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ON COEFFICIENT POLYNOMIALS OF THE SKEIN POLYNOMIAL OF AN ORIENTED LINK

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0. Introduction

For an oriented link L, we denote by $P_L(\ell, m)$ the skein (= two variable Jones, HOMFLY or FLYPMOTH) polynomial (cf. [5]) in the convention of Lickorish/Millett [14]. Let r denote the component number of L. We consider

$$P_{\#}(L;\ell,m) = (\ell m)^{r-1}P_L(\ell,m).$$

Then $P_{\#}(L;\ell,m)$ can be written as

$$\sum_{n=0}^{+\infty} p_{2n}(L;\ell)m^{2n}.$$

where $p_{2n}(L;\ell)$ is a Laurent polynomial in ℓ^2 and 0 except a finite number of n. We denote $-\ell^2$ and $-m^2$ by x and y, respectively, and then $P_{\#}(L;\ell,m)$ and $p_{2n}(L;\ell)$ by $C_{\#}(L;x,y)$ and $c_n(L;x)(-x)^n$, respectively. Clearly, $c_n(L;x)$ is a Laurent polynomial in x and we have

$$C_{\#}(L;x,y) = \sum_{n=0}^{+\infty} c_n(L;x)(xy)^n.$$

We define $c_n(L;x) = 0$ for $n < 0$. We call the polynomial $c_n(L;x)$ the *n-th coefficient polynomial* of the skein polynomial $P_L(L;\ell,m)$ (or simply of the link L). In this paper, we investigate these coefficient polynomials $c_n(L;x)$, $n \in Z$. In particular, we consider the following integral invariants of L:

$$\begin{aligned}
\tau_n(L) &= c_n(L;1),\\
\rho_n(L) &= nc_n(L;1) + c'_n(L;1),\\
\tau^*_n(L) &= c_n(L;-1),\\
\rho^*_n(L) &= -nc_n(L;-1) + c'_n(L;-1),
\end{aligned}$$

where $c'_n(L;\pm1)$ denotes the derivative of $c_n(L;x)$ at $x = \pm1$. Clearly, $\tau_n(L) \equiv \tau^*_n(L)$ (mod 2) and $\rho_n(L) \equiv \rho^*_n(L)$ (mod 2). It turns out that among the $c_n(L;x)$'s,

$\tau_n(L), \rho_n(L)$ give informations on local relations, but $\tau_n^*(L), \rho_n^*(L)$ give informations on global relations. We shall observe that $\tau_0(L) = \delta_{1,r}$, $\rho_0(L) = \delta_{2,r}$, where $\delta_{i,j}$ denotes the Kronecker's delta, and for $n > 0$, $\tau_n(L)$ and $\rho_n(L)$ are determined by the $c_k(L; x)$'s for all k with $k \leq n-1$. When $r = 1$, that is, L is a knot K, we shall show that the conditions $\tau_0(K) = 1$ and $\rho_0(K) = 0$ characterize the zeroth coefficient polynomial $c_0(K; x)$. As a consequence, we can characterize the zeroth coefficient polynomial $c_0(L; x)$ for all links L. For any knot K, we have

$$\tau_0^*(K) - 1 \equiv 0 \pmod 4.$$

It is shown that the Arf invariant of K is trivial, that is, $\tau_1(K) \equiv 0 \pmod 2$ if and only if

$$\tau_0^*(K) - 1 \equiv 0 \pmod 8.$$

Further, it is shown that if the Z_2-Alexander polynomial of K is trivial, that is, $\tau_n(K) \equiv 0 \pmod 2$ for all $n \geq 1$, then

