CHARACTERISTIC GENERA OF CLOSED ORIENTABLE 3-MANIFOLDS

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ABSTRACT

A complete invariant defined for (closed connected orientable) 3-manifolds is an invariant defined for the 3-manifolds such that any two 3-manifolds with the same invariant are homeomorphic. Further, if the 3-manifold itself can be reconstructed from the data of the complete invariant, then it is called a characteristic invariant defined for the 3-manifolds. In a previous work, a characteristic lattice point invariant defined for the 3-manifolds was constructed by using an embedding of the prime links into the set of lattice points. In this paper, a characteristic rational invariant defined for the 3-manifolds called the characteristic genus defined for the 3-manifolds is constructed by using an embedding of a set of lattice points called the PDelta set into the set of rational numbers. The characteristic genus defined for the 3-manifolds is also compared with the Heegaard genus, the bridge genus and the braid genus defined for the 3-manifolds. By using this characteristic rational invariant defined for the 3-manifolds, a smooth real function with the definition interval $(-1,1)$ called the charactericstic genus function is constructed as a characteristic invariant defined for the 3-manifolds.

Mathematical Subject Classification 2010: 57M25, 57M27

Keywords: Braid, 3-manifold, Prime link, Characteristic genus, Characteristic function

1. Introduction

It is classically well-known[1] that every closed connected orientable surface F is characterized by the maximal number, say $n(\geqq 0)$ of mutually disjoint simple loops ω_i $(i = 1, 2, , n)$ in F such that the complement $F \setminus \cup_{i=1}^{n} \omega_i$ is connected. This number n is called the *genus* of F . We consider the union L^0 of n mutually disjoint 0-spheres S_i^0 $(i = 1, 2, \dots, n)$ in the 2-sphere S^2 (namely, the set of $2n$ points in S^2) as an S^0-*link with* n *components*. Then the surface characterization stated above is dual to the statement that the surface F of genus n is obtained as the 1-handle surgery manifold $\chi(L^0)$ of S^2 along an S^0-link L^0 with n components. Let $\mathbb{M}^2$ be the set of (the unoriented types of) closed connected orientable surfaces, and $\mathbb{L}^0$ the set of (unoriented types of) S^0-links. Since any two S^0-links with the same number of components belong to the same type, we have a well-defined bijection

$$\alpha^0 : \mathbb{M}^2 \to \mathbb{L}^0$$

sending a surface $F \in \mathbb{M}^2$ to an S^0-link $L^0 \in \mathbb{L}^0$ such that $\chi(L^0) = F$. Further, let $\mathbb{X}^0$ be the set of non-negative integers, and $\mathbb{G}^0$ the set of (the isomorphism classes of) "the link groups"$\pi_1(S^2 \setminus L^0)$ of all S^0-links $L^0 \in \mathbb{L}^0$. Then we have further two natural bijections

$$\sigma^0 : \mathbb{L}^0 \to \mathbb{X}^0, \quad \pi^0 : \mathbb{L}^0 \to \mathbb{G}^0$$

such that $\sigma^0(L^0) = n$ and $\pi^0(L^0) = \pi_1(S^2 \setminus L^0)$ for an S^0-link L^0 with n components, respectively, so that we have the composite bijections

$$g^0 = \sigma^0_\alpha = \sigma^0\alpha^0 : \mathbb{M}^2 \to \mathbb{X}^0, \quad \pi^0_\alpha = \pi^0\alpha^0 : \mathbb{M}^2 \to \mathbb{G}^0.$$

For every surface $F \in \mathbb{M}^2$, the number $g^0(F) = n$ is equal to the genus of F , and the group $\pi^0_\alpha(F)$ is a free group of rank $2n - 1$ (if $n \geqq 1$) or the trivial group $\{1\}$ (if $n = 0$). Thus, the genus $g^0(F)$ determines the S^0-link $\alpha^0(F)$, the group $\pi^0_\alpha(F)$ and the surface F itself. As we discussed in the paper [5], an analogous argument is possible for closed connected orientable 3-manifolds, although the existence of non-trivial links in the 3-sphere S3 makes the classification complicated. Here, for convenience we explain an idea of this argument of [5] briefly. Let $\mathbb{M}$ be the set of (unoriented types of) closed connected orientable 3-manifolds. Let $\mathbb{L}$ be the set of (unoriented types of) links in S^3 (including the knots as one-component links). A *lattice point* of *length* n is an element $\mathbf{x}$ of $\mathbb{Z}^n$ for the natural number n where $\mathbb{Z}$ denotes the set of integers.

In this paper, the empty lattice point ϕ of length 0 and the empty knot ϕ are also considered. Let $\mathbb{X}$ be the set of all lattice points. We have a canonical map

$$\mathrm{cl}\beta : \mathbb{X} \to \mathbb{L}$$

[1] cf. B. von Kerékjártó [15].

sending a lattice point $\mathbf{x}$ to a closed braid diagram $\mathrm{cl}\beta(\mathbf{x})$, which is surjective by the Alexander theorem (cf. J. S. Birman [1]). It was shown in [5] that every well-order of the set $\mathbb{X}$ induces an injection

$$\sigma : \mathbb{L} \to \mathbb{X}$$

which is a right inverse of the map $\mathrm{cl}\beta$. In particular, by taking the caninical well-order which is explained in § 2, we consider the subset $\mathbb{L}^p \subset \mathbb{L}$ consisting of prime links as a well-ordered set with the order inherited from $\mathbb{X}$ by σ, where the two-component trivial link is excluded from $\mathbb{L}^p$. The length $\ell(L)$ of a prime link $L \in \mathbb{L}^p$ is the length $\ell(\sigma(L))$ of the lattice point $\sigma(L)$. Let $\mathbb{G}$ be the set of (isomorphism types of) the link groups $\pi_1(S^3 \setminus L)$ for all links L in S^3. Let $\pi : \mathbb{L} \to \mathbb{G}$ be the map sending a link L to the link group $\pi_1(S^3 \setminus L)$. Let $\mathbb{L}^\pi$ be the subset of $\mathbb{L}^p$ consisting of a π-*minimal link*, that is, a prime link L such that L is the initial element of the subset

$$\{L' \in \mathbb{L}^p |\, \pi_1(S^3 \setminus L') = \pi_1(S^3 \setminus L)\}.$$

We are interested in this subset $\mathbb{L}^\pi$ because it has a crucial property that the restriction of π to $\mathbb{L}^\pi$ is injective. Since the restriction of σ to $\mathbb{L}^\pi$ is also injective, we can consider $\mathbb{L}^\pi$ as a well-ordered set by the order induced from the order of $\mathbb{X}$. In [4], we showed that the set

$$\mathbb{L}^\pi(M) = \{L \in \mathbb{L}^\pi |\, \chi(L, 0) = M\}$$

is not empty for every 3-manifold $M \in \mathbb{M}$, where $\chi(L, 0)$ denotes the 0-surgery manifold of S^3 along L and we define $\chi(L, 0) = S^3$ when L is the empty knot ϕ. By R. Kirby's theorem [16] on the Dehn surgeries of framed links, we note that the set $\mathbb{L}^\pi(M)$ is defined in terms of only links so that any two π-minimal links in $\mathbb{L}^\pi(M)$ are related by two kinds of Kirby moves and choices of orientations of S^3. Sending every 3-manifold M to the initial element of $\mathbb{L}^\pi(M)$ induces an embedding

$$\alpha : \mathbb{M} \to \mathbb{L}$$

with $\chi(\alpha(M), 0) = M$ for every 3-manifold $M \in \mathbb{M}$, which further induces two embeddings

$$\sigma_\alpha = \sigma\alpha : \mathbb{M} \to \mathbb{X}, \quad \pi_\alpha = \pi\alpha : \mathbb{M} \to \mathbb{G}.$$

By a special featur of the 0-surgery, the S^0-link $\alpha(M) \cap S^2$ in S^2 produces a surface $\chi(\alpha(M) \cap S^2)$ naturally embedded in M with $\alpha^0(\chi(\alpha(M) \cap S^2)) = \alpha(M) \cap S^2$ for every 2-sphere S^2 in S^3 meeting the link $\alpha(M)$ transversely. In this sense, the embedding α is an extension of the embedding α^0. In this construction, we can reconstruct the link $\alpha(M)$, the group $\pi_\alpha(M)$ and the 3-manifold M itself from the lattice point $\sigma(M) \in \mathbb{X}$. Thus, we have constructed the embeddings σ, σ_α and π_α analogous to the embeddings σ, σ_α and π_α, respectively. The length $\ell(M)$ of a 3-manifold $M \in \mathbb{M}$ is the length

$\ell(\sigma_\alpha(M))$ of the lattice point $\sigma_\alpha(M)$. In [14], the 3-manifolds of lengths $\leqq 10$ are classified (see also [9, 11, 12]). In this process, the prime links and their exteriors of lengths $\leqq 10$ have been earlier classified (See [6, 7, 8, 10]). In general, an invariant Inv defined for a family of topological objects is *complete* if any two members A and A' with $\mathrm{Inv}(A) = \mathrm{Inv}(A')$ are homeomorphic. The complete invariant $\mathrm{Inv}(A)$ is a *characteristic* invariant if the object A can be reconstructed from data of $\mathrm{Inv}(A)$. For example, the group invariant $\pi_\alpha(M)$ is a complete invariant defined for the 3-manifolds $M \in \mathbb{M}$ taking the value in finitely presented groups and the lattice point $\sigma_\alpha(M)$ is a characteristic invariant defined for the 3-manifolds $M \in \mathbb{M}$ taking the value in lattice points. For an interval $I \subset \mathbb{R}$, we put $I_\mathbb{Q} = I \cap \mathbb{Q}$, where $\mathbb{R}$ and $\mathbb{Q}$ denote the sets of real numbers and rational numbers, respectively.

