S}}_1}{d\tau} = \epsilon^2 \frac{m_2}{r^2_{12}} \left[\left(2\vec{v}_2 - \frac{3}{2}\vec{v}_1\right) \times \vec{n}_{12}\right] \times \vec{\mathcal{S}}_1 + \mathcal{O}(\epsilon^3),$$

$$F^i_{1\mathrm{QO}} = \epsilon^4 \frac{3}{2 r^4_{12}} \left(m_1 I_2^{\langle kl \rangle} + m_2 I_1^{\langle kl \rangle}\right) \left(2\delta^{il} n^k_{12} - 5 n^i_{12} n^k_{12} n^l_{12}\right)$$

See Tagoshi, Ohashi & Owen (2001) for 1PN SO force.

3.5 PN monopole EOM in a circular orbit in CMF.

$$\frac{dV^i}{d\tau} = -\Omega^2 r_{12}^i (+A_{RR}^i)$$

$$\begin{aligned} m^2\Omega^2 &= \gamma^3 \left[1 + \gamma(-3+\nu) + \gamma^2 \left(6 + \frac{41}{4}\nu + \nu^2\right)\right. \\ &\left. + \gamma^6 \left(-10 + \left(\frac{-2375}{24} + \frac{41\pi^2}{64}\right)\nu + \frac{19}{2}\nu^2 + \nu^3\right)\right] \end{aligned}$$

where V^i is a relative velocity,

A_{RR} is the 2.5 PN + 3.5 PN radiation reaction acceleration.

$m = m_1 + m_2, \nu = m_1 m_2/m^2, \gamma = m/r_{12}$

There is no undetermined coefficient!

3 PN monopole Conserved Energy in a circular orbit in CMF

$$
\begin{aligned}
E(x) &= -\frac{m\nu x}{2}\left[1+\left(-\frac{3}{4}-\frac{1}{12}\nu\right)x+\left(-\frac{27}{8}+\frac{19}{8}\nu-\frac{1}{24}\nu^2\right)x^2\right. \\
&\quad \left.+\left(-\frac{675}{64}+\left\{\frac{34445}{576}-\frac{205\pi^2}{96}\right\}\nu-\frac{155}{96}\nu^2-\frac{35}{5184}\nu^3\right)x^3\right].
\end{aligned}
$$

$x=(m\Omega)^{3/2}$, Ω is the orbital angular frequency.

c.f. Blanchet and Faye (2000)

$$
\begin{aligned}
E_{BF}(x) &= -\frac{m\nu x}{2}\left[1+\left(-\frac{3}{4}-\frac{1}{12}\nu\right)x+\left(-\frac{27}{8}+\frac{19}{8}\nu-\frac{1}{24}\nu^2\right)x^2\right. \\
&\quad +\left.\left(-\frac{675}{64}+\left\{\frac{209323}{4032}-\frac{205\pi^2}{96}-\frac{110}{9}\lambda\right\}\nu-\frac{155}{96}\nu^2-\frac{35}{5184}\nu^3\right)x^3\right].
\end{aligned}
$$

3.5 PN Result

Our EOM

- is based on energy-momentum conservation laws,
- is based on EIH-like surface integral approach,
- multipoles (mass, spin,…) in it are defined to include gravitational energy (t_{LL}).
- is hence applicable to self-gravitating (regular) star modulo some scalings on density, rotational velocity, and radius,
- is Lorentz invariant in perturbative sense (Blanchet & Faye 2001),
- admits conserved orbital energy when excluding radiation reaction forces,
- is unambiguous (no undetermined coefficients),
- includes Leading order SO, SS, QO and spin precession equations.
- physically (modulo gauge and center of mass definition) agrees with other works.

3.5 PN Result cont'd

Able to derive the unambiguous 3.5 PN EOM
→ Fix one undetermined parameter λ in the 3 PN waveform

$$\lambda = -1987/3080$$

in consistent with the dim. reg. work (Damour, Jaranowski, and Schäfer (2001)).

→ Confirm the DJS work and indirectly support the use of dim. reg. in ADMTT gauge.

From BF/BDE-F, we can say a strongly self-gravitating spherical star in a binary follows a dimensionaly regularized geodesic even in such a dynamical space-time up to and including 3.5 PN order.
(THIS IS NON-TRIVIAL.)

It was not possible to derive all the 3 PN gravitational field.
→ Need to study other techniques if 4 PN is (really) necessary.