$$\tau_0^*(K) - 1 \equiv 0 \pmod{16}.$$

A *canonical Seifert surface* for a link L is a Seifert surface of L obtained from a link diagram of L by Seifert's algorithm. The *canonical genus* of L, denoted by $g_c(L)$, is the minimal genus of connected canonical Seifert surfaces for L. The *genus* of L, denoted by $g(L)$, is the minimal genus of connected Seifert surfaces for L. Clearly, $g_c(L) \geq g(L)$. It is known that $g_c(L) = g(L)$ when L is an alternating link (cf. Murasugi [22], [3, p.228]). We see from a result of Moriah [17] considering the free genus that for any positive integer s, there exists a knot (in fact, a *twist* knot along a knot) K' with $g_c(K') - g(K') \geq s$. By a technical reason, the Alexander polynomial of the knot K' considered by Moriah must be non-trivial. In this paper, we shall construct a knot K' with trivial Alexander polynomial such that $g_c(K') - g(K') = 2s$ for any positive integer s. By a basic result, we show that if K' is any finitely many iterated (twisted or untwisted) double of a knot K with $\tau_0^*(K) \neq 1$, then $g(K') = 1$ but $g_c(K') \geq 3$.

Before concluding this introduction, we note the relation of the coefficient polynomials $c_n(L; x)$ with Conway polynomial (cf. Conway [4])

$$\nabla(L; z) = P(L; \sqrt{-1}, -\sqrt{-1}z)$$

and Jones polynomial (cf. [6])

$$V(L; t) = P(L; \sqrt{-1}t^{-1}, -\sqrt{-1}(\sqrt{t} - \sqrt{t^{-1}})).$$

Letting

$$\nabla_{\#}(L; z) = P_{\#}(L; \sqrt{-1}, -\sqrt{-1}z) = z^{r-1}\nabla(L; z)$$

and
$$V_{\#}(L;t)=P_{\#}(L;\sqrt{-1}t^{-1},-\sqrt{-1}(\sqrt{t}-\sqrt{t^{-1}})),$$
we have the following:

$$\nabla_{\#}(L;z)=\sum_{n=0}^{+\infty}c_n(L;1)z^{2n}=\sum_{n=0}^{+\infty}\tau_n(L)z^{2n}, \tag{0.1}$$

$$V_{\#}(L;t^{-1})=\sum_{n=0}^{+\infty}c_n(L;t^2)t^n(t-1)^{2n}. \tag{0.2}$$

We discuss some properties of the coefficient polynomials in §1 and τ_n-, ρ_n-invariants in §2 and τ_n^*-, ρ_n^*-invariants in §4. In §3, we characterize the zeroth coefficient polynomial. In Appendix A, we show that a $Z[Z]$-homology equivalence for links induces an S-equivalence for Seifert matrices. In Appendix B, we establish a natural relation between the homologies of the infinite cyclic covering and any finite cyclic covering of a link.

1. Properties of the coefficient polynomials

Let L_+, L_- and L_0 be oriented links, identical except near one point being  ,  ,  , respectively. Then we have the identity
$$\ell P(L_+;\ell,m)+\ell^{-1}P(L_-;\ell,m)+mP(L_0;\ell,m)=0.$$
This is equivalent to the identity
$$-\ell^2P_{\#}(L_+;\ell,m)-P_{\#}(L_-;\ell,m)=(\ell^2m^2)^{\delta}P_{\#}(L_0;\ell,m)$$
with $\delta=(r_+-r_0+1)/2(=0$ or $1)$, where r_+, r_0 denote the component numbers of L_+,L_0, respectively. Since $P_{\#}(O^r;\ell,m)=(-1)^{r-1}(\ell^2+1)^{r-1}$ for a trivial r-component link O^r, we see that $P_{\#}(L;\ell,m)$ can be written as
$$\sum_{n=0}^{+\infty}p_{2n}(L;\ell)m^{2n}$$
for Laurent polynomials $p_{2n}(L;\ell)$ in ℓ^2 which are 0 except a finite number of n (cf. [14]). Writing $-\ell^2,-m^2,P_{\#}(L;\ell,m)$ and $p_{2n}(L;\ell)$ as $x,y,C_{\#}(L+x,y)$ and $c_n(L;x)(-x)^n$, respectively, we have
$$C_{\#}(L;x,y)=\sum_{n=0}^{+\infty}c_n(L;x)(xy)^n,$$
$$xC_{\#}(L_+;x,y)-C_{\#}(L_-;x,y)=(xy)^{\delta}C_{\#}(L_0;x,y),$$
$$C_{\#}(O^r;x,y)=(x-1)^{r-1}.$$