In this paper, we consider a lattice point set $P\Delta$ called the *PDelta set* such that

$$\sigma_\alpha(\mathbb{M}) \subset \sigma(\mathbb{L}^p) \subset P\Delta \subset \mathbb{X}.$$

An embedding $g : P\Delta \to [0, +\infty)_\mathbb{Q}$ called the *characteristic genus* is constructed so that the image $g(\mathbb{S})$ of every subset $\mathbb{S} \subset P\Delta$ containing the empty lattice point $\emptyset$ and the zero lattice point $\mathbf{0} \in \mathbb{Z}$ (called a *PDelta subset*) is a characteristic invariant defined for the set $\mathbb{S}$. By taking $\mathbb{S} = \sigma(\mathbb{L}^p)$, the *characteristic genus $g(L)$ defined for the prime links $L \in \mathbb{L}^p$* is obtained. By taking $\mathbb{S} = \sigma_\alpha(\mathbb{M})$, the *characteristic genus $g(M)$ defined for the 3-manifolds $M \in \mathbb{M}$* is obtained.

An explanation of the PDelta set is made in § 2. A construction of the embedding g is done in § 3. In § 4, some properties of the characteristic genera of the 3-manifolds are stated together with the calculation results of the 3-manifolds of lengths $\leqq 7$. In particular, the characteristic genus $g(M)$ for a 3-manifold M is compared with the Heegaard genus $g_h(M)$, the bridge genus $g_b(M)$ and the braid genus $g_{br}(M)$. In § 5, from the characteristic genus g, we construct a smooth real function $G_\mathbb{S}(t)$ with the definition interval $(-1, 1)$ for every PDelta subset $\mathbb{S}$ which is a characteristic invariant defined for the set $\mathbb{S}$. By taking $\mathbb{S} = \sigma(\mathbb{L}^p)$, the *characteristic prime link function $G_{\mathbb{L}^p}(t)$* is obtained as a characteristic invariant defined for the prime link set $\mathbb{L}^p$. By taking $\mathbb{S} = \sigma_\alpha(\mathbb{M})$, the *characteristic genus function $G_\mathbb{M}(t)$* is obtained as a characteristic invariant defined for the 3-manifold set $\mathbb{M}$.

Concluding this introductory section, we mention here some analogous invariants derived from different viewpoints. Y. Nakagawa defined in [18] a family of integer-valued characteristic invariants of the set of knots by using R. W. Ghrist's universal template (although a generalization to oriented links appears difficult). Also, J. Milnor and W. Thurston defined in [17] a non-negative real-valued invariant defined for the closed connected 3-manifolds with the property that if $\tilde{N} \to N$ is a degree $n(\geqq 2)$ connected covering of a closed connected 3-manifold N, then the invariant of $\tilde{N}$ is n times the invariant of N, so that it does not classify lens spaces.

2. The range of the prime links in the set of lattice points

To investigate the image $\sigma(\mathbb{L}^p) \subset \mathbb{X}$, we need some notations on lattice points in [5, 6, 7, 8, 9, 10, 11, 12, 14]. For a lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ of length $\ell((\mathbf{x}) = n$, we denote the lattice points $(x_n, \ldots, x_2, x_1)$ and $(|x_1|, |x_2|, \ldots, |x_n|)$ by $\mathbf{x}^T$ and $|\mathbf{x}|$, respectively. Let $|\mathbf{x}|_N$ be a permutation $(|x_{j_1}|, |x_{j_2}|, \ldots, |x_{j_n}|)$ of the coordinates $|x_j|\,(j = 1, 2, \ldots, n)$ of $|\mathbf{x}|$ such that

$$|x_{j_1}| \leqq |x_{j_2}| \leqq \cdots \leqq |x_{j_n}|.$$

Let $\min|\mathbf{x}| = \min_{1 \leqq i \leqq n} |x_i|$ and $\max|\mathbf{x}| = \max_{1 \leqq i \leqq n} |x_i|$. The *dual* lattice point of $\mathbf{x}$ is given by $\delta(\mathbf{x}) = (x'_1, x'_2, \ldots, x'_n)$ where $x'_i = \mathrm{sign}(x_i)(\max_{|}\mathbf{x}| + 1 - |x_i|)$ and $\mathrm{sign}(0) = 0$ by convention.

Defining $\delta^0(\mathbf{x}) = \mathbf{x}$ and $\delta^n(\mathbf{x}) = \delta(\delta^{n-1}(\mathbf{x}))$ inductively, we note that $\delta^2(\mathbf{x}) \neq \mathbf{x}$ in general, but $\delta^{n+2}(\mathbf{x}) = \delta^n(\mathbf{x})$ for all $n \geqq 1$. For a lattice point $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ of length m, we denote by $(\mathbf{x}, \mathbf{y})$ the lattice point

$$(x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m).$$

of length $n + m$. For an integer m and a natural number n, we denote by m^n the lattice point $(m, m, \ldots, m)$ of length n. Also, we take $-m^n = (-m)^n$. A reason why we do not consider $\mathbb{L}$ but $\mathbb{L}^p$ is because we can use the following lemma which is shown in [5]:

Lemma 2.1 We have $\mathrm{cl}\beta(\mathbf{x}) = \mathrm{cl}\beta(\mathbf{y})$ in $\mathbb{L}$ modulo split additions of trivial links if and only if $\mathbf{y}$ is obtained from $\mathbf{x}$ by a finite number of the following transformations:
(1) $(\mathbf{x}, 0) \leftrightarrow \mathbf{x}$.
(2) $(\mathbf{x}, \mathbf{y}, -\mathbf{y}^T) \leftrightarrow \mathbf{x}$.
(3) $(\mathbf{x}, y) \leftrightarrow \mathbf{x}$ when $|y| > \max|\mathbf{x}|$.
(4) $(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leftrightarrow (\mathbf{x}, \mathbf{z}, \mathbf{y})$ when $\min|y| > \max|z| + 1$ or $\min|z| > \max|y| + 1$.
(5) $(\mathbf{x}, \pm y, y + 1, y) \leftrightarrow (\mathbf{x}, y + 1, y, \pm(y + 1))$ when $y(y + 1) \neq 0$.
(6) $(\mathbf{x}, \mathbf{y}) \leftrightarrow (\mathbf{y}, \mathbf{x})$.
(7) $\mathbf{x} \leftrightarrow \mathbf{x}^T \leftrightarrow -\mathbf{x} \leftrightarrow -\mathbf{x}^T$.
(8) $\mathbf{x} \leftrightarrow \mathbf{x}'$ when $\mathrm{cl}\beta(\mathbf{x})$ is a disconnected link and $\mathrm{cl}\beta(\mathbf{x}')$ is obtained from $\mathrm{cl}\beta(\mathbf{x})$ by changing the orientation of a component of $\mathrm{cl}\beta(\mathbf{x})$.

There is an algorithm to obtain $\mathrm{cl}\beta(\mathbf{x}')$ from $\mathrm{cl}\beta(\mathbf{x})$ in (8).

