CHARACTERISTIC GENERA OF CLOSED ORIENTABLE 3-MANIFOLDS

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ABSTRACT

A *complete invariant* for (closed, connected, orientable) 3-manifolds is an invariant for the 3-manifolds such that the invariants of any two 3-manifolds are equal if and only if the 3-manifolds are homeomorphic. Further, if we can reconstruct the 3-manifold itself from the data of the complete invariant, then we call it a *characteristic* invariant. In a previous paper, the author introduced a characteristic lattice point invariant for the 3-manifolds, which makes the 3-manifolds a well-ordered set. The purpose of this paper is to construct a characteristic rational invariant from the characteristic lattice point, which we call the *characteristic genus*. We also construct the *characteristic function*, a two-variable holomorphic function which theoretically classifies all the 3-manifolds.

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1. Introduction

It is classically well-known (cf. B. von Kerékjártó [11]) that every closed connected orientable surface F is characterized by the maximal number, say $n(\geqq 0)$ of mutually disjoint simple loops ω_i $(i = 1, 2, \dots, n)$ in F such that the complement $F - \cup_{i=1}^{n}\omega_i$ is connected. This number n is called the *genus* of F. We consider the union L^0 of n

mutually disjoint 0-spheres S_i^0 $(i = 1, 2, \dots, n)$ in the 2-sphere S^2 (namely, the set of $2n$ points in S^2) as an S^0-*link with* n *components*. Then the surface characterization stated above is dual to the statement that the surface F of genus n is obtained as the 1-handle surgery manifold $\chi(L^0)$ of S^2 along an S^0-link L^0 with n components. Let $\mathbb{M}^2$ be the set of (the unoriented types of) closed connected orientable surfaces, and $\mathbb{L}^0$ the set of (unoriented types of) S^0-links. Since any two S^0-links with the same number of components belong to the same type, we have a well-defined bijection

$$\alpha^0 : \mathbb{M}^2 \longrightarrow \mathbb{L}^0$$

sending a surface $F \in \mathbb{M}^2$ to an S^0-link $L^0 \in \mathbb{L}^0$ such that $\chi(L^0) = F$. Further, let $\mathbb{X}^0$ be the set of non-negative integers, and $\mathbb{G}^0$ the set of (the isomorphism classes of) "the link groups"$\pi_1(S^2 - L^0)$ of all S^0-links $L^0 \in \mathbb{L}^0$. Then we have further two natural bijections

$$\sigma^0 : \mathbb{L}^0 \longrightarrow \mathbb{X}^0, \tag{1}$$

$$\pi^0 : \mathbb{L}^0 \longrightarrow \mathbb{G}^0 \tag{2}$$

such that $\sigma^0(L^0) = n$ and $\pi^0(L^0) = \pi_1(S^2 - L^0)$ for an S^0-link L^0 with n components, respectively, so that we have the composite bijections

$$g = \sigma_\alpha^0 = \sigma^0\alpha^0 : \mathbb{M}^2 \longrightarrow \mathbb{X}^0, \tag{3}$$

$$\pi_\alpha^0 = \pi^0\alpha^0 : \mathbb{M}^2 \longrightarrow \mathbb{G}^0. \tag{4}$$

For every surface $F \in \mathbb{M}^2$, the number $g(F) = n$ is equal to the genus of F, and the group $\pi_\alpha^0(F)$ is a free group of rank $2n-1$ (if $n \geqq 1$) or the trivial group $\{1\}$ (if $n = 0$). Thus, the genus $g(F)$ determines the S^0-link $\alpha^0(F)$, the group $\pi_\alpha^0(F)$ and the surface F itself. As we discussed in the paper [6], an analogous argument is possible for closed connected orientable 3-manifolds, although the existence of non-trivial links in the 3-sphere S^3 makes the classification complicated. Here, for convenience we explain an idea of this argument of [6] here briefly. Let $\mathbb{M}$ be the set of (unoriented types of) closed connected orientable 3-manifolds. Let $\mathbb{L}$ be the set of (unoriented types of) links in S^3 (including knots as one-component links). The *set* $\mathbb{X}$ *of lattice points* is the disjoint union of $\mathbb{Z}^n$ for all $n = 1, 2, 3, \dots$ where $\mathbb{Z}$ is the set of integers. An element $\mathbf{x} \in \mathbb{Z}^n$ is called a *lattice point* of *length* $\ell(\mathbf{x}) = n$. We have a canonocal map

$$\mathrm{cl}\beta : \mathbb{X} \longrightarrow \mathbb{L}$$

sending a lattice point $\mathbf{x}$ to a closed braid diagram $\mathrm{cl}\beta(\mathbf{x})$, which is surjective by the Alexander theorem (cf. J. S. Birman [1]). It was shown in in [6] that every well-order of $\mathbb{X}$ induces an injection

$$\sigma : \mathbb{L} \longrightarrow \mathbb{X}$$

which is a right inverse of $\mathrm{cl}\beta$. In particular, by taking the caninical well-order which is explained in §2, we consider the subset $\mathbb{L}^{\mathrm{p}} \subset \mathbb{L}$ consisting of prime links as a well-ordered set with the order inherited from $\mathbb{X}$ by σ, where the two-component trivial link is excluded from $\mathbb{L}^{\mathrm{p}}$. The *length* $\ell(L)$ of a prime link $L \in \mathbb{L}^{\mathrm{p}}$ is the length $\ell(\sigma(L))$ of the lattice point $\sigma(L)$. Let $\mathbb{G}$ be the set of (isomorphism types of) the link groups $\pi_1(S^3 - L)$ for all links L in S^3. Let $\pi : \mathbb{L} \to \mathbb{G}$ be the map sending a link L to the link group $\pi_1(S^3 - L)$. Let $\mathbb{L}^{\pi}$ be the subset of $\mathbb{L}^{\mathrm{p}}$ consisting of π-*minimal links*, that is, prime links L such that L is the initial element of the subset

$$\{L' \in \mathbb{L}^{\mathrm{p}} | \pi_1(S^3 - L') = \pi_1(S^3 - L)\} \subset \mathbb{L}^{\mathrm{p}}.$$