The existence of the skein polynomial $P(L;\ell,m)$ is equivalent to saying the following:

THEOREM 1.1. *For each oriented link L, there is a set of Laurent polynomial invariants in x, $\{c_n(L;x) \mid n \in Z\}$ of L, determined uniquely by the following identities*:

(1.1.1) $$c_n(O^1;x) = \begin{cases} 0 & \text{for } n \neq 0, \\ 1 & \text{for } n = 0. \end{cases}$$

(1.1.2) $$xc_n(L_+;x) - c_n(L_-;x) = c_{n-\delta}(L_0;x) \text{ with } \delta = (r_+ - r_0 + 1)/2.$$

Since the following (1.2)–(1.5) are easy exercises on the known properties of $P(L;\ell,m)$ (cf. [5], [14]), we omit the proofs. (We can deduce them directly from Theorem 1.1.)

(1.2). *Let K_i, $i = 1,2,\dots,r$, be the components of L, and λ the total linking number of L, i.e., $\lambda = \sum_{i<j} \mathrm{Link}(K_i,K_j)$. Then*

$$c_0(L;x) = (x-1)^{r-1}x^{-\lambda}c_0(K_1;x)c_0(K_2;x)\cdots c_0(K_r;x).$$

(1.3). *Let $L_1 \circ L_2$ and $L_1\#L_2$ be a split union and a connected sum of links L_1, L_2, respectively. Then*

$$c_n(L_1 \circ L_2;x) = (x-1)c_n(L_1\#L_2;x),$$
$$c_n(L_1\#L_2;x) = \sum_{p+q=n} c_p(L_1;x)c_q(L_2;x).$$

(1.4). *Let $\overline{L}$ be the mirror image of L. Then*

$$c_n(\overline{L};x) = (-1)^{r-1}x^{r-1-2n}c_n(L;x^{-1}).$$

(1.5). *For any r-component link L, we have*

$$\sum_{n=0}^{+\infty} c_n(L;x)(1-x)^{2n} = (x-1)^{r-1}.$$

The identity in (1.5) is equivalent to the Lickorish/Millett identity $P(L;\ell,-(\ell+\ell^{-1})) = 1$. J. Przytycki proposed the identity $P(L;\ell,\ell+\ell^{-1}) = (-1)^{r-1}$. Writing it in terms of $c_n(L;x)$, we see that it is also equivalent to (1.5). For our estimate of the difference $g_c(K)-g(K)$, we shall use an inequality of Morton [18] implying the following:

(1.6) $$c_n(L;x) = 0 \quad \text{for} \quad n > g_c(L) + r - 1.$$

For any integers $g(\geq 0)$, $r(\geq 1)$, there is an r-component link L with $g_c(L) = g$ and $c_{g+r-1}(L;x) \neq 0$, showing that (1.6) is best possible. For example, let L be a connected sum of an $(r-1)$-fold connected sum of the Hopf link L_H and the g-fold connected sum of the trefoil knot $K(3_1)$. By changing the space orientation if necessary, we have

$$c_n(L_H;x) = \begin{cases} x(x-1) & \text{for } n = 0, \\ -1 & \text{for } n = 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$c_n(K(3_1);x) = \begin{cases} x(2-x) & \text{for } n = 0, \\ 1 & \text{for } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Then we see that the link L has r components and by (1.3) $c_{g+r-1}(L;x) = (-1)^{r-1}$ and $g_c(L) = g$. For this link L, we see also from (1.3) that $c_n(L;x)$ is divided by $(x-1)^{r-1-n}$ but not divided by $(x-1)^{r-n}$ for any n with $0 \leq n < r-1$. This means that the following lemma is best possible:

Lemma 1.7. *For any $r(> 1)$-component link L and any integer n with $0 \leq n < r-1, (x-1)^{r-1-n}$ divides $c_n(L;x)$.*

Proof. Regard L as L_+ or L_- at a crossing point of two components of L. Then we have $xc_n(L_+;x) - c_n(L_-;x) = c_{n-1}(L_0;x)$. By induction on $r, (x-1)^{(r-1)-1-(n-1)} = (x-1)^{r-1-n}$ divides $c_{n-1}(L_0;x)$. Hence $(x-1)^{r-1-n}$ divides $c_n(L_+;x)$ if and only if $(x-1)^{r-1-n}$ divides $c_n(L_-;x)$. This implies that $(x-1)^{r-1-n}$ divides $c_n(L;x)$ if $(x-1)^{r-1-n}$ divides $c_n(L^0;x)$ of the split union L^0 of the components, K_i, $i = 1,2,\ldots,r$, of L. By (1.3), $c_n(L^0;x) = (x-1)^{r-1}c_n(K_1\#K_2\#\cdots\#K_r;x)$. Ths completes the proof.

The reversing formula for the Jones polynomial in [16], [19] is translated by (0.2) as follows:

(1.8). *Let L' be a link obtained from a link L by reversing the orientation of a component K of L and λ be the linking number of K and $L-K$. Then*

$$\sum_{n=0}^{+\infty} c_n(L';t^2)t^n(t-1)^{2n} = t^{3\lambda}\sum_{n=0}^{+\infty} c_n(L;t^2)t^n(t-1)^{2n}.$$

2. The τ_n- and ρ_n-invariants

For a link L in S^3, let $N(L)$ be a tubular neighborhood of L in S^3, and $E(L)$ the exterior (i.e., $E(L) = S^3 - \operatorname{int} N(L)$). Let $\widetilde{E}(L)$ be the infinite cyclic covering space of $E(L)$ (associated with the epimorphism $\pi_1(E(L)) \to \langle t \rangle$ sending each meridian of L to t).

DEFINITION. A map $f : (S^3, L^*) \to (S^3, L)$ for links L^*, L is a $Z[Z]$-*homology equivalence* if all of the following (1)–(3) are satisfied:

(1) f preserves the orientations on S^3, L^* and L,
(2) For some tubular neighborhoods $N(L^*)$, $N(L)$ of L^*, L and the exteriors $E(L^*)$, $E(L)$, the restriction $f \mid N(L^*)$ gives a diffeomorphism $N(L^*) \cong N(L)$ and $f(E(L^*)) = E(L)$.
(3) The infinite cyclic covering lift $\widehat{f}_E : \widetilde{E}(L^*) \to \widetilde{E}(L)$ of $f_E = f \mid E(L^*) : E(L^*) \to E(L)$ induces a homology isomorphism $\widetilde{f}_{E*} : H_1(\widetilde{E}(L^*); Z) \cong H_1(\widetilde{E}(L); Z)$.

For example, any link imitation map discussed in [7], [8] is a $Z[Z]$-equivalence.

LEMMA 2.1.
(1) *If there is a $Z[Z]$-homology equivalence $f : (S^3, L^*) \to (S^3, L)$, then $\tau_n(L^*) = \tau_n(L)$ for all n,*
(2) *If $n < r-1$ or $n > g(L) + r - 1$, then $\tau_n(L) = 0$, so that*

$$\nabla_{\#}(L; z) = \sum_{n=r-1}^{g(L)+r-1} \tau_n(L) z^{2n}.$$

PROOF. In Appendix A, it is shown that any Seifert matrices M^*, M associated with connected Seifert surfaces for L^*, L are S-equivalent (See [3, 13.34] for the definition of S-equivalence). Let $z = t^{1/2} - t^{-1/2}$. Then

$$\begin{aligned}\nabla(L^*; z) &= \det(t^{-1/2} M^* - t^{1/2} M^{*\prime}) \\ &= \det(t^{-1/2} M - t^{1/2} M') = \nabla(L; z).\end{aligned}$$