Approximate analytic solutions to 3 PN EOM

3PN monopole: Memmesheimer, Gopakumar & Schäfer (2004),
Leading order SO: Königsdörfer & Gopakumar (2005)

Kepler motion (Parametric solution to Newtonian EOM)

$$R = a(1 - e\cos u),$$

$$\phi - \phi_0 = v \equiv 2\arctan\left[\left(\frac{1+e}{1-e}\right)^{1/2}\tan\frac{u}{2}\right],$$

$$l \equiv n(t - t_0) = u - e\sin u,$$

parametric solution to 3 PN EOM (excluding 2.5 PN radiation reaction).

$$r = a_r(1 - e_r\cos u),$$

$$l \equiv n(t - t_0) = u - e_t\sin u + \left(\frac{g_{4t}}{c^4} + \frac{g_{6t}}{c^6}\right)(v - u) + \left(\frac{f_{4t}}{c^4} + \frac{f_{6t}}{c^6}\right)\sin v + \frac{i_{6t}}{c^6}\sin 2v + \frac{h_{6t}}{c^6}\sin 3v,$$

$$\frac{2\pi}{\Phi}(\phi - \phi_0) = v + \left(\frac{f_{4\phi}}{c^4} + \frac{f_{6\phi}}{c^6}\right)\sin 2v + \left(\frac{g_{4\phi}}{c^4} + \frac{g_{6\phi}}{c^6}\right)\sin 3v + \frac{i_{6\phi}}{c^6}\sin 4v + \frac{h_{6\phi}}{c^6}\sin 5v,$$

Waveform

Waveform up to 3.5 PN order (Blanchet's living review paper.)

$$h_{+,\times} = \frac{2G\mu x}{c^2 R}\left\{H^{(0)}_{+,\times} + x^{1/2}H^{(1/2)}_{+,\times} + xH^{(1)}_{+,\times} + x^{3/2}H^{(3/2)}_{+,\times} + x^2 H^{(2)}_{+,\times} + x^{5/2}H^{(5/2)}_{+,\times} + \mathcal{O}\left(\frac{1}{c^6}\right)\right\}.$$

$$H^{(0)}_{\times} = -2c_i \sin 2\psi, \quad H^{(0)}_{+} = -(1+c_i^2)\cos 2\psi,$$

$$\psi = \phi - \frac{2GM\omega}{c^3}\ln\left(\frac{\omega}{\omega_0}\right) \qquad x = (m\omega)^{2/3}.$$

$$\begin{aligned}\phi = -\frac{x^{-5/2}}{32\nu}\Bigg\{ & 1 + \left(\frac{3715}{1008} + \frac{55}{12}\nu\right)x - 10\pi x^{3/2} + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\nu + \frac{3085}{144}\nu^2\right)x^2 \\ & + \left(\frac{38645}{1344} - \frac{65}{16}\nu\right)\pi x^{5/2}\ln\left(\frac{x}{x_0}\right) \\ & + \left[\frac{12348611926451}{18776862720} - \frac{160}{3}\pi^2 - \frac{1712}{21}C - \frac{856}{21}\ln(16\,x)\right. \\ & \quad \left. + \left(-\frac{15737765635}{12192768} + \frac{2255}{48}\pi^2\right)\nu + \frac{76055}{6912}\nu^2 - \frac{127825}{5184}\nu^3\right]x^3 \\ & + \left(\frac{77096675}{2032128} + \frac{378515}{12096}\nu - \frac{74045}{6048}\nu^2\right)\pi x^{7/2} + \mathcal{O}\left(\frac{1}{c^8}\right)\Bigg\},\end{aligned}$$

Additional phase:

$$\Psi_e(f) = -\frac{7065}{187136}[\pi\mathcal{M}(1+z)]^{-5/3} e_0^2 f_0^{19/9} f^{-34/9}.$$

Due to orbital eccentricity: Królak, Kokkotas & Schäfer (1995)

$$\Psi_\beta(f) + \Psi_\sigma(f) = \frac{3}{4}(8\pi\mathcal{M}(1+z)f)^{-5/3}[4\beta y^{3/2} - 10\sigma y^2].$$

Due to leading order Spin-Orb. & SS

Vecchio (2004).

Appendix. Gauge condition and EOM, relations among multipole moments.

$$
\begin{aligned}
0 = h^{\tau\mu}{}_{,\mu} &= 4\epsilon^4 \sum_{A=1,2} \left[\frac{\dot{P}^\tau_A}{r_A} + \frac{r^i_A}{r^3_A} \left(P^\tau_A v^i_A + \epsilon^2 \dot{D}^i_A - P^i_A \right) \right] \\
&+ \sum_{A=1,2} \oint_{\partial B_A} \frac{dS_i}{|\vec{x}-\vec{y}|} \left(\Lambda^{\tau i}_N - v^i_A \Lambda^{\tau\tau}_N \right) \cdots, \\
0 = h^{i\mu}{}_{,\mu} &= 4\epsilon^4 \sum_{A=1,2} \frac{\dot{P}^i_A}{r_A} + \sum_{A=1,2} \oint_{\partial B_A} \frac{dS_j}{|\vec{x}-\vec{y}|} \left(\Lambda^{ij}_N - v^j_A \Lambda^{\tau i}_N \right) + \cdots,
\end{aligned}
$$

Harmonic gauge condition

$\Updownarrow$

Mass-energy relation, momentum-velocity relation, EOM, other relations among multipole moments