We are interested in this subset $\mathbb{L}^{\pi}$ because it has a crucial property that the restriction of π to $\mathbb{L}^{\pi}$ is injective. Since the restriction of σ to $\mathbb{L}^{\pi}$ is also injective, we can consider $\mathbb{L}^{\pi}$ as a well-ordered set by the order induced from the order of $\mathbb{X}$. In [5], we showed that the set

$$\mathbb{L}^{\pi}(M) = \{L \in \mathbb{L}^{\pi} | \chi(L, 0) = M\}$$

is not empty for every 3-manifold $M \in \mathbb{M}$, where $\chi(L, 0)$ denotes the 0-surgery manifold of S^3 along L . By R. Kirby's theorem [12] on the Dehn surgery of framed links, we note that the set $\mathbb{L}^{\pi}(M)$ is defined in terms of only links so that any two π-minimal links in $\mathbb{L}^{\pi}(M)$ are related by two kinds of Kirby moves and choices of orientations of S^3, although the definition of $\mathbb{L}^{\pi}(M)$ above depends on homeomorphisms on 3-manifolds. Sending M to the initial element of $\mathbb{L}^{\pi}(M)$, we have an embedding

$$\alpha : \mathbb{M} \longrightarrow \mathbb{L}$$

with $\chi(\alpha(M), 0) = M$ for every 3-manifold $M \in \mathbb{M}$ which induces two embeddings

$$\sigma_{\alpha} = \sigma\alpha : \mathbb{M} \longrightarrow \mathbb{X}, \tag{5}$$

$$\pi_{\alpha} = \pi\alpha : \mathbb{M} \longrightarrow \mathbb{G}. \tag{6}$$

By a special featur of the 0-surgery, the S^0-link $\alpha(M) \cap S^2$ in S^2 produces a surface $\chi(\alpha(M) \cap S^2)$ naturally embedded in M with $\alpha^0(\chi(\alpha(M) \cap S^2)) = \alpha(M) \cap S^2$ for every 2-sphere S^2 in S^3 meeting the link $\alpha(M)$ transversely. In this sense, the embedding α is an extension of the embedding α^0. In this construction, we can reconstruct the link $\alpha(M)$, the group $\pi_{\alpha}(M)$ and the 3-manifold M itself from the lattice point $\sigma_{\alpha}(M) \in \mathbb{X}$. Thus, we have constructed the embeddings α, σ_{α}, and π_{α} analogous to the embeddings α^0, σ^0_{α}, and π^0_{α}, respectively. The *length* $\ell(M)$ of a 3-manifold $M \in \mathbb{M}$ is the length $\ell(\sigma_{\alpha}(M))$ of the lattice point $\sigma_{\alpha}(M)$. To calculate $\sigma_{\alpha}(M)$, we proposed a program on the classification problem of $\mathbb{M}$(see J. Hempel [3]) and classified the 3-manifolds of lengths $\leqq 7$ in [6] (See [10] for some more classifications of 3-manifolds). We also classified the prime links of lengths $\leqq 9$ in [7] and of lengths $\leqq 10$ in [9] and the prime link exteriors of length $\leqq 9$ in [8]. In general, we say that an invariant Inv for topological objects is *complete* if any two topological objects A and A' with $\mathrm{Inv}(A) = \mathrm{Inv}(A')$ are homeomorphic. The complete invariant $\mathrm{Inv}(A)$ is *characteristic* if the topological object A can be reconstructed from the data of $\mathrm{Inv}(A)$. For example, $\pi_{\alpha}(M)$ is a complete invariant for $M \in \mathbb{M}$ taking the value in groups and $\sigma_{\alpha}(M)$ is a characteristic invariant for $M \in \mathbb{M}$ taking the value in lattice points. For an interval $I \subset \mathbb{R}$, we put $I_{\mathbb{Q}} = I \cap \mathbb{Q}$, where $\mathbb{R}$ and $\mathbb{Q}$ denote the sets of real numbers and rational numbers, respectively. In this paper, we propose a characteristic rational invariant with value in $[0,+\infty)_{\mathbb{Q}}$ for all $M \in \mathbb{M}$, which we call the *characteristic genus* of M and denoted by $g(M)$. Our idea of this construction is to define a subset $\mathcal{B} \subset \mathbb{X}$ containing the image $\sigma(\mathbb{L}^{\mathrm{p}})$ which we call the B-set and then to embed the B-set $\mathcal{B}$ into the set $\mathbb{Q}$ in order to use the characteristic lattice point invariant $\sigma_\alpha(M)$. An explanation of the B-set $\mathcal{B}$ is made in §2. The embedding of $\mathcal{B}$ into $\mathbb{Q}$ is done in §3. In §4, we list the characteristic genera of all the 3-manifolds with lengths $\leqq 7$. Let $g_h(M)$ denote the Heegaard genus of M, namely the minimal genus of a surface which splits M into two handlebodies. Some properties of this characteristic genus are as follows (see §5 for more detailed properties):

Properties of the characteristic genus.
(1) We have $g(S^3) = g_h(S^3) = 0$, $g(S^1 \times S^2) = g_h(S^1 \times S^2) = 1$, and $g(M) \geqq g_h(M)+1$ for every $M \in \mathbb{M}$ with $M \neq S^3, S^1 \times S^2$.
(2) For every $M \in \mathbb{M}$ with $M \neq S^3, S^1 \times S^2$, we have $\ell(M) \geqq 3$ and $g(M) \in (\ell(M) - \frac{1}{2}, \ell(M) + \frac{1}{2})_{\mathbb{Q}}$.
(3) For every integer $n \geqq 3$, there are only finitely many 3-manifolds $M \in \mathbb{M}$ such that $g(M) \in (n - \frac{1}{2}, n + \frac{1}{2})_{\mathbb{Q}}$.
(4) From the value of $g(M)$, the lattice point $\sigma_\alpha(M)$, the link $\alpha(M)$, the group $\pi_\alpha(M)$ and the 3-manifold M itself are reconstructed.

In §6, as a variant of the characteristic genus, we construct a *characteristic function* $\mu(\ell, z)$, a holomorphic function with the absolute convergence domain containing the 4-disk $D^4 = \{(\ell, z) \in \mathbb{C}^2 | |\ell|^2 + |z|^2 \leqq 1\}$, which classifies all the 3-manifolds $M \in \mathbb{M}$. Concluding this introductory section, we mention here some analogous invariants derived from different viewpoints. Y. Nakagawa defined in [14] a family of integer-valued characteristic invariants of the set of knots by using R. W. Ghrist's universal template (although a generalization to oriented links appears difficult). Also, J. Milnor and W. Thurston defined in [13] a non-negative real-valued invariant for the closed connected 3-manifolds by the property that if $\tilde{N} \to N$ is a degree $n(\geqq 2)$ connected covering of a closed connected 3-manifold N, then the invariant of $\tilde{N}$ is n times the invariant of N, so that it does not classify lens spaces.