Hence $\nabla_{\#}(L^*, z) = \nabla_{\#}(L; z)$, showing (1). We show (2). For $n < r-1$, $\tau_n(L) = 0$ follows from Lemma 1.7. Let M be a Seifert matrix associated with a connected surface of genus $g = g(L)$. Then

$$\begin{aligned}\nabla_{\#}(L; z) &= \sum_{n=0}^{+\infty} \tau_n(L)(t + t^{-1} - 2)^n \\ &= (t^{1/2} - t^{-1/2})^{r-1} \det(t^{-1/2} M - t^{1/2} M')\end{aligned}$$

$$= t^{-g-r+1}(t-1)^{r-1}\det(M - tM').$$

This implies that $\tau_n(L) = 0$ for $n > g(L) + r - 1$. This completes the proof.

The following (2.2) follows from (1.3):

$$\tau_n(L_1 \circ L_2) = 0,$$
$$\tau_n(L_1 \# L_2) = \rho_n(L_1 \circ L_2) = \sum_{p+q=n} \tau_p(L_1)\tau_q(L_2),$$
$$\rho_n(L_1 \# L_2) = \sum_{p+q=n} (\rho_p(L_1)\tau_q(L_2) + \tau_p(L_1)\rho_q(L_2)). \tag{2.2}$$

The following (2.3) follows from (1.4):

$$\tau_n(\overline{L}) = (-1)^{r-1}\tau_n(L),$$
$$\rho_n(\overline{L}) = (-1)^r \rho_n(L) + (-1)^{r-1}(r-1)\tau_n(L). \tag{2.3}$$

Let $c_n^{(k)}(L;x)$ be the k-fold derivative of $c_n(L;x)$, and $d_n^{(k)}(L) = c_n^{(k)}(L;1)/k!$. Note that $d_n^{(k)}(L)$ is an integer and $\tau_n(L) = d_n^{(0)}(L)$, and $\rho_n(L) = nd_n^{(0)}(L) + d_n^{(1)}(L)$. The following lemma shows that $\tau_0(L) = \delta_{1,r}$ and $\rho_0(L) = \delta_{2,r}$, and $\tau_n(L)$, $\rho_n(L)$ $(n > 0)$ are determined by the $c_k(L;x)$'s for all k with $k \leq n-1$.

LEMMA 2.4.

$$\tau_n(L) = -\sum_{k=0}^{n-1} d_k^{(2n-2k)}(L) + \delta_{2n+1,r},$$

$$\rho_n(L) = -\sum_{k=0}^{n-1}(nd_k^{(2n-2k)}(L) + d_k^{(2n-2k+1)}(L)) + n\delta_{2n+1,r} + \delta_{2n+2,r}.$$

PROOF. Consider the Tayler expansion

$$c_n(L;x) = \sum_{k=0}^{+\infty} d_n^{(k)}(L)(x-1)^k$$

around $x = 1$. By (1.5), we have

$$(x-1)^{r-1} = \sum_{n=0}^{+\infty}\left(\sum_{k=0}^{+\infty} d_n^{(k)}(L)(x-1)^k\right)(x-1)^{2n}$$
$$= \sum_{s=0}^{+\infty}\left(\sum_{2n+k=s} d_n^{(k)}(L)\right)(x-1)^s.$$

That is,

$$\sum_{2n+k=s} d_n^{(k)}(L) = \delta_{s,r-1} = \delta_{s+1,r} \quad (s = 0, 1, 2, \dots).$$

Hence

$$d_n^{(0)}(L) = -\sum_{k=0}^{n-1} d_k^{(2n-2k)}(L) + \delta_{2n+1,r},$$
$$d_n^{(1)}(L) = -\sum_{k=0}^{n-1} d_k^{(2n-2k+1)}(L) + \delta_{2n+2,r}.$$

Then we obtain the desired identities, completing the proof.