The *canonical order* of $\mathbb{X}$ is a well-order determined as follows: Namely, the well-order in $\mathbb{Z}$ is defined by $0 < 1 < -1 < 2 < -2 < 3 < -3 < \ldots$, and this order of $\mathbb{Z}$

is extended to a well-order in $\mathbb{Z}^n$ for every $n \geqq 2$ so that for $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{Z}^n$ we define $\mathbf{x}_1 < \mathbf{x}_2$ if we have one of the following conditions (1)-(3):

(1) $|\mathbf{x}_1|_N < |\mathbf{x}_2|_N$ by the lexicographic order (on the natural number order).
(2) $|\mathbf{x}_1|_N = |\mathbf{x}_2|_N$ and $|\mathbf{x}_1| < |\mathbf{x}_2|$ by the lexicographic order (on the natural number order).
(3) $|\mathbf{x}_1| = |\mathbf{x}_2|$ and $\mathbf{x}_1 < \mathbf{x}_2$ by the lexicographic order on the well-order of $\mathbb{Z}$ defined above.

Finally, for any two lattice points $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{X}$ with $\ell(\mathbf{x}_1) < \ell(\mathbf{x}_2)$, we define $\mathbf{x}_1 < \mathbf{x}_2$.

For a subset $\mathbb{S} \subset \mathbb{X}$ and a non-negative integer n, let

$$\mathbb{S}^{(n)} = \{\mathbf{x} \in \mathbb{S} |\, \ell(x) \leqq n\}$$

and call it the *n-fragment* of $\mathbb{S}$.

The *Delta set* is the subset Δ of $\mathbb{X}$ consisting of $\emptyset, \mathbf{0}$ and all lattice points $\mathbf{x}$ of lengths $n \geqq 2$ satisfying $x_1 = 1$ and

$$1 \leqq \min \mathbf{x} \leqq \max |\mathbf{x}| \leqq \frac{n}{2}.$$ [2]

An important property of the Delta set Δ is that the n-fragment $\Delta^{(n)}$ of the Delta set Δ is a finite set for every non-negative integer n.

In our argument, the special lattice point $\mathbf{a}_n$ of length n defined for every even integer $n = 2m \geqq 4$ is important. This lattice point $\mathbf{a}_n$ is defined inductively as follows: Let $\mathbf{a}_4 = (1, -2, 1, -2)$. Assuming that $\mathbf{a}_n = (\mathbf{a}'_n, (-1)^{m-1}m)$ is defined, we define

$$\mathbf{a}_{n+2} = (\mathbf{a}'_n, (-1)^m(m+1), (-1)^{m-1}m, (-1)^m(m+1)).$$

It is noted that the nth coordinate of $\mathbf{a}_n$ is $(-1)^{m-1}m$ and $\mathrm{cl}\beta(\mathbf{a}_n)$ is a 2-bridge knot or a 2-bridge link according to whether m is even or odd, respectively. The *PDelta set* $P\Delta$ is the subset of the Delta set Δ consisting of

$$\emptyset, \mathbf{0}, 1^2, \mathbf{a}_n \,(\text{for any even}\, n \geqq 4)$$

and all lattice points $\mathbf{x}$ of lengths $n \geqq 3$ satisfying $x_1 = 1$ and

$$1 \leqq \min |\mathbf{x}| \leqq \max |\mathbf{x}| < \frac{n}{2}.$$

A *sublattice point* of a lattice point $\mathbf{x}$ is a lattice point $\mathbf{x}'$ such that $\mathbf{x} = (\mathbf{u}, \mathbf{x}', \mathbf{v})$ for some lattice points $\mathbf{u}, \mathbf{v}$ (which may be the empty lattice point). When we write

[2] Further restricted subsets of the present Delta set are called Delta sets in [5, 6, 8, 9, 11, 12, 14].

$|\mathbf{x}|_N = (1^{e_1}, 2^{e_2}, \ldots, m^{e_m})$ for $m = \max|\mathbf{x}|$, the non-negative integer e_k is called the *exponent* of k in $\mathbf{x}$ and denoted by $\exp_k(\mathbf{x})$.

The *DeltaStar set* Δ^* is the subset of $P\Delta$ consisting of

$$\emptyset, \mathbf{0}, 1^n \text{ (for any } n \geqq 2), \mathbf{a}_n \text{ (for any even } n \geqq 4)$$

and all the lattice points $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ $(n \geqq 5)$ which have all the following conditions (1)-(8):

(1) $x_1 = 1$, $2 \leqq |x_n| \leqq \max|\mathbf{x}| < \frac{n}{2}$.
(2) $\exp_k(\mathbf{x}) \geqq 2$ for every k with $1 \leqq k \leqq \max|\mathbf{x}|$.
(3) Every lattice point obtained from $\mathbf{x}$ by permuting the coordinates of $\mathbf{x}$ cyclically is not of the form $(\mathbf{x}', \mathbf{x}'')$ where $1 \leqq \max|\mathbf{x}'| < \min|\mathbf{x}''|$.
(4) For every $i < n$, one of the following identities or inequality holds: $|x_i|-1 = |x_{i+1}|$, $x_i = x_{i+1}$ or $|x_i| < |x_{i+1}|$.
(5) For a sublattice point $\mathbf{x}'$ of $\mathbf{x}$ such that $|\mathbf{x}'| = (k, (k+1)^e, k)$ and $\exp_k \mathbf{x} = 2$ for some $k, e \geqq 1$ or such that $|\mathbf{x}'| = (k^e, k+1, k)$ or $(k, k+1, k^e)$ and $\exp_k(\mathbf{x}) = e+1$ for some $k, e \geqq 1$, then $\mathbf{x}' = \pm(k, -\varepsilon(k+1)^e, k)$, $\pm(\varepsilon k^e, -(k+1), k)$ or $\pm(k, -(k+1), \varepsilon k^e)$ for some $\varepsilon = \pm 1$, respectively. Further, if $e = 1$, then $\varepsilon = 1$.
(6) For a sublattice point $\mathbf{x}'$ of $\mathbf{x}$ with $|\mathbf{x}'| = (k+1, k^e, k+1)$ for some $k, e \geqq 1$, then $\mathbf{x}' = \pm(k+1, \varepsilon k^e, k+1)$ for some $\varepsilon = \pm 1$. Further if $e = 1$, then $\varepsilon = -1$.
(7) $\mathbf{x}$ is the initial element of the set of the lattice points obtained from every lattice point of $\pm\mathbf{x}$, $\pm\mathbf{x}^T$, $\pm\delta(\mathbf{x})$ and $\pm\delta(\mathbf{x})^T$ by permuting the coordinates cyclically.
(8) $|\mathbf{x}|$ is not of the form $(|\mathbf{x}'|, k+1, k, (k+1)^e, k)$ or $(|\mathbf{x}'|, k+1, k^2, k+1, k)$ for $e \geqq 1$, $k \geqq 2$ and $\max|\mathbf{x}'| \leqq k$.

The following lemma is important to our argument:

Lemma 2.3. $\sigma_\alpha(\mathbb{M}) \subset \sigma(\mathbb{L}^p) \subset \Delta^* \subset P\Delta$.

This lemma means that the collections of the links $\mathrm{cl}\beta(\mathbf{x})$ and the 3-manifolds $\chi(\mathrm{cl}\beta(\mathbf{x}, 0)$ for all lattice points $\mathbf{x} \in P\Delta$ contain all the prime links and all the 3-manifolds, respectively.

Proof of Lemm 2.3. In [5], the inclusions $\sigma_\alpha(\mathbb{M}) \subset \sigma(\mathbb{L}^p) \subset \Delta$ are shown except counting the property (8). In [8, Lemma 3.6], we showed that $\sigma(\mathbb{L}^p)$ has (8). Then to complete the proof, it is sufficient to show that if $\mathbf{x} \in \sigma(\mathbb{L}^p)$ has $\ell(\mathbf{x}) = n \geqq 4$ and $\max|\mathbf{x}| = \frac{n}{2}$, then we have $\mathbf{x} = \mathbf{a}_n$. Since $\mathbf{x}$ is in Δ, we see that $|\mathbf{x}|_N = (1^2, 2^2, \ldots, m^2)$. By the transformations (1)-(7) in Lemma 2.1, we see that unless $|\mathbf{x}| = |\mathbf{a}_n|$, we can transform $\mathbf{x}$ into a smaller lattice point $\mathbf{x}'$. Then considering $\mathbf{x}$ itself, we conclude that unless $\mathbf{x} = \mathbf{a}_n$, the lattice point $\mathbf{x}$ is transformed into a smaller lattice point $\mathbf{x}''$. □

The DeltaStar set Δ^* approximates the prime link lattice point set $\sigma(\mathbb{L}^p)$, but they are different. For example, the lattice point $(1^2, 2, -1^2, 2) \in \Delta^*$ does not belong to the prime link subset $\sigma(\mathbb{L}^p)$. In fact, the prime link $L = \mathrm{cl}\beta(1^2, 2, -1^2, 2) = 6^3_3$ appears as a smaller lattice point $(1^2, 2, 1^2, 2)$ in the tables of [5, 8, 12, 14].