2. The range of prime links in the set of lattice points

To investigate the image $\sigma(\mathbb{L}) \subset \mathbb{X}$, we need some notations on lattice points in [6, 8, 9, 10]. For a lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of length n, we denote the lattice points $(x_n, \ldots, x_2, x_1)$ and $(|x_1|, |x_2|, \ldots, |x_n|)$ by $\mathbf{x}^T$ and $|\mathbf{x}|$, respectively. Let $|\mathbf{x}|_N$ be a permutation $(|x_{j_1}|, |x_{j_2}|, \ldots, |x_{j_n}|)$ of the coordinates $|x_j|$ $(j = 1, 2, \ldots, n)$ of $|\mathbf{x}|$ such that

$$|x_{j_1}| \leqq |x_{j_2}| \leqq \cdots \leqq |x_{j_n}|.$$

Let $\min|\mathbf{x}| = \min_{1 \leqq i \leqq n} |x_i|$ and $\max|\mathbf{x}| = \max_{1 \leqq i \leqq n} |x_i|$. The *dual* lattice point $\delta(\mathbf{x}) = (x'_1, x'_2, \ldots, x'_n)$ of $\mathbf{x}$ is defined by $x'_i = \mathrm{sign}(x_i)(\max|\mathbf{x}| + 1 - |x_i|)$ where $\mathrm{sign}(0) = 0$ by convention. Defining $\delta^0(\mathbf{x}) = \mathbf{x}$ and $\delta^n(\mathbf{x}) = \delta(\delta^{n-1}(\mathbf{x}))$ inductively, we note that $\delta^2(\mathbf{x}) \neq \mathbf{x}$ in general, but $\delta^{n+2}(\mathbf{x}) = \delta^n(\mathbf{x})$ for all $n \geqq 1$. For a lattice point $\mathbf{y} = (y_1,\, y_2, \ldots, y_m)$ of length m, we denote by $(\mathbf{x}, \mathbf{y})$ the lattice point

$$(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m)$$

of length $n + m$. For an integer m and a positive integer n, we denote by m^n the lattice point $(m, m, \ldots, m)$ of length n. Also, we take $-m^n = (-m)^n$. A reason why we do not consider $\mathbb{L}$, but $\mathbb{L}^{\mathrm{p}}$ is because we can use the following lemma which is shown in [6]:

Lemma 2.1 We have $\mathrm{cl}\beta(\mathbf{x}) = \mathrm{cl}\beta(\mathbf{y})$ in $\mathbb{L}$ modulo split additions of trivial links if and only if $\mathbf{y}$ is obtained from $\mathbf{x}$ by a finite number of the following transformations:

(1) $(\mathbf{x}, 0) \leftrightarrow \mathbf{x}$.
(2) $(\mathbf{x}, \mathbf{y}, -\mathbf{y}^T) \leftrightarrow \mathbf{x}$.
(3) $(\mathbf{x}, y) \leftrightarrow \mathbf{x}$ when $|y| > \max|\mathbf{x}|$.
(4) $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leftrightarrow (\mathbf{x}, \mathbf{z}, \mathbf{y})$ when $\min|\mathbf{y}| > \max|\mathbf{z}| + 1$ or $\min|\mathbf{z}| > \max|\mathbf{y}| + 1$.
(5) $(\mathbf{x}, \pm y, y + 1, y) \leftrightarrow (\mathbf{x}, y + 1, y, \pm(y + 1))$ when $y(y + 1) \neq 0$.
(6) $(\mathbf{x}, \mathbf{y}) \leftrightarrow (\mathbf{y}, \mathbf{x})$.
(7) $\mathbf{x} \leftrightarrow \mathbf{x}^T \leftrightarrow -\mathbf{x} \leftrightarrow -\mathbf{x}^T$.
(8) $\mathbf{x} \leftrightarrow \mathbf{x}'$ when $\mathrm{cl}\beta(\mathbf{x})$ is a disconnected link and $\mathrm{cl}\beta(\mathbf{x}')$ is obtained from $\mathrm{cl}\beta(\mathbf{x})$ by changing the orientation of a component of $\mathrm{cl}\beta(x)$.

There is an algorithm to obtain $\mathrm{cl}\beta(\mathbf{x}')$ from $\mathrm{cl}\beta(\mathbf{x})$ for (8). The *canonical order* of $\mathbb{X}$ is a well-order determined as follows: Namely, the well-order in $\mathbb{Z}$ is defined by $0 < 1 < -1 < 2 < -2 < 3 < -3 < \ldots$, and this order of $\mathbb{Z}$ is extended to a well-order in $\mathbb{Z}^n$ for every $n \geqq 2$ so that for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{Z}^n$ we define $\mathbf{x}_1 < \mathbf{x}_2$ if we have one of the following conditions (1)-(3):

(1) $|\mathbf{x}_1|_N < |\mathbf{x}_2|_N$ by the lexicographic order (on the natural number order).
(2) $|\mathbf{x}_1|_N = |\mathbf{x}_2|_N$ and $|\mathbf{x}_1| < |\mathbf{x}_2|$ by the lexicographic order (on the natural number order).
(3) $|\mathbf{x}_1| = |\mathbf{x}_2|$ and $\mathbf{x}_1 < \mathbf{x}_2$ by the lexicographic order on the well-order of $\mathbb{Z}$ defined above.

Finally, for any two lattice points $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{X}$ with $\ell(\mathbf{x}_1) < \ell(\mathbf{x}_2)$, we define $\mathbf{x}_1 < \mathbf{x}_2$.

By definition, we have $\sigma_\alpha(\mathbb{M}) \subset \sigma(\mathbb{L}^{\mathrm{p}}) \subset \mathbb{X}$. For every even integer $n = 2m \geqq 4$, we use a specific lattice point $\mathbf{a}_n$ of length n with the nth coordinate $(-1)^{m-1}m$ which is defined inductively as follows: Let $\mathbf{a}_4 = (1, -2, 1, -2)$. Assuming that $\mathbf{a}_n = (\mathbf{a}'_n, (-1)^{m-1}m)$ is defined, we define

$$\mathbf{a}_{n+2} = (\mathbf{a}'_n, (-1)^m(m + 1), (-1)^{m-1}m, (-1)^m(m + 1)).$$

A *sublattice point* of a lattice point $\mathbf{x}$ is a lattice point $\mathbf{x}'$ such that $\mathbf{x} = (\mathbf{u}, \mathbf{x}', \mathbf{v})$ for some lattice points $\mathbf{u}, \mathbf{v}$ (which may be the empty lattice point). When we write $|\mathbf{x}|_N = (1^{e_1}, 2^{e_2}, \ldots, m^{e_m})$ for $m = \max|\mathbf{x}|$, the integer e_k is called the *exponent* of k in $\mathbf{x}$ and denoted by $\exp_k(\mathbf{x})$. In this paper, we define the following subset of $\mathbb{X}$ containing $\sigma(\mathbb{L}^{\mathrm{p}})$:

Definition 2.2. The B-*set* is a subset $\mathcal{B}$ of $\mathbb{X}$ consisting of

$$0(\in \mathbb{Z}), \quad 1^n(\text{for } n \geqq 2), \quad \mathbf{a}_n \,(\text{for even } n \geqq 4)$$

and all the lattice points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ $(n \geqq 5)$ which have all the following conditions (1)-(8):

(1) $x_1 = 1$, $2 \leqq |x_n| \leqq \max|\mathbf{x}| < n/2$.
(2) $\exp_k(\mathbf{x}) \geqq 2$ for every k with $1 \leqq k \leqq \max|\mathbf{x}|$.
(3) Every lattice point obtained from $\mathbf{x}$ by permuting the coordinates of $\mathbf{x}$ cyclically is not of the form $(\mathbf{x}', \mathbf{x}'')$ where $1 \leqq \max|\mathbf{x}'| < \min|\mathbf{x}''|$.
(4) For every $i < n$, one of the following identities or inequality holds. $|x_i| - 1 = |x_{i+1}|$, $x_i = x_{i+1}$ or $|x_i| < |x_{i+1}|$.
(5) For a sublattice point $\mathbf{x}'$ of $\mathbf{x}$ with $|\mathbf{x}'| = (k, (k+1)^e, k)$, $(k^e, k+1, k)$ or $(k, k+1, k^e)$ for some $k, e \geqq 1$ and $\exp_k \mathbf{x} = 2$, then $\mathbf{x}' = \pm(k, -\varepsilon(k+1)^e, k)$, $\pm(\varepsilon k^e, -(k+1), k)$ or $\pm(k, -(k+1), \varepsilon k^e)$ for some $\varepsilon = \pm 1$, respectively. Further, if $e = 1$, then $\varepsilon = 1$.
(6) For a sublattice point $\mathbf{x}'$ of $\mathbf{x}$ with $|\mathbf{x}'| = (k+1, k^e, k+1)$ for some $k, e \geqq 1$, then $\mathbf{x}' = \pm(k+1, \varepsilon k^e, k+1)$ for some $\varepsilon = \pm 1$. Further if $e = 1$, then $\varepsilon = -1$.
(7) $\mathbf{x}$ is the initial element of the set of the lattice points obtained from every lattice point of $\pm\mathbf{x}$, $\pm\mathbf{x}^T$, $\pm\delta(\mathbf{x})$ and $\pm\delta(\mathbf{x})^T$ by permuting the coordinates cyclically.
(8) $|\mathbf{x}|$ is not of the form $(|\mathbf{x}'|, k+1, k, (k+1)^e, k)$ or $(|\mathbf{x}'|, k+1, k^2, k+1, k)$ for $e \geqq 1$, $k \geqq 2$ and $\max|\mathbf{x}'| \leqq k$.

We note that the delta set Δ in [6] is a subset of $\mathbb{X}$ obtained from Definition 2.2 by relaxing only the condition (1) to the following (1)$'$.

(1)$'$ $x_1 = 1$, $2 \leqq |x_n| \leqq \max|\mathbf{x}| \leqq n/2$.

We show the following lemma:

Lemma 2.3. $\sigma(\mathbb{L}^{\mathrm{p}}) \subset \mathcal{B}$.

Proof. In [6], it is proved that $\sigma(\mathbb{L}^{\mathrm{p}}) \subset \Delta$ except for (8). In [9], we showed that $\sigma(\mathbb{L}^{\mathrm{p}})$ has (8). By these facts, it is sufficient to prove that if $\mathbf{x} \in \sigma(\mathbb{L}^{\mathrm{p}})$ has $\ell(\mathbf{x}) = n \geqq 4$ and $\max|\mathbf{x}| = \frac{n}{2}$, then we have $\mathbf{x} = \mathbf{a}_n$. Since $\mathbf{x}$ is in Δ, we see that $|\mathbf{x}|_N = (1^2, 2^2, \ldots, m^2)$. By the transformations (1)-(7) in Lemma 2.1, we see that unless $|\mathbf{x}| = |\mathbf{a}_n|$, we can transform $\mathbf{x}$ into a smaller lattice point $\mathbf{x}'$. Then considering $\mathbf{x}$ itself, we conclude that unless $\mathbf{x} = \mathbf{a}_n$, the lattice point $\mathbf{x}$ is transformed into a smaller lattice point $\mathbf{x}''$. $\square$

3. Embedding the diamond set into the set of rational numbers

For a lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{B}$, we define the rational number

$$\zeta(\mathbf{x}) = n + \frac{x_2}{n} + \cdots + \frac{x_n}{n^{n-1}}.$$

By definition, $\zeta(0) = 1$. We show the following lemma:

Lemma 3.1. The map $\mathbf{x} \mapsto \zeta(\mathbf{x})$ induces an embedding

$$\zeta : \mathcal{B} \longrightarrow \mathbb{Q}$$

such that for every $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathcal{B}$ with $n \geqq 3$ we have the following properties (1)-(3):

(1) $|\zeta(\mathbf{x}) - n| < \frac{1}{2}$.
(2) The lattice point $\mathbf{x} \in \mathcal{B}$ is reconstructed from the value of $\zeta(\mathbf{x})$.
(3) For any integer $n \geqq 3$, there are only finitely many $\mathbf{x} \in \mathcal{B}$ with

$$\zeta(\mathbf{x}) \in (n - \frac{1}{2}, n + \frac{1}{2})_{\mathbb{Q}}.$$

Proof. To show (1), first assume that $\mathbf{x} \in \mathcal{B}$ has $|x_i| < n/2$ for all i. Then $|x_i| \leqq (n-1)/2$ Then we have

$$\begin{aligned}
|\zeta(\mathbf{x}) - n| &\leqq \frac{n-1}{2}\left(\frac{1}{n} + \cdots + \frac{1}{n^{n-1}}\right) \\
&= \frac{n-1}{2} \cdot \frac{1}{n^{n-1}}\left(1 + n + \cdots + n^{n-2}\right) \\
&= \frac{n-1}{2} \cdot \frac{1}{n^{n-1}} \cdot \frac{n^{n-1}-1}{n-1} \\
&= \frac{1}{2}(1 - \frac{1}{n^{n-1}}) < \frac{1}{2}.
\end{aligned}$$