For a knot K, we have $\tau_0(K) = 1$, $\rho_0(K) = 0$ by Lemma 2.4. For the knot connected sum $K_1\#K_2$, we obtain from (2.2) that

$$\tau_1(K_1\#K_2) = \tau_1(K_1) + \tau_1(K_2), \quad \rho_1(K_1\#K_2) = \rho_1(K_1) + \rho_1(K_2).$$

By (2.3), we note that

$$\tau_n(\overline{K}) = \tau_n(K), \quad \rho_n(\overline{K}) = -\rho_n(K).$$

By Lemma 2.4, we have the following two identities:

$$\tau_1(K) = -c_0''(K;1)/2 = p_0''(K;\sqrt{-1})/8,$$
$$\rho_1(K) = -c_0''(K;1)/2 - c_0'''(K;1)/6.$$

The first identity was observed by Lickorish/Millett in [14]. Azuma [1] observed (without establishing this second identity) that the right hand side of the second identity is additive on connected sum operation and (-1)-multiplicative on mirror image operation. The following is obtained from (1.8) with $t = -1$ taken and Lemma 2.1(2):

(2.5). *Let L' be a link obtained from a link L by reversing the orientation of a component K of L and λ be the linking number of K and $L - K$. Then we have*

$$\sum_{n=r-1}^{g(L')+r-1} (-4)^n \tau_n(L') = (-1)^\lambda \sum_{n=r-1}^{g(L)+r-1} (-4)^n \tau_n(L).$$

3. A characterization of the zeroth coefficient polynomial

By Lemma 2.4, any knot K has $\tau_0(K) = c_0(K;1) = 1$, $\rho_0(K) = c_0'(K;1) = 0$. The following theorem shows that these conditions on $c_0(K;x)$ are complete:

THEOREM 3.1. *For any Laurent polynomial $f(x)$ with $f(1) = 1$ and $f'(1) = 0$, there exists an unknotting number one knot K with $c_0(K;x) = f(x)$.*

Lemma 2.4 also means that $\tau_1(K) = 0$ if and only if $c_0''(K;1) = 0$ for any knot K. The following corollary to Theorem 3.1 gives a characterization of $c_0(K;x)$ for all knots K with $\tau_1(K) = 0$:

COROLLARY 3.2. *For any Laurent polynomial $f(x)$ with $f(1) = 1$, $f'(1) = f''(1) = 0$, there exists an unknotting number one knot K with $\tau_1(K) = 0$ and $c_0(K;x) = f(x)$.*

One may ask whether every Laurent polynomial $f(x)$ with $f(1) = 1$, $f'(1) = f''(1) = 0$ is realizable as $c_0(K;x)$ of a knot K with trivial Alexander polynomial (i.e., $\tau_n(K) = 0$ for any $n \geq 1$), but the answer is negative because for any such knot K, it will be shown in Lemma 4.9 that $c_0(K;-1) = \tau_0^*(K) \equiv 1 \pmod{16}$. For an r-component link L, we see from (1.2) that $c_0(L;x)$ can be written as $(x-1)^{r-1}g(x)$ for some Laurent polynomial $g(x)$ with $g(1) = 1$. The following gives a characterization of $c_0(L;x)$ for $r(\geq 2)$-component links L.

COROLLARY 3.3. *For any $r \geq 2$ and any Laurent polynomial $g(x)$ with $g(1) = 1$, there exists an r-component link L with $c_0(L;x) = (x-1)^{r-1}g(x)$.*

PROOF. Let $\lambda = g'(1)$ and $f(x) = x^{-\lambda}g(x)$. Then $f(1) = 1$ and $f'(1) = 0$. By Theorem 3.1, there is a knot K with $c_0(K;x) = f(x)$. We take an r-component link L which is a union of K and $r-1$ trivial knots with total linking number $-\lambda$. By (1.2), we have $c_0(L;x) = (x-1)^{r-1}x^{\lambda}f(x) = (x-1)^{r-1}g(x)$, as desired.