3. Embedding the PDelta set into the set of rational numbers

For a lattice point $\mathbf{x} = (x_1, x_2, \dots, x_n) \in P\Delta$ with $n \geqq 2$, we define the rational numbers

$$\begin{aligned} \tau(x) &= \frac{1}{n^{n-1}}(x_2 + x_3 n + \cdots + x_n n^{n-2}), \\ g(\mathbf{x}) &= n + \tau(\mathbf{x}). \end{aligned}$$

For example, we have

$$\tau(1^2) = \frac{1}{2},\ g(1^2) = 2 + \frac{1}{2}.$$

By convention, we put:

$$\tau(\emptyset) = g(\emptyset) = 0, \quad \tau(\mathbf{0}) = 0, \quad g(\mathbf{0}) = 1.$$

The rational number $g(\mathbf{x})$ is called the *characteristic genus* or simply the *genus* of $\mathbf{x}$, and $\tau(\mathbf{x})$ the *decimal part* of the characteristic genus $g(\mathbf{x})$ or the *decimal torsion* of $\mathbf{x}$. According to whether the last coordinate x_n is positive or negavtive, the lattice point $\mathbf{x}$ is called to be *ending-positive* or *ending-negative*, respectively. We show the following theorem:

Theorem 3.1. The map $\mathbf{x} \mapsto g(\mathbf{x})$ induces an embedding

$$g : P\Delta \to [0, +\infty)_{\mathbb{Q}}$$

such that for every $\mathbf{x} = (x_1, x_2, \dots, x_n) \in P\Delta$ with $n \geqq 3$ we have the following properties (1)-(3):

(1) According to whether $\mathbf{x}$ is ending-positive or ending-negative, we have respectively

$$g(\mathbf{x}) \in (n, n + \frac{1}{2})_{\mathbb{Q}} \quad \text{or} \quad g(\mathbf{x}) \in (n - \frac{1}{2}, n)_{\mathbb{Q}}$$

In particular, the length $\ell(\mathbf{x})$ is equal to the maximal integer not exceeding the number $g(\mathbf{x}) + \frac{1}{2}$.

(2) The lattice point $\mathbf{x} \in P\Delta$ is reconstructed from the value of $g(\mathbf{x})$.
(3) There are only finitely many $\mathbf{x} \in P\Delta$ with

$$g(\mathbf{x}) \in (n - \frac{1}{2}, n + \frac{1}{2})_{\mathbb{Q}}.$$

Here is a note on the values on $\emptyset$, $\mathbf{0}$ and 1^2.

Remark 3.2. The values $\tau(\emptyset) = g(\emptyset) = 0$, $\tau(\mathbf{0}) = 0$ and $g(\mathbf{0})$ are not definite values. For example, As another choice, by a geometric meaning on the braids, the zero lattice point $\mathbf{0}$ may be considered as the lattice point $(1, -1)$ where the values $\tau(1, -1) = -\frac{1}{2}$ and $g(1, -1) = 2 - \frac{1}{2} = 1 + \frac{1}{2}$ are taken. On the other hand, the lattice points $(1, -1)$ and 1^2 are considered as exceptional ones in the sense that the characteristic genus does not determine the decimal torsion uniquely as follows:

$$g(1, -1) = 2 - \frac{1}{2} = 1 + \frac{1}{2} \quad \text{and} \quad g(1^2) = 2 + \frac{1}{2} = 3 - \frac{1}{2}.$$

Proof of Theorem 3.1. To show the first half of (1), first consider a lattice point $\mathbf{x} \in P\Delta$ with $|x_i| < \frac{n}{2}$ for all i. Then we have $|x_i| \leqq \frac{n-1}{2}$ and

$$\begin{aligned} |\tau(\mathbf{x}) - \frac{x_n}{n}| &\leqq \frac{n-1}{2} \cdot \frac{1}{n^{n-1}}(1 + n + \cdots + n^{n-3}) \\ &= \frac{n-1}{2} \cdot \frac{1}{n^{n-1}} \cdot \frac{n^{n-2} - 1}{n-1} \frac{1}{2}(\frac{1}{n} - \frac{1}{n^{n-1}}) < \frac{1}{2n}. \end{aligned}$$

Hence

$$-\frac{1}{2n} < \tau(\mathbf{x}) - \frac{x_n}{n} < \frac{1}{2n}.$$

Since $x_n \neq 0$, this shows the assertion of (1) except for the lattice points $\mathbf{a}_n$. Let $\mathbf{a}_n = (a_1, a_2, \ldots, a_n)$. It is directly checked that $|g(\mathbf{a}_n) - n| < \frac{1}{2}$ and $|\tau(\mathbf{a}_n) - \frac{a_n}{n}| < \frac{1}{2n}$ for $n = 4$. Let $n \geqq 6$ be even. Since $|a_i| < \frac{n}{2}$ for all i except $|a_{n-2}| = |a_n| = \frac{n}{2}$ and $|a_{n-1}| = \frac{n-2}{2}$, we have

$$\begin{aligned} |\tau(\mathbf{a}_n) - \left(\frac{a_{n-2}}{n^3} + \frac{a_{n-1}}{n^2} + \frac{a_n}{n}\right)| &\leqq \frac{n-1}{2} \cdot \frac{1}{n^{n-1}} \left(1 + n + \cdots + n^{n-5}\right) \\ &= \frac{n-1}{2} \cdot \frac{1}{n^{n-1}} \cdot \frac{n^{n-4} - 1}{n-1} = \frac{1}{2n^3} - \frac{1}{2n^{n-1}} < \frac{1}{2n^3}. \end{aligned}$$

For the sign ε of a_n, we have

$$\frac{a_{n-2}}{n^3} + \frac{a_{n-1}}{n^2} + \frac{a_n}{n} = \varepsilon(\frac{1}{2n^2} - \frac{n-2}{2n^2} + \frac{1}{2} = \frac{\varepsilon(n-1)(n+1)}{2n^2},$$

so that

$$-\frac{1}{2n^3} < \tau(\mathbf{a}_n) - \frac{\varepsilon(n-1)(n+1)}{2n^2} < \frac{1}{2n^3}.$$

This shows that the assertion of (1) holds for the lattice points $\mathbf{a}_n$.

To show that g is an embedding, let $\ell(\mathbf{x}) = n \geqq 3$. Then $g(\mathbf{x})$ is distinct from $g(\emptyset) = 0$, $g(\mathbf{0}) = 1$ and $g(1^2) = 2 + \frac{1}{2}$. If the value of $g(\mathbf{x})$ is given, then the length $n(\geqq 3)$ of $\mathbf{x}$ is uniquely determined by (1). For $\mathbf{x}' = (x'_1, x'_2, \ldots, x'_n) \in P\Delta$, assume that

$$g(\mathbf{x}) = g(\mathbf{x}') = n + \frac{x'_2}{n^{n-1}} + \cdots + \frac{x'_n}{n}.$$

If $\max|\mathbf{x}| < \frac{n}{2}$ or $\max|\mathbf{x}'| < \frac{n}{2}$, then we have inductively

$$x'_i - x_i \equiv 0 \pmod{n} \text{ and } |x'_i - x_i| \leqq |x'_i| + |x_i| < \frac{n}{2} + \frac{n}{2} = n$$

for all i $(i = 1, 2, \ldots, n)$. Thus, we must have $x'_i - x_i = 0$ $(i = 1, 2, \ldots, n)$ and $\mathbf{x} = \mathbf{x}'$. If $\max|\mathbf{x}| = \frac{n}{2}$ or $\max|\mathbf{x}'| = \frac{n}{2}$, then we obtain by definition and the argument above $\mathbf{x} = \mathbf{x}' = \mathbf{a}_n$, showing (2). Since there are only finitely many lattice points with length n in $P\Delta$, we have (3) by(1). □

The *decimal torsion* and the *characteristic genus* of a prime link $L \in \mathbb{L}^p$ is defined to be $\tau(L) = \tau(\sigma(L))$ and $g(L) = g(\sigma(L))$, respectively. Then $g(L) = \ell(L) + \tau(L)$. For the empty knot ϕ, the trivial knot O and the Hopf link 2^2_1, we have

$$\tau(\phi) = g(\phi) = 0,\ \tau(O) = 0,\ g(O) = 1,\ \tau(2^2_1) = \frac{1}{2},\ g(2^2_1) = 2 + \frac{1}{2}.$$

Further, for every prime link L with $\ell(L) \geqq 3$, we have

$$g(L) \in (\ell(L) - \frac{1}{2}, \ell(L) + \frac{1}{2})_{\mathbb{Q}}$$

by Theorem 3.1. The decimal torsion and the characteristic genus of a 3-manifold $M \in \mathbb{M}$ is defined to be $\tau(M) = \tau(\sigma_\alpha(M))$ and $g(M) = g(\sigma_\alpha(M))$, respectively, whose properties will be discussed in § 4.