It is directly checked that $|\zeta(\mathbf{a}_n) - n| < \frac{1}{2}$ for $n = 4, 6$. Let $n \geqq 8$ be even. Then

$$\begin{aligned}
|\zeta(\mathbf{a}_n) - n| &\leqq \frac{2}{n} + \frac{1}{n^2} + \frac{n-1}{2}\left(\frac{1}{n^3} + \cdots + \frac{1}{n^{n-4}} + \frac{1}{n^{n-2}}\right) + \frac{n}{2}\left(\frac{1}{n^{n-3}} + \frac{1}{n^{n-1}}\right) \\
&= \frac{n-1}{2} \cdot \frac{1}{n^{n-1}}\left(1 + n + \cdots + n^{n-2}\right) \\
&= \frac{3n^{n-5} + 4n^{n-4} + 1}{2n^{n-3}} < \frac{1}{2},
\end{aligned}$$

showing (1). To show that ζ is an embedding, for $\mathbf{x} \neq 0$ we note that $\zeta(\mathbf{x}) \neq \zeta(0) = 0$ and that if the value of $\zeta(\mathbf{x})$ is given, then the length $n(\geqq 2)$ of $\mathbf{x}$ is determined uniquely from the denominator of $\zeta(\mathbf{x})$ by (1). For $n = 2, 3$, we obtain the result. Let $n \geqq 4$. For $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_n) \in \mathcal{B}$ with $|x'_i| < n/2$ $(i = 1, 2, \ldots, n)$, assume that

$$\zeta(\mathbf{x}) = \zeta(\mathbf{x}') = n + \frac{x'_2}{n} + \cdots + \frac{x'_n}{n^{n-1}}.$$

Then inductively we have

$$x'_i - x_i \equiv 0 \pmod{n} \quad (i = 1, 2, \ldots, n).$$

Since

$$|x'_i - x_i| \leqq |x'_i| + |x_i| < \frac{n}{2} + \frac{n}{2} = n,$$

we must have $x'_i - x_i = 0\,(i = 1, 2, \ldots, n)$ and $\mathbf{x} = \mathbf{x}'$. If $|x_i| = \frac{n}{2}$ for some i, then $\mathbf{x} = \mathbf{x}' = \mathbf{a}_n$, showing (2). Since there are only finitely many lattice points with length n in $\mathcal{B}$, we have (3) by (1). $\square$

4. The table of genera of 3-manifolds with up to 7 lengths

By the classification of [6], if $\ell(M) = 1, 2$, then we have $M = S^1 \times S^2$, S^3, respectively. The reason why S^3 occurs by $\ell(M) = 2$ is because we take S^3 as the 0-surgery manifold of S^3 along the Hopf link 2^2_1 and we have $\sigma_\alpha(S^3) = 1^2$. However, *we can also take S^3 as the 3-manifold without 0-surgery of S^3 along a link.* In this paper, we adopt this viewpoint and introduce the empty lattice point $\boldsymbol{\emptyset} \in \mathcal{B} \subset \mathbb{X}$ of length 0, the empty link $\emptyset \in \mathbb{L}$ with bridge index 0 to define

$$\alpha(S^3) = \emptyset, \quad \sigma_\alpha(S^3) = \boldsymbol{\emptyset}, \quad \zeta(\boldsymbol{\emptyset}) = 0.$$

Also, we have $\pi_\alpha(S^3) = \{1\}$ by introducing the trivial group $\{1\}$ to the set $\mathbb{G}$ of link groups. In this viewpoint, we note that *there is no 3-manifold $M \in \mathbb{M}$ with $\ell(M) = 2$.* Then we make the following definition of characteristic genus:

Definition 4.1. The *characteristic genus* $g(M)$ of a 3-manifold $M \in \mathbb{M}$ is given by the identity

$$g(M) = \zeta(\sigma_\alpha(M)).$$

It is direct that $g(S^3) = 0$ and $g(S^1 \times S^2) = 1$. We shall confirm in Theorem 5.4 that $g(M)$ is a characteristic invariant. Since $\sigma_\alpha(\mathbb{M}) \subset \mathcal{B}$ and every initial segment of $\mathcal{B}$ is a finite set, there are only finitely many 3-manifolds with length n for every $n \geqq 0$. By convention, we put $M_{0,0} = S^3$. According to the canonical well-order of $\mathbb{X}$, we enumerate the 3-manifolds of length $n \geqq 1$ as follows:

$$M_{n,1} < M_{n,2} < \cdots < M_{n,m_n}$$

for a non-negative integer m_n depending only on n. Let $\mathbf{x}_{n,i} = \sigma_\alpha(M_{n,i}) \in \mathcal{B}$ and $g_{n,i} = g(M_{n,i})$. By [6], we reconstruct from $\mathbf{x}_{n,i}$ the link $\alpha(M_{n,i}) = L_{n,i} \in \mathbb{L}$, the group $\pi_\alpha(M_{n,i}) = G_{n,i} \in \mathbb{G}$ and the 3-manifold $M_{n,i}$ itself. By (1) of Lemma 3.1, we reconstruct $\mathbf{x}_{n,i}$ from $g_{n,i}$, so that we can construct from $g_{n,i}$ the lattice point $\mathbf{x}_{n,i}$, the link $L_{n,i}$, the group $G_{n,i}$ and the 3-manifold $M_{n,i}$ itself. In [6], we listed all the lattice points $\mathbf{x}_{n,i}$ and the links $L_{n,i}$ identified with the notations in D. Rolfsen's table [15] for all $n \leqq 7$. In the following table, we list all the genera $g_{n,i}$ together with $\mathbf{x}_{n,i}$ and $L_{n,i}$ for all $n \leqq 7$, where we note that there is no 3-manifold of length 2 by the reason stated above (different from the list of [6] at this point). For convenience, we also include the homological data in this table by letting $H_{n,i} = H_1(M_{n,i};\mathbb{Z})$.

Table 4.2.
$g_{0,0} = 0$, $\mathbf{x}_{0,0} = \boldsymbol{\emptyset}$, $L_{0,0} = \emptyset$, $H_{0,0} = 0$.

$g_{1,1} = 1$, $\mathbf{x}_{1,1} = 0$, $L_{1,1} = O$, $H_{1,1} = \mathbb{Z}$.

$g_{3,1} = 3 + \frac{4}{9} = 3.44444444\ldots$, $\mathbf{x}_{3,1} = 1^3$, $L_{3,1} = 3_1$, $H_{3,1} = \mathbb{Z}$.

$g_{4,1} = 4 + \frac{21}{64} = 4.328125$, $\mathbf{x}_{4,1} = 1^4$, $L_{4,1} = 4_1^2$, $H_{4,1} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
$g_{4,2} = 4 - \frac{15}{32} = 3.53125$, $\mathbf{x}_{4,2} = (1,-2,1,-2)$, $L_{4,2} = 4_1$, $H_{4,2} = \mathbb{Z}$.