COROLLARY 3.4. *For any $r \geq 1$ and any Laurent polynomial $h(x)$, there exists an r-component link L such that*

$$\sum_{n=1}^{+\infty} c_n(L;x)(x-1)^{2n-2} = \begin{cases} (x-1)^{r-2}h(x) & \text{for } r \geq 2, \\ h(x) & \text{for } r = 1. \end{cases}$$

PROOF. By Theorem 3.1 and Corollary 3.3, we have an r-component link L such that

$$c_0(L;x) = \begin{cases} (x-1)^{r-1}[1-(x-1)h(x)] & \text{for } r \geq 2, \\ 1-(x-1)^2h(x) & \text{for } r = 1. \end{cases}$$

Combining this identity with the identity (1.5), we obtain the desired identity, completing the proof.

We shall provide two lemmas to prove Theorem 3.1. For any Laurent polynomial $f(x)$, we denote the integer $-f''(1)/2$ by $d(f)$ and the Laurent polynomial degree by $\deg f$.

LEMMA 3.5. *Let $f(x)$, $g(x)$ and $h(x)$ be Laurent polynomials and d be an integer such that*

$$f(x) = xg(x) - (x-1)x^d h(x).$$

Then the following three conditions are equivalent:

(1) $f(1) = g(1) = 1,\ f'(1) = g'(1) = 0$ *and* $d = d(f) - d(g)$,

(2) $f(1) = h(1) = 1$ *and* $f'(1) = h'(1) = 0$,

(3) $g(1) = h(1) = 1$ *and* $g'(1) = h'(1) = 0$.

PROOF. These equivalences are proved by the following identities (easily obtained by taking derivatives at $x = 1$): $f(1) = g(1)$, $f'(1) = g(1) + g'(1) - h(1)$ and $d(f) = -g'(1) + d(g) + h(1)d + h'(1)$. This completes the proof

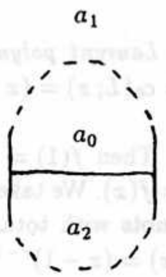

Fig. 1

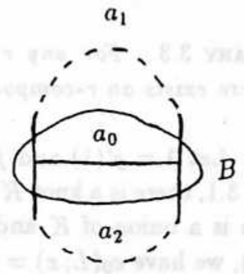

Fig. 2

LEMMA 3.6. *For any knots K', K'', K''' and any integer d, there exists a knot K such that*

$$c_0(K;x) = xc_0(K';x) - (x-1)x^d c_0(K'';x)c_0(K''';x).$$

PROOF. By a result of Kinoshita in [11], [12], there is a θ-curve $a_0 \cup a_1 \cup a_2$ in S^3 with $\partial a_0 = \partial a_1 = \partial a_2$ such that $a_1 \cup a_2$, $a_0 \cup a_1$, $a_0 \cup a_2$ are isotopic to the knots K', K'', K''', respectively (cf. Fig. 1).

Choose a 3-ball neighborhood B of a_0 such that $B \cap (a_0 \cup a_1 \cup a_2)$ is a trivial H-graph (cf. Fig. 2). Replace this H-graph by a 2-string tangle with $2e+2$ crossings indicated in Fig. 3 to obtain an oriented knot K so that if K is regarded as K_- at the point p indicated in Fig. 3, then K_+ is isotopic to K' and K_0 is a

2-component link with components isotopic to K'', K'''. We choose the full twist number e in Fig. 3 so that the linking number of the components of K_0 is $-d$.

By (1.1.2) and (1.2), we have

$$c_0(K;x) = xc_0(K';x) - (x-1)x^d c_0(K'';x)c_0(K''';x).$$

This completes the proof.