It is also noted that there are many embeddings similar to g. For example, for a lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Delta$, we define the rational number

$$g'(\mathbf{x}) = n + \frac{x_2}{(n+1)^{n-1}} + \cdots + \frac{x_n}{n+1}.$$

By convention, we have $g'(\emptyset) = 0$ and $g'(\mathbf{0}) = 1$. The following embedding result is essentially a consequence of Theorem 3.1 and observed earlier in [8] (, although the Delta set was taken as a smaller set).

Corollary 3.3. The map $\mathbf{x} \mapsto g'(\mathbf{x})$ induces an embedding

$$g' : \Delta \to [0, +\infty)_{\mathbb{Q}}$$

such that for every $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \Delta$ with $n \geqq 2$ we have the following properties (1)-(3):

(1) $|g'(\mathbf{x}) - n| < \frac{1}{2}$.

(2) The lattice point $\mathbf{x} \in \Delta$ is reconstructed from the value of $g'(\mathbf{x})$.

(3) There are only finitely many $\mathbf{x} \in \Delta$ with

$$g'(\mathbf{x}) \in (n - \frac{1}{2}, n + \frac{1}{2})_{\mathbb{Q}}.$$

In fact, this corollary is shown by an analogous argument of Theorem 3.1 taking a lattice point $\mathbf{x}$ of length n as a lattice point $(\mathbf{x}, 0)$ of length $n+1$. Our argument also goes well by using Corollary 3.2, but there is a demerit that the denominator of the rational value becomes further large.

In the forthcoming paper [13], a joint work with T. Tayama, a subset of the Delta set Δ, called the *ADelta set* $A\Delta$ which is different from the PDelta set $P\Delta$ discussed here, is discussed as a complex number version of this paper by representing every lattice point of $A\Delta$ in the complex number plane with norm smaller than or equal to $\frac{1}{2}$.

4. Properties of the characteristic genus of a 3-manifold

Table 4.1: The characteristic genera of 3-manifolds with lengths up to 7

M	$\mathbf{x}$	g
$M_{0,1} = \chi(\phi, 0) = S^3$	ϕ	0
$M_{1,1} = \chi(O, 0) = S^1 \times S^2$	$\mathbf{0}$	1
$M_{3,1} = \chi(3_1, 0)$	1^3	$3 + \frac{4}{9} = 3.44444444\ldots$
$M_{4,1} = \chi(4_1^2, 0)$	1^4	$4 + \frac{21}{64} = 4.328125$
$M_{4,2} = \chi(4_1, 0)$	$(1, -2, 1, -2)$	$4 - \frac{15}{32} = 3.53125$
$M_{5,1} = \chi(5_1, 0)$	1^5	$5 + \frac{156}{625} = 5.\ldots$
$M_{5,2} = \chi(5_1^2, 0)$	$(1^2, -2, 1, -2)$	$5 - \frac{234}{625} = 4.\ldots$
$M_{6,1} = \chi(6_1^2, 0)$	1^6	$6 + \frac{1555}{7776} = 6.199974279$
$M_{6,2} = \chi(5_2, 0)$	$(1^3, 2, -1, 2)$	$6 + \frac{2455}{7776} = 6.31571502\ldots$
$M_{6,3} = \chi(6_2, 0)$	$(1^3, -2, 1, -2)$	$6 - \frac{2441}{7776} = 5.68608539\ldots$
$M_{6,4} = \chi(6_3^3, 0)$	$(1^2, 2, 1^2, 2)$	$6 + \frac{2857}{7776} = 6.367412551\ldots$
$M_{6,5} = \chi(6_1^3, 0)$	$(1^2, -2, 1^2, -2)$	$6 - \frac{2351}{7776} = 5.697659465\ldots$
$M_{6,6} = \chi(6_3, 0)$	$(1^2, -2, 1, -2^2)$	$6 - \frac{2999}{7776} = 5.614326131\ldots$
$M_{6,7} = \chi(6_2^3, 0)$	$(1, -2, 1, -2, 1, -2)$	$6 - \frac{611}{1944} = 5.685699588\ldots$
$M_{6,8} = \chi(6_3^2, 0)$	$(1, -2, 1, 3, -2, 3)$	$6 + \frac{223}{486} = 6.458847736\ldots$
$M_{7,1} = \chi(7_1, 0)$	1^7	$7 + \frac{19608}{117649} = 7.16666525$
$M_{7,2} = \chi(6_2^2, 0)$	$(1^4, 2, -1, 2)$	$7 + \frac{31956}{117649} = 7.271621518\ldots$
$M_{7,3} = \chi(7_1^2, 0)$	$(1^4, -2, 1, -2)$	$7 - \frac{31842}{117649} = 6.729347465\ldots$
$M_{7,4} = \chi(7_4^2, 0)$	$(1^3, -2, 1^2, -2)$	$7 - \frac{30960}{117649} = 6.736844342\ldots$
$M_{7,5} = \chi(7_2^2, 0)$	$(1^3, -2, 1, -2^2)$	$7 - \frac{38163}{117649} = 6.675619852\ldots$
$M_{7,6} = \chi(7_5^2, 0)$	$(1^2, -2, 1^2, -2^2)$	$7 - \frac{38037}{117649} = 6.676690834\ldots$
$M_{7,7} = \chi(7_6^2, 0)$	$(1^2, -2, 1, -2, 1, -2)$	$7 - \frac{31863}{117649} = 6.729168968\ldots$
$M_{7,8} = \chi(6_1, 0)$	$(1^2, 2, -1, -3, 2, -3)$	$7 - \frac{46682}{117649} = 6.603209548\ldots$
$M_{7,9} = \chi(7_6, 0)$	$(1^2, -2, 1, 3, -2, 3)$	$7 + \frac{46684}{117649} = 7.396807452\ldots$
$M_{7,10} = \chi(7_7, 0)$	$(1, -2, 1, -2, 3, -2, 3)$	$7 + \frac{46555}{117649} = 7.39571097\ldots$
$M_{7,11} = \chi(7_1^3, 0)$	$(1, -2, 1, 3, -2^2, 3)$	$7 + \frac{45085}{117649} = 7.383216176\ldots$

By the classification of [5], if $\ell(M) = 1, 2$, then we have $M = S^1 \times S^2, S^3$, respectively. The reason why S^3 occurs by $\ell(M) = 2$ is because we take S^3 as the 0-surgery manifold of S^3 along the Hopf link 2_1^2 and we have $\sigma_\alpha(S^3) = 1^2$. However, we can also take S^3 as the 3-manifold without 0-surgery of S^3 along a link. This is the reason why the empty lattice point $\emptyset \in P\Delta \subset \mathbb{X}$ of length 0 and the empty knot $\phi \in \mathbb{L}^p$ with bridge index 0 are introduced.We assume

$$\alpha(S^3) = \phi,\ \sigma_\alpha(S^3) = \emptyset,\ \ell(\emptyset) = 0,\ g(\emptyset) = 0,$$

so that $g(S^3) = 0$. Also, we have the group invariant $\pi_\alpha(S^3) = \{1\}$ by introducing the trivial group $\{1\}$ to the set $\mathbb{G}$ of link groups. Under this consideration, *there is no* 3-*manifold* $M \in \mathbb{M}$ *with* $\ell(M) = 2$. Since $\sigma_\alpha(M) \subset P\Delta$ and the n-fragment of $P\Delta$ for every n is a finite set, there are only finitely many 3-manifolds with length n for every $n \geqq 0$. According to the canonical well-order of $\mathbb{X}$, the 3-manifolds of length $n \geqq 1$ are enumerated as follows:

$$M_{n,1} < M_{n,2} < \cdots < M_{n,m_n}$$

for a non-negative integer m_n depending only on n. By the introduction of the empty knot $\phi \in \mathbb{L}^p$, we put $M_{0,1} = S^3$. By [5], we reconstruct from the lattice point $\sigma_\alpha(M_{n,i})$ the link $\alpha(M_{n,i}) \in \mathbb{L}^p$, the group $\pi_\alpha(M_{n,i}) \in \mathbb{G}$ and the 3-manifold $M_{n,i}$ itself. By (2) of Theorem 3.1, we reconstruct the lattice point $\sigma_\alpha(M_{n,i})$ from the characteristic genus $g(M_{n,i})$, so that we can construct from $g(M_{n,i})$ the lattice point $\sigma_\alpha(M_{n,i})$, the link $\alpha(M_{n,i})$, the group $\pi_\alpha(M_{n,i})$ and the 3-manifold $M_{n,i}$ itself.