$g_{5,1} = 5 + \frac{156}{625} = 5.2496$, $\mathbf{x}_{5,1} = 1^5$, $L_{5,1} = 5_1$, $H_{5,1} = \mathbb{Z}$.
$g_{5,2} = 5 + \frac{78}{625} = 5.1248$, $\mathbf{x}_{5,2} = (1^2,-2,1,-2)$, $L_{5,2} = 5_1^2$, $H_{5,2} = \mathbb{Z} \oplus \mathbb{Z}$.

$g_{6,1} = 6 + \frac{1555}{7776} = 6.1999742\ldots$, $\mathbf{x}_{6,1} = 1^6$, $L_{6,1} = 6_1^2$, $H_{6,1} = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.
$g_{6,2} = 6 + \frac{395}{1944} = 6.2031893\ldots$, $\mathbf{x}_{6,2} = (1^3,2,-1,2)$, $L_{6,2} = 5_2$, $H_{6,2} = \mathbb{Z}$.
$g_{6,3} = 6 + \frac{361}{1944} = 6.1856995\ldots$, $\mathbf{x}_{6,3} = (1^3,-2,1,-2)$, $L_{6,3} = 6_2$, $H_{6,3} = \mathbb{Z}$.
$g_{6,4} = 6 + \frac{443}{1944} = 6.2278806\ldots$, $\mathbf{x}_{6,4} = (1^2,2,1^2,2)$, $H_{6,4} = \mathbb{Z}_2$.
$g_{6,5} = 6 + \frac{113}{972} = 6.1162551\ldots$, $\mathbf{x}_{6,5} = (1^2,-2,1^2,-2)$, $L_{6,5} = 6_1^3$, $H_{6,5} = \mathbb{Z}_2$.
$g_{6,6} = 6 + \frac{443}{3888} = 6.1139403\ldots$, $\mathbf{x}_{6,6} = (1^2,-2,1,-2^2)$, $L_{6,6} = 6_3$, $H_{6,6} = \mathbb{Z}$.
$g_{6,7} = 6 - \frac{611}{1944} = 5.6856995\ldots$, $\mathbf{x}_{6,7} = (1,-2,1,-2,1,-2)$, $L_{6,7} = 6_2^3$, $H_{6,7} = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$.
$g_{6,8} = 6 - \frac{253}{864} = 5.7071759\ldots$, $\mathbf{x}_{6,8} = (1,-2,1,3,-2,3)$, $L_{6,8} = 6_3^2$, $H_{6,8} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

$g_{7,1} = 7 + \frac{19608}{117649} = 7.1666652\ldots$, $\mathbf{x}_{7,1} = 1^7$, $L_{7,1} = 7_1$, $H_{7,1} = \mathbb{Z}$.
$g_{7,2} = 7 + \frac{19644}{117649} = 7.1669712\ldots$, $\mathbf{x}_{7,2} = (1^4,2,-1,2)$, $L_{7,2} = 6_2^2$, $H_{7,2} = \mathbb{Z}_3 \oplus \mathbb{Z}_3$.
$g_{7,3} = 7 + \frac{19458}{117649} = 7.1653902\ldots$, $\mathbf{x}_{7,3} = (1^4,-2,1,-2)$, $L_{7,3} = 7_1^2$, $H_{7,3} = 0$.
$g_{7,4} = 7 + \frac{18576}{117649} = 7.1578933\ldots$, $\mathbf{x}_{7,4} = (1^3,-2,1^2,-2)$, $L_{7,4} = 7_4^2$, $H_{7,4} = \mathbb{Z} \oplus \mathbb{Z}$.
$g_{7,5} = 7 + \frac{18555}{117649} = 7.1577148\ldots$, $\mathbf{x}_{7,5} = (1^3,-2,1,-2^2)$, $L_{7,5} = 7_2^2$, $H_{7,5} = 0$.
$g_{7,6} = 7 + \frac{12381}{117649} = 7.1052367\ldots$, $\mathbf{x}_{7,6} = (1^2,-2,1^2,-2^2)$, $L_{7,6} = 7_5^2$, $H_{7,6} = \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
$g_{7,7} = 7 + \frac{12255}{117649} = 7.1041657\ldots$, $\mathbf{x}_{7,7} = (1^2,-2,1,-2,1,-2)$, $L_{7,7} = 7_6^2$, $H_{7,7} = \mathbb{Z}\oplus\mathbb{Z}$.
$g_{7,8} = 7 + \frac{21130}{117649} = 7.179602\ldots$, $\mathbf{x}_{7,8} = (1^2,2,-1,-3,2,-3)$, $L_{7,8} = 6_1$, $H_{7,8} = \mathbb{Z}$.
$g_{7,9} = 7 + \frac{12484}{117649} = 7.1061122\ldots$, $\mathbf{x}_{7,9} = (1^2,-2,1,3,-2,3)$, $L_{7,9} = 7_6$, $H_{7,9} = \mathbb{Z}$.
$g_{7,10} = 7 - \frac{31763}{117649} = 6.7300189\ldots$, $\mathbf{x}_{7,10} = (1,-2,1,-2,3,-2,3)$, $L_{7,10} = 7_7$, $H_{7,10} = \mathbb{Z}$.
$g_{7,11} = 7 - \frac{30293}{117649} = 6.7425137\ldots$, $\mathbf{x}_{7,11} = (1,-2,1,3,-2^2,3)$, $L_{7,11} = 7_1^3$, $H_{7,11} = \mathbb{Z}_2$.

5. Properties of the characteristic genus

For every 3-manifold $M \in \mathbb{M}$ with $M \neq S^3, S^1 \times S^3$, we have $\ell(M) \geqq 3$. Let x_2 be the second coordinate of the lattice point $\sigma_\alpha(M) \in \mathcal{B}$. By the definition of $\mathcal{B}$, we have $x_2 \neq 0$ for $\ell(M) \geqq 3$. We say that a 3-manifold $M \in \mathbb{M}$ with $\ell(M) \geqq 3$ is *positive* or *negative*, respectively, according to if $x_2 > 0$ or $x_2 < 0$. Every 3-manifold $M \in \mathbb{M}$ has a Heegaard splitting, i.e., a union of two handlebodies by pasting along the boundaries. The *Heegaard genus*, $g_h(M)$ of M is the minimum of the genera of such handlebodies. The following relationship between a bridge presentation of a link $L \in \mathbb{L}$(see [4]) and Heegaard splittings of the Dehn surgery manifolds along L is a folk result (although we did not find a reference).