In [KTB] the lattice points of the 3-manifolds $M_{n,i}$ together with the geometric structures for all $n \leqq 10$ are listed. In the following table, the characteristic genera $g(M_{n,i})$ for all $n \leqq 7$ are given together with the data of the lattice point $\sigma_\alpha(M_{n,i})$ and the link $\alpha(M_{n,i})$ identified with a knot or a link in D. Rolfsen's table [20], where it is noted that there is no 3-manifold of length 2 by the reason stated above and at this point the table is different from the tables of [5, 11, 12, 14].

For every 3-manifold $M \in \mathbb{M}$ with $M \neq S^3, S^1 \times S^3$, we have $\ell(M) \geqq 3$. Every 3-manifold $M \in \mathbb{M}$ has a Heegaard splitting, i.e., a union of two handlebodies by pasting along the boundaries. The Heegaard genus, $g_h(M)$ of M is the minimum of the genera of such handlebodies. The following lemma gives a relationship between a bridge presentation of a link $L \in \mathbb{L}$ (see [3] for an explanation of bridge presentation) and Heegaard splittings of the Dehn surgery manifolds along L.

Lemma 4.2. Let a link $L \in \mathbb{L}$ have a g-bridge presentation. Then every Dehn surgery manifold M of S^3 along L admits a Heegaard splitting of genus g.

Proof. Since S^3 is a union of two 3-balls B, B' pasting along the boundary spheres such that $T = L \cap B$ and $T' = L \cap B'$ are trivial tangles of g proper arcs in B and B', respectively. Let $N(T)$ be a tubular neighborhood of T in B, $V = \mathrm{cl}(B \setminus N(T))$, and $V' = B' \cup N(T)$. By construction, V and V' are handlebodies of genus g and forms a Heegaard splitting of S^3. To complete the proof, it suffices to show that the Dehn surgery from S^3 to M along L just changes V' into another handlebody V'', so that V and V'' forms a Heegaard splitting of M of genus g. Since T' is a trivial tangle in B' of g proper arcs, there are $g-1$ proper disks D_i $(i = 1, 2, \ldots, g-1)$ in B' which split B' into a 3-manifold regarded as a tubular neighborhood $N(T')$ of T' in B'. Then the union $N(L) = N(T) \cup N(T')$ is regarded as a tubular neighborhood of L in S^3. The Dehn surgery from S^3 to M along L just changes $N(L)$ into the union of solid tori obtained from $N(L)$ by the Dehn surgery without changing the boundary $\partial N(L)$. Thus, we obtain the desired handlebody V'' by pasting along the disks corresponding to D_i $(i = 1, 2, \ldots, g-1)$. □

Let $g_b(M)$ and $g_{br}(M)$ denote respectively the *bridge genus* and the *braid genus* of M, namely the minimal bridge index and the minimal braid index for links whose 0-surgery manifolds are M. We define $g_b(S^3) = g_{br}(S^3) = 0$ by considering that S^3 is obtained from S^3 by the 0-surgery along the empty knot ϕ. The 3-manifold M with $\ell(M) \geqq 3$ is *ending-positive* or *ending-negative*, respectively, according to whether $\sigma_\alpha(M)$ is ending-positive or ending-negative. Then we have the following lemma:

Lemma 4.3. For every $M \in \mathbb{M}$ with $\ell(M) \geqq 3$, we have

$$2g_h(M) - 2 \leqq 2g_b(M) - 2 \leqq 2g_{br}(M) - 2 \leqq \ell(M) < g(M) + \mathrm{end}(M),$$

where $\mathrm{end}(M)$ is 0 or $\frac{1}{2}$, respectively, according to whether M is ending-positive or ending-negative.

Proof. By Lemmas 2.3 and 4.2, we have

$$g_h(M) \leqq g_b(M) \leqq g_{br}(M) \leqq \frac{\ell(M)}{2} + 1.$$

By Theorem 3.1 (1), according to whether M is ending-positive or ending-negative, the inequality $\ell(M) < g(M)$ or $\ell(M) < g(M) + \frac{1}{2}$ holds, respectively, from which the result follows. □

We show the following theorem:

Theorem 4.4. The characteristic genus $g(M)$ of every $M \in \mathbb{M}$ is a characteristic invariant defined for $\mathbb{M}$ such that

$$g_h(S^3) = g_b(S^3) = g_{br}(S^3) = g(S^3) = \ell(S^3) = 0,$$
$$g_h(S^1 \times S^3) = g_b(S^1 \times S^3) = g_{br}(S^1 \times S^3) = g(S^1 \times S^3) = \ell(S^1 \times S^3) = 1$$

and every $M \in \mathbb{M}$ with$M \neq S^3, S^1 \times S^3$ has the following properties:

(1) The 3-manifold M itself, the lattice point $\sigma_\alpha(M)$, the link $\alpha(M)$ and the group $\pi_\alpha(M)$ are reconstructed from the value of $g(M)$.
(2) According to whether M is ending-positive or ending-negative, the characteristic genus $g(M)$ belongs to $(n, n+\frac{1}{2})_{\mathbb{Q}}$ or $(n-\frac{1}{2}, n)_{\mathbb{Q}}$ for $n = \ell(M)$.
(3) There are only finitely many 3-manifolds $M \in \mathbb{M}$ such that

$$g(M) \in (n - \frac{1}{2}, n + \frac{1}{2})_{\mathbb{Q}}.$$

(4) The inequalities

$$2g_h(M) - 2 \leqq 2g_b(M) - 2 \leqq 2g_{br}(M) - 2 \leqq \ell(M) < g(M) + \mathrm{end}(M)$$

hold, where $\mathrm{end}(M)$ is 0 or $\frac{1}{2}$, respectively, according to whether M is ending-positive or ending-negative.

Proof. By definition, we have the values of S^3 and $S^1 \times S^2$.By the property of σ_α in [5] and Theorem 3.1, it is seen that $g(M)$ is a characteristic rational invariant defined for $\mathbb{M}$ and the properties (1)-(3) hold. (4) is obtained in Lemma 4.3. □

The following corollary is direct from Theorem 4.5 (3).

Corollary 4.5. For any infinite subset $\mathbb{M}' \subset \mathbb{M}$, we have

$$\sup\{\ell(M) |\, M \in \mathbb{M}'\} = +\infty.$$

For every integer $n > 1$, since there are infinitely many 3-manifolds $M \in \mathbb{M}$ with $g_{br}(M) \leqq n$, we see from Corollary 4.5 that there are lots of 3-manifolds $M \in \mathbb{M}$ such that the difference $\ell(M) - g_{br}(M)$ is sufficiently large. However, exact calculations of the invariants $g_b(M)$, $g_{br}(M)$, $\ell(M)$ for most 3-manifolds are not known and remain as an open problem. Here are some elementary examples.

Example 4.6. (1) Let $M = \chi(3_1, 0) = M_{3,1}$ for the trefoil knot 3_1. Since the braid index of 3_1 is 2 and M is not the lens space, we see from Table 4.1 that

$$g_h(M) = g_b(M) = g_{br}(M) = 2 < \frac{\ell(M)}{2} + 1 = 2.5 \text{ and } g(M) = 3 + \frac{4}{9} = 3.444\ldots.$$

(2) Let $M = \chi(4_1^2, 0) = M_{4,1}$ for the $(2,4)$-torus link 4_1^2. Since the braid index of 4_1^2 is 2 and the first integral homology $H_1(M)$ has exactly 2 generators, we see from Table 4.1 that

$$g_h(M) = g_b(M) = g_{br}(M) = 2 < \frac{\ell(M)}{2} + 1 = 3 \text{ and } g(M) = 4 + \frac{21}{64} = 4.328\ldots.$$

(3) Let $M = \chi(4_1, 0) = M_{4,2}$ for the figure eight knot 4_1. Since the bridge index of 4_1 is 2 and M is not any lens space, we see that $g_h(M) = g_b(M) = 2$. If M is obtained from a knot or link of braid index 2, then M would be obtained from a $(2k+1)$-half-twist knot $K(k)$ by 0-surgery. However, this is impossible because the Alexander polynomial of the homology handles M and $M(k) = \chi(K(k), 0)$ are

$$A_M(t) = t^2 - 3t + 1, \quad A_{M(k)} = \frac{t^{2k+1} + 1}{t + 1}$$

and they are distinct. These results and Table 4.1 mean that

$$g_h(M) = g_b(M) = 2 < g_{br}(M) = \frac{\ell(M)}{2} + 1 = 3 < g(M) = 4 - \frac{15}{32} = 3.531\ldots.$$

We note here that the bridge genus behaves differently from the Heegaard genus, although $g_h(M) = g_b(M)$ in Example 4.6. For example, if M is a lens space except S^3 and $S^1 \times S^2$, then we have $g_b(M) \geqq 3$ whereas $g_h(M) = 1$. In fact, the first homology $H_1(M)$ is a non-trivial finite cyclic group. Onthe other hand, if $1 \leqq g_b(M) \leqq 2$, then $H_1(M)$ would be isomorphic to the infinite cyclic group $\mathbb{Z}$ or a direct double $\mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/m\mathbb{Z}$ for some $m \geqq 0$, which is a contradiction. Concretly, the pro—ective 3-space $M = P^3$ has $\sigma_\alpha(M) = (1^2, 2, 1^2, 2)$ (see [5, 14]) and hence $g_b(M) = 3$. By developing a similar consideration, S. Okazaki[19] has observed a linear independence on the Heegaard genus $g_h(M)$, the bridge genus $g_b(M)$ and the braid genus $g_{br}(M)$.

5. Constructing a characteristic smooth real function defined for the PDelta set

A *PDelta subset* is a subset $\mathbb{S}$ of the PDelta set $P\Delta$ containing the lattice points $\emptyset$ and $\mathbf{0}$.[3] Let a and t be real numbers such that either $-1 \leqq a \leqq 1$ and $-1 < t < 1$ or $-1 < a < 1$ and $-1 \leqq t \leqq 1$. Then the linear fraction

$$B(t;a) = \frac{t-a}{1-at}$$

is considered. If $|t| < 1$ and $|a| < 1$, then $|B(t;a)| < 1$, because we have

$$1 - |B(t;a)|^2 = \frac{(1-t^2)(1-a^2)}{(1-at)^2}.$$

If $|a| = 1$ or $|t| = 1$, then it is easily checked that $|B(t;a)| = 1$. In fact, we have $B(t;\pm1) = B(\mp1, a) = \mp1$.

Noting that the decimal torsions of $\emptyset$, $\mathbf{0}$ and 1^2 are not definite values as it is explained in Remark 3.2, we put the following definition for any $\mathbf{x} \in P\Delta$:

$$G_{\mathbf{x}}(t) = \begin{cases} B(t;\tau(\mathbf{x})) & (\ell(\mathbf{x}) \geqq 3) \\ B(t;1) = -1 & (\mathbf{x} = 1^2) \\ B(t;-1) = 1 & (\mathbf{x} = \emptyset,\, \mathbf{0}) \end{cases}$$

For every n-fragment $\mathbb{S}^{(n)}$ of a PDelta subset $\mathbb{S} \subset P\Delta$, the function

$$G^{(n)}_{\mathbb{S}}(t) = \prod_{\mathbf{x} \in \mathbb{S}^{(n)}} G_{\mathbf{x}}(t)$$

[3] This condition is imposed for simplicity.

is called a finite *Blaschke product*[4] whose zero's are precisely the decimal torsions $\tau(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{S}^{(n)}$ except $\emptyset, \mathbf{0}$ and 1^2. By the assumption of the set $\mathbb{S}$, we have

$$G_{\mathbb{S}}^{(0)}(t) = G_{\mathbb{S}}^{(1)}(t) = 1.$$

Further, according to whether the lattice point 1^2 belongs to $\mathbb{S}$ or not, we have $G_{\mathbb{S}}^{(2)}(t) = -1$ or 1, respectively. For example, when we take $\mathbb{S} = \mathbb{L}^p$, the functions $G_{\mathbb{L}^p}^{(n)}(t)$ for $n = 0, 1, 2, 3, 4, 5$ are calculated as follows:

$$\begin{aligned}
G_{\mathbb{L}^p}^{(0)}(t) &= 1, \\
G_{\mathbb{L}^p}^{(1)}(t) &= 1, \\
G_{\mathbb{L}^p}^{(2)}(t) &= -1, \\
G_{\mathbb{L}^p}^{(3)}(t) &= -G_{1^3}(t) = -B(t; \frac{4}{9}), \\
G_{\mathbb{L}^p}^{(4)}(t) &= -G_{1^3}(t)Q_{1^4}(t)G_{(1,-2,1,-2)}(t) = -B(t; \frac{4}{9})B(t; \frac{21}{64})B(t; \frac{-15}{32}), \\
G_{\mathbb{L}^p}^{(5)}(t) &= -G_{1^3}(t)G_{1^4}(t)G_{(1,-2,1,-2)}(t)G_{1^5}(t)G_{(1^2,-2,1,-2)}(t) \\
&= -B(t; \frac{4}{9})B(t; \frac{21}{64})B(t; \frac{-15}{32})B(t; \frac{156}{625})B(t; \frac{-234}{625}).
\end{aligned}$$

We obtain the following theorem.

Theorem 5.1. For every PDelta subset $\mathbb{S}$, the series function

$$G_{\mathbb{S}}(t) = \sum_{n=0}^{+\infty} G_{\mathbb{S}}^{(n)}(t) t^n$$

is a smooth real function defined on the interval $(-1, 1)$ which is a characteristic invariant defined for the set $\mathbb{S}$.

Proof. Since $|G_{\mathbb{S}}^{(n)}(t)| \leqq 1$ for any n, we have

$$|G_{\mathbb{S}}(t)| \leqq \sum_{n=0}^{+\infty} |t|^n = \frac{1}{1 - |t|}.$$

This means that the series $G_{\mathbb{S}}(t)$ defined on $(-1, 1)$ is uniformly convergent in the wide sense. Using that the function $G_{\mathbb{S}}^{(n)}(t)$ $(t \in (-1, 1))$ is uniformly convergent in the wide sense, we see from the Weierstrass double series theorem that the series

[4] See Blaschke [2]. The author thanks to K. Sakan for suggesting the Blaschke product.

function $G_{\mathbb{S}}(t)$ is a smooth real function defined on $(-1,1)$. To see that the function $G_{\mathbb{S}}(t)$ is characteristic for $\mathbb{S}$, it suffices to see by induction on $n \geqq 2$ that the set of the decimal torsions $\tau(\mathbf{x})$ for all lattice points $\mathbf{x} \in \mathbb{S}^{(n)}$ except $\emptyset$, $\mathbf{0}$ is determined by the function $G_{\mathbb{S}}(t)$. According to whether 1^2 is in $\mathbb{S}$ or not, the second derivative $\frac{d^2}{dt^2}G_{\mathbb{S}}(0)$ is -2 or 2, respectively. Thus, $\mathbb{S}^{(2)}$ is determined by the function $G_{\mathbb{S}}(t)$. Assume that all the lattice points of $\mathbb{S}^{(n-1)}$ $(n-1 \geqq 2)$ are determined by the function $G_{\mathbb{S}}(t)$. Let

$$\bar{G}_{\mathbb{S}}^{(n)}(t) = G_{\mathbb{S}}(t) - \sum_{i=0}^{n-1} G_{\mathbb{S}}^{(i)}(t)t^i.$$

The function $\bar{G}_{\mathbb{S}}^{(n)}(t)$ has the following splitting form:

$$\bar{G}_{\mathbb{S}}^{(n)}(t) = G_{\mathbb{S}}^{(n)}(t) \cdot \tilde{G}(t) \cdot t^n,$$

where

$$\tilde{G}(t) = 1 + \tilde{G}_{\mathbb{S}}^{(n+1)}(t)t + \tilde{G}_{\mathbb{S}}^{(n+2)}(t)t^2 + \tilde{G}_{\mathbb{S}}^{(n+3)}(t)t^3 + \dots$$

for some finite Blaschke products $\tilde{G}_{\mathbb{S}}^{(n+i)}(t)$ with

$$G_{\mathbb{S}}^{(n)}(t) \cdot \tilde{G}_{\mathbb{S}}^{(n+i)}(t) = G_{\mathbb{S}}^{(n+i)}(t)$$

for all i $(i = 1, 2, 3, \dots)$. We show that the function $\tilde{G}(t)$ has no zero's in the interval $(-\frac{1}{2}, \frac{1}{2})$. In fact, we have

$$|\tilde{G}(t)| \geqq 1 - \sum_{i=1}^{+\infty} |t|^i = \frac{1-2|t|}{1-|t|} > 0$$

for any t with $|t| < \frac{1}{2}$. This means that the decimal torsions $\tau(\mathbf{x})$ for all lattice points $\mathbf{x} \in \mathbb{S}^{(n)}$ except $\emptyset$, $\mathbf{0}$ and 1^2 are characterized by the zero's of the function $\bar{G}_{\mathbb{S}}^{(n)}(t)$ in the interval $(-\frac{1}{2}, \frac{1}{2}) \setminus \{0\}$. □