Lemma 5.1. Let a link $L \in \mathbb{L}$ have a g-bridge presentation. Then every Dehn surgery manifold M of S^3 along L admits a Heegaard splitting of genus g.

Proof. Since S^3 is a union of two 3-balls B, B' pasting along the boundary spheres such that $T = L \cap B$ and $T' = L \cap B'$ are trivial tangles of g proper arcs in B and B', respectively. Let $N(T)$ be a tubular neighborhood of T in B, $V = \mathrm{cl}(B - N(T))$, and $V' = B' \cup N(T)$. By construction, V and V' are handlebodies of genus g and forms a Heegaard splitting of S^3. To complete the proof, it suffices to show that the Dehn surgery from S^3 to M along L just changes V' into another handlebody V'', so that V and V'' forms a Heegaard splitting of M of genus g. Since T' is a trivial tangle in B' of g proper arcs, there are $g-1$ proper disks D_i $(i = 1, 2, \dots, g-1)$ in B' which split B' into a 3-manifold regarded as a tubular neighborhood $N(T')$ of T' in B'. Then the union $N(L) = N(T) \cup N(T')$ is regarded as a tubular neighborhood of L in S^3. The Dehn surgery from S^3 to M along L just changes $N(L)$ into the union of solid tori obtained from $N(L)$ by the Dehn surgery without changing the boundary $\partial N(L)$. Thus, we obtain the desired handlebody V'' by pasting along the disks corresponding to D_i $(i = 1, 2, \dots, g-1)$. □

Let $g_b(M)$ and $g_{br}(M)$ denote the *bridge genus* and the *bridge genus* of M, namely the minimal bridge index and the minimal braid index for links whose 0-surgery manifolds are M. We define $g_b(S^3) = g_{br}(S^3) = 0$ by considering that S^3 is obtained from S^3 by the 0-surgery along the empty link $\emptyset$. Let $g_\alpha(M) = \max|\sigma_\alpha(M)| + 1$. By convention, we have $\max|\emptyset| = -1$, so that $g_\alpha(S^3) = 0$. Then we have the following lemma:

Lemma 5.2. For every $M \in \mathbb{M}$, we have

$$g_h(M) \leqq g_b(M) \leqq g_{br}(M) \leqq g_\alpha(M) \leqq \frac{\ell(M)}{2} + 1.$$

Proof. For $M \neq S^3$, the link $\alpha(M)$ has a $g_\alpha(M)$-bridge presentation by definition, so that we obtain the desired inequalities from Lemmas 2.3 and 5.1. For $M = S^3$, we have

$$g_h(S^3) = g_b(S^3) = g_{br}(S^3) = g_\alpha(S^3) = \ell(S^3) = 0$$

by convention. $\square$

We do not know whether there is an $M \in \mathbb{M}$ with $g_{br}(M) < g_\alpha(M)$. Here are some examples.

Example 5.3. (1) Let $M = \chi(3_1, 0) = M_{3,1}$ for the trefoil knot 3_1. Since the bridge index of 3_1 is 2 and M is no lens space, we see from Table 4.2 that

$$g_h(M) = g_b(M) = g_{br}(M) = g_\alpha(M) = 2 < \frac{\ell(M)}{2}+1 = 2.5 \text{ and } g(M) = 3+\frac{4}{9} = 3.44....$$

(2) Let $M = \chi(4_1^2, 0) = M_{4,1}$ for the (2, 4)-torus link 4_1^2. Since the bridge index of 4_1^2 is 2 and the first homology $H_1(M)$ has exactly 2 generators, we see from Table 4.2 that

$$g_h(M) = g_b(M) = g_{br}(M) = g_\alpha(M) = 2 < \frac{\ell(M)}{2}+1 = 3 \text{ and } g(M) = 4+\frac{21}{64} = 4.32....$$

(3) Let $M = \chi(4_1, 0) = M_{4,2}$ for the figure eight knot 4_1. Since the bridge index of 4_1 is 2 and M is not any lens space, we see that $g_h(M) = g_b(M) = 2$. If M is obtained from a braid index 2 knot or link, then M would be obtained from a $(2k+1)$-half-twist knot K_k by 0-surgery. The Alexander polynomial of the homology handle M is t^2-3t+1 and different from the Alexander polynomial $\frac{t^{2k+1}+1}{t+1}$ of the homology handle $\chi(K_k, 0)$. This result and Table 4.2 mean that $g_{br}(M) = g_\alpha(M) = 3 = \frac{\ell(M)}{2}+1$. Also, by Table 4.2 we have $g(M) = 4 - \frac{15}{32} = 3.53....$

Incidentally, we note here that the bridge genus behaves differently from the Heegaard genus, although $g_h(M) = g_b(M)$ in Example 5.3. If M is a lens space except S^3 and $S^1 \times S^2$, then we have $g_b(M) \geqq 3$ whereas $g_h(M) = 1$. In fact, the first homology $H_1(M)$ is a finite cyclic group, but if $(1 \leqq) g_b(M) \leqq 2$, then $H_1(M)$ would be isomorphic to Z or $Z_m \oplus Z_m$ for some $m \geqq 2$, which is a contradiction. Concretly, the projective 3-space M has $\sigma_\alpha(M) = (1^2, 2, 1^2, 2)$ (see [6]) and hence $g_b(M) = g_\alpha(M) = 3$. We show the following theorem:

Theorem 5.4 The characteristic genus $g(M)$ of every $M \in \mathbb{M}$ is a characteristic invariant such that

$$\begin{aligned}&g_h(S^3) = g_b(S^3) = g(S^3) = \ell(S^3) = 0,\\ &g_h(S^1 \times S^2) = g_b(S^1 \times S^2) = g(S^1 \times S^2) = \ell(S^1 \times S^2) = 1\end{aligned}$$

and every $M \in \mathbb{M}$ with $M \neq S^3, S^1 \times S^2$ has the following properties:

(1) Not only the 3-manifold M itself but also the lattice point $\sigma_\alpha(M)$, the link $\alpha(M)$, and the group $\pi_\alpha(M)$ are reconstructed from the value of $g(M)$.
(2) According to if M is positive or negative, $g(M)$ belongs to $(n, n+\frac{1}{2})_{\mathbb{Q}}$ or $(n-\frac{1}{2}, n)_{\mathbb{Q}}$ for $n = \ell(M) \geqq 3$. In particular, $\ell(M)$ is equal to the maximal integer not exceeding the number $g(M) + \frac{1}{2}$.

(3) For every integer $n \geqq 3$, there are only finitely many 3-manifolds $M \in \mathbb{M}$ such that $g(M) \in (n-\frac{1}{2}, n+\frac{1}{2})_{\mathbb{Q}}$.
(4) $g_h(M)+1 \leqq g_b(M)+1 < g(M)$.

Proof. By definition, we have $g_h(S^3) = g_b(S^3) = g(S^3) = \ell(S^3) = 0$ and $g_h(S^1 \times S^2) = g_b(S^1 \times S^2) = g(S^1 \times S^2) = \ell(S^1 \times S^2) = 1$. By the property of σ_α in [6] and Lemma 3.1, we see that $g(M)$ is a characteristic rational invariant and the properties (1) and (3) hold. To show (2), let $\mathbf{x} = \sigma_\alpha(M) = (x_1, x_2, \dots, x_n)$ have $|x_i| < n/2$ for all i. Then $|x_i| \leqq (n-1)/2$ and we have

$$\begin{aligned} |\zeta(\mathbf{x}) - n - \frac{x_2}{n}| &\leqq \frac{n-1}{2}\left(\frac{1}{n^2} + \cdots + \frac{1}{n^{n-1}}\right) \\ &= \frac{n-1}{2} \cdot \frac{1}{n^{n-1}}\left(1 + n + \cdots + n^{n-3}\right) \\ &= \frac{1}{2}(\frac{1}{n} - \frac{1}{2n^{n-2}}) < \frac{1}{2n}. \end{aligned}$$

Hence $\mathrm{sign}(\zeta(\mathbf{x}) - n) = \mathrm{sign}(x_2)$. Since the second coordinate of $\mathbf{a}_n$ is -2 and $\mathrm{sign}(\zeta(\mathbf{a}_n) - n) = -1$ is directly checked for $n = 4, 6$. For $n \geqq 8$,

$$\begin{aligned} |\zeta(\mathbf{a}_n) - n - \frac{-2}{n}| &\leqq \frac{n-1}{2} \cdot \left(\frac{1}{n^2} + \cdots + \frac{1}{n^{n-4}} + \frac{1}{n^{n-2}}\right) + \frac{n}{2} \cdot \left(\frac{1}{n^{n-3}} + \frac{1}{n^{n-1}}\right) \\ &= \frac{1}{2n} + \frac{1}{2n^{n-3}} \end{aligned}$$

showing $\mathrm{sign}(\zeta(\mathbf{a}_n) - n) = -1$ and we obtain (2). To see (4), Example 5.3 shows that the inequality $g_b(M)+1 < g(M)$ holds for $\ell(M) < 5$. Let $\ell(M) \geqq 5$. If M is positive, then we have

$$g(M) > \ell(M) = (\frac{\ell(M)}{2} + 1) + 1 + (\frac{\ell(M)}{2} - 2) > g_b(M) + 1,$$

by Lemma 5.2 and $\ell(M) \geqq 5$. If M is negative and $\ell(M)$ is odd, then we write $\ell(M) = 2s+1$ for $s \geqq 2$. Since $g_b(M) \leqq s+1$ and $g(M) + \frac{1}{2} > 2s+1$, we have

$$g(M) \geqq 2s + \frac{1}{2} = (s+1) + 1 + (s - \frac{3}{2}) > g_b(M) + 1.$$

If M is negative and $\ell(M)$ is even, then we write $\ell(M) = 2s$ for $s \geqq 3$. Since $g_b(M) \leqq s+1$ and $g(M) + \frac{1}{2} > 2s$, we have

$$g(M) \geqq 2s - \frac{1}{2} = (s+1) + 1 + (s - \frac{5}{2}) > g_b(M) + 1.$$

□

For every integer $n > 1$, since there are infinitely many 3-manifolds $M \in \mathbb{M}$ with $g_b(M) = n$, we see from the property (3) of Theorem 5.4 that there is a 3-manifold $M \in \mathbb{M}$ with $g_b(M) = n$ such that the difference $g(M) - g_b(M)$ is sufficiently large.

6. Constructing the characteristic function

The *torsion genus* of a 3-manifold $M \in \mathbb{M}$ is the rational number

$$t(M) = g(M) - \ell(M) \in (-\frac{1}{2}, \frac{1}{2})_{\mathbb{Q}}$$

for $M \neq S^3, S^1 \times S^2$ and $t(M) = 1$ for $M = S^3, S^1 \times S^2$. The pair $(\ell(M), t(M))$ is also a characteristic invariant of $\mathbb{M}$. Let $t_{n,i} = t(M_{n,i})$. For non-negative integers n, i, we define

$$b_{n,i} = \begin{cases} t_{n,i} & \text{if } t_{n,i} \text{ exists} \\ 0 & \text{otherwise.} \end{cases}$$

For any complex numbers ℓ and z, we define the formal power series

$$\begin{aligned} \mu(\ell, z) &= \sum_{n,i=0}^{+\infty} b_{n,i} \ell^i z^n \\ &= 1 + \ell z + \frac{4}{9}\ell z^3 + \frac{21}{64}\ell z^4 - \frac{15}{32}\ell^2 z^4 + \frac{156}{625}\ell z^5 + \frac{78}{625}\ell^2 z^5 + \dots \end{aligned}$$

Then we have the following theorem.

Theorem 6.3. The formal power series $\mu(\ell, z)$ is a holomorphic function on the unit 4-disk $D^4 = \{(\ell, z) \in \mathbb{C}^2 | \, |\ell|^2 + |z|^2 \leqq 1\}$ which classifies all the closed connected orientable 3-manifolds.

Proof. If $\ell = 0$ or $z = 0$, then $\mu(\ell, z) = 1$. For every $(\ell, z) \in D^4$ with $\ell z \neq 0$, since $|b_{n,i}| \leqq 1$ and $|\ell| < 1, |z| < 1$, we have

$$0 \leqq \sum_{n,i=0}^{+\infty} b_{n,i} |\ell^i| \cdot |z^n| \leqq \sum_{n,i=0}^{+\infty} |\ell|^i \cdot |z|^n = (\sum_{n=0}^{+\infty} |\ell|^i)(\sum_{i=0}^{+\infty} |z|^n) = \frac{1}{(1 - |\ell|)(1 - |z|)},$$

showing that $\mu(\ell, z)$ is absolutely convergent for every $(\ell, z) \in D^4$. To see that $\mu(\ell, z)$ classifies $\mathbb{M}$, it suffices to take the derivative $\frac{\partial^{i+n}}{\partial \ell^i \partial z^n}\mu(0, 0) = (i!)(n!)b_{n,i}$, which is $(i!)(n!)t_{n,i}$ if $b_{n,i} \neq 0$. □

Finally, we have one question: *Does the formal sum*

$$\mu(x) = \sum_{M \in \mathbb{M}} x^{g(M)}$$

converge for $0 < x < 1$?

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