It is noted that the series function $G_{\mathbb{S}}(t)$ does not converge for $t = \pm 1$. This is because

$$\lim_{n \to +\infty} |G_{\mathbb{S}}^{(n)}(\pm 1) \cdot (\pm 1)^n| = 1 \neq 0.$$

The function $G_{\mathbb{S}}(t)$ is called the *characteristic genus function* defined for the PDelta subset $\mathbb{S}$. For example, for $\mathbb{S} = \{\emptyset, \mathbf{0}\}$, we have

$$G_{\mathbb{S}}(t) = 1 + t + t^2 + t^3 + \cdots = \frac{1}{1-t}.$$

For $\mathbb{S} = \{\emptyset, \mathbf{0}, 1^2\}$, we have

$$G_{\mathbb{S}}(t) = 1 + t - (t^2 + t^3 + t^4 + \dots) = 1 + t - \frac{t^2}{1-t}.$$

For a finite set $\mathbb{S}$ with the maximal length n,

$$G_{\mathbb{S}}(t) = \sum_{i=0}^{n-1} G_{\mathbb{S}}^{(i)}(t) t^i + G_{\mathbb{S}}^{(n)}(t) \frac{t^n}{1-t}.$$

For the subset $\mathbb{S} = \sigma(\mathbb{L}^p)$, we denote $G_{\mathbb{S}}^{(n)}(t)$ and $G_{\mathbb{S}}(t)$ by $G_{\mathbb{L}^p}^{(n)}(t)$ and $G_{\mathbb{L}^p}(t)$, respectively. The following corollary is direct from Theorem 5.1.

Corollary 5.2. The series function

$$\begin{aligned} G_{\mathbb{L}^p}(t) &= \sum_{n=0}^{+\infty} G_{\mathbb{L}^p}^{(n)}(t) t^n \\ &= 1 + t - t^2 - B(t, \frac{4}{9})t^3 - B(t, \frac{4}{9})B(t, \frac{21}{64})B(t, \frac{-15}{32})t^4 \\ &\qquad - B(t, \frac{4}{9})B(t, \frac{21}{64})B(t, \frac{-15}{32})B(t, \frac{156}{625})B(t, \frac{-234}{625})t^5 + \dots \end{aligned}$$

is a smooth real function defined on the interval (-1, 1) which is a characteristic invariant defined for the prime link set $\mathbb{L}^p$.

For example, let $\mathbb{L}(2, *)$ be the set of $(2, n)$-torus links regarding the $(2, 0)$-torus link as the empty knot ϕ. Since

$$\sigma(\mathbb{L}(2, *)) = \{1^n | \, n = 0, 1, 2, 3, \dots\},$$

where $1^0 = \phi$, $1 = 0$ and $\tau(1^n) = \frac{1}{n-1} - \frac{1}{n^n - n^{n-1}}$ for $n \geqq 3$, we have:

$$G_{\mathbb{L}(2,*)}(t) = 1 + t - t^2 - \sum_{n=3}^{+\infty} \left(\prod_{k=3}^{n} B\left(t, \frac{1}{k-1} - \frac{1}{k^k - k^{k-1}}\right)\right) t^n.$$

For the subset $\mathbb{S} = \sigma_\alpha(\mathbb{M})$, we denote $G_{\mathbb{S}}^{(n)}(t)$ and $G_{\mathbb{S}}(t)$ by $G_{\mathbb{M}}^{(n)}(t)$ and $G_{\mathbb{M}}(t)$, respectively. Noting that the lattice point 1^2 is excluded from $\sigma(\mathbb{M})$ (by the reason that the empty lattice point $\emptyset$ is introduced), we have the following corollary obtained from Theorem 5.1.

Corollary 5.3 The series function

$$\begin{aligned} G_{\mathbb{M}}(t) &= \sum_{n=0}^{+\infty} G_{\mathbb{M}}^{(n)}(t)t^n \\ &= 1 + t + t^2 + B(t; \frac{4}{9})t^3 + B(t; \frac{4}{9})B(t; \frac{21}{64})B(t; \frac{-15}{32})t^4 \\ &\qquad + B(t; \frac{4}{9})B(t; \frac{21}{64})B(t; \frac{-15}{32})B(t; \frac{156}{625})B(t, \frac{-234}{625})t^5 + \dots. \end{aligned}$$

is a smooth real function defined on the interval $(-1, 1)$ which is a characteristic invariant defined for the 3-manifold set $\mathbb{M}$.

Acknowledgements. This work was supported by JSPS KAKENHI Grant Number 24244005.

References

[1] J. S. Birman, Braids, links, and mapping class groups, Ann. Math. Studies, 82 (1974), Princeton Univ. Press.

[2] W. Blaschke, Eine Erweiterung des Satzes von Vitali über Folgen analytischer Funktionen, Berichte Math.-Phys. Kl., Sächs. Gesell. der Wiss. Leipzig, 67 (1915), 194-200.

[3] A. Kawauchi, A survey of knot theory, (1996), Birkhäuser.

[4] A. Kawauchi, Topological imitation of a colored link with the same Dehn surgery manifold, in: Proceedings of Topology in Matsue 2002, Topology Appl., 146-147 (2005), 67-82.

[5] A. Kawauchi, A tabulation of 3-manifolds via Dehn surgery, Boletin de la Sociedad Matematica Mexicana (3), 10 (2004), 279-304.

[6] A. Kawauchi and I. Tayama, Enumerating the prime knots and links by a canonical order, in: Proc. 1st East Asian School of Knots, Links, and Related Topics (Seoul, Jan. 2004), (2004), 307-316. (http://www.sci.osaka-cu.ac.jp/~kawauchi/index.htm)

[7] A. Kawauchi and I. Tayama, Enumerating the exteriors of prime links by a canonical order, in: Proc. Second East Asian School of Knots, Links, and Related Topics in Geometric Topology (Darlian, Aug. 2005), (2005), 269-277. (http://www.sci.osaka-cu.ac.jp/~kawauchi/index.htm)

[8] A. Kawauchi and I. Tayama, Enumerating prime links by a canonical order, Journal of Knot Theory and Its Ramifications, 15 (2006), 217-237.

[9] A. Kawauchi and I. Tayama, Enumerating 3-manifolds by a canonical order, Intelligence of low dimensional topology 2006, Series on Knots and Everything, 40 (2007), 165-172.

[10] A. Kawauchi and I. Tayama, Enumerating prime link exteriors with lengths up to 10 by a canonical order, Proceedings of the joint conference of Intelligence of Low Dimensional Topology 2008 and the Extended KOOK Seminar (Osaka, Oct. 2008), (2008), 135-143. (http://www.sci.osaka-cu.ac.jp/ kawauchi/index.htm)

[11] A. Kawauchi and I. Tayama, Enumerating homology spheres with lengths up to 10 by a canonical order, Proceedings of Intelligence of Low-Dimensional Topology 2009 in honor of Professor Kunio Murasugi's 80th birthday (Osaka, Nov. 2009), (2009), 83-92. (http://www.sci.osaka-cu.ac.jp/ kawauchi/index.htm)

[12] A. Kawauchi and I. Tayama, Enumerating 3-manifolds with lengths up to 9 by a canonical order, Topology Appl., 157 (2010), 261-268.

[13] A. Kawauchi and I. Tayama, Representing 3-manifolds in the complex number plane, preprint. (http://www.sci.osaka-cu.ac.jp/˜kawauchi/index.htm)

[14] A. Kawauchi, I. Tayama and B. Burton, Tabulation of 3-manifolds of lengths up to 10, Proceedings of International Conference on Topology and Geometry 2013, joint with the 6th Japan-Mexico Topology Symposium, Topology and its Applications (to appear). http://dx.doi.org/10.1016/j.topol.2015.05.036

[15] B. von Kerékjártó, Vorlesungen über Topologie, Spinger, Berlin, 1923.

[16] R. Kirby, A calculus for framed links in S^3, Invent. Math., 45(1978), 35-56.

[17] J. Milnor and W. Thurston, Characteristic numbers of 3-manifolds, Enseignment Math., 23(1977), 249-254.

[18] Y. Nakagawa, A family of integer-valued complete invariants of oriented knot types, J. Knot Theory Ramifications, 10(2001), 1160-1199.

[19] S. Okazaki, On Heegaard genus, bridge genus and braid genus for a 3-manifold, J. Knot Theory Ramifications, 20 (2011), 1217-1227. DOI: 10.1142/S0218216511009145

[20] D. Rolfsen, Knots and links, (1976), Publish or Perish.