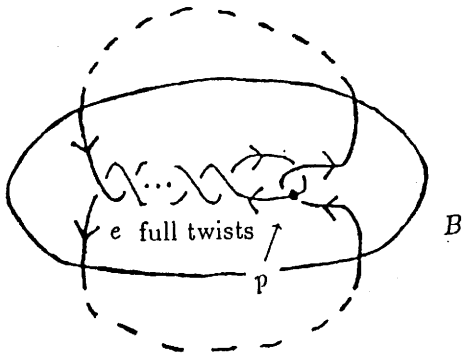

Fig. 3

Proof of Theorem 3.1. We first prove the existence of K whose unknotting number $u(K)$ may be greater than one by induction on $\deg f$. If $\deg f = 1$, then $f(x)$ is written as $(1+b)x^b - bx^{b+1}$ for an integer b. Taking $K(b)$ to be the torus knot of type $(2, -2b-1)$, we have $c_0(K(b);x) = f(x)$. If $\deg f = 2$, then $f(x)$ is written as

$$\begin{aligned} &b'x^{b+2} - (2b'+b)x^{b+1} + (1+b'+b)x^b \\ &= x[bx^{b-1} - (b-1)x^b] - (x-1)x^{b-b'}[(1+b')x^{b'} - b'x^{b'+1}]. \end{aligned}$$

for integers b, b'. Since $c_0(K(b-1);x) = bx^{b-1} - (b-1)x^b$ and $c_0(K(b');x) = (1+b')x^{b'} - b'x^{b'+1}$, we obtain from Lemma 3.6 with $K' = K(b-1)$, $K'' = K(b')$, $K''' = O^1$, $d = b - b'$ a knot $K(b,b')$ with $c_0(K(b,b');x) = f(x)$. Assume that $\deg f \geq 3$ and write

$$f(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_s x^s \quad (n - s \geq 3)$$

for integers $b_n, b_{n-1}, \cdots, b_s$. We consider a Laurent polynomial $g(x)$ determined by

$$f(x) = xg(x) - (x-1)x^{n-2}c_0(k(-1,-b_n);x),$$

where note that $c_0(K(-1,-b_n);x) = (1+2b_n) - b_n(x+x^{-1})$. Since $n-3 \geq s$, it follows that $\deg g \leq \deg f - 1$. By Lemma 3.5, we have $g(1) = 1$ and $g'(1) = 0$.

By induction, we have a knot K' with $c_0(K';x) = g(x)$. Applying Lemma 3.6 with $K' = K'$, $K'' = K(-1,-b_n)$, $K''' = O^1$, $d = n-2$, we obtain a knot K with $c_0(K;x) = f(x)$. Next, we show that we can choose as such a knot an unknotting number one knot. If $f(x) = 1$, then for example the knot 8_{14} is a desired knot. Assume $f(x) \neq 1$. Using $f(1) = 1$, we consider a Laurent polynomial $h(x)$ determined by

$$f(x) = x \cdot 1 - (x-1)x^{d(f)}h(x).$$

By Lemma 3.5, $h(x)$ has $h(1) = 1$ and $h'(1) = 0$. We have a knot K_h with $c_0(K_h;x) = h(x)$. Applying Lemma 3.6 with $K' = O^1$, $K'' = O^1$, $K''' = K_h$, $d = d(f)$, we have a knot K with $c_0(K;x) = f(x)$ and $u(K) = 1$. This completes the proof of Theorem 3.1.

4. The τ_n^*- and ρ_n^*-invariants

The following are obtained from (1.3):

$$(4.1)\qquad \begin{aligned} \tau_n^*(L_1 \circ L_2) &= -2\tau_n^*(L_1 \# L_2), \\ \tau_n^*(L_1 \# L_2) &= \sum_{p+q=n} \tau_p^*(L_1)\tau_q^*(L_2), \\ \rho_n^*(L_1 \circ L_2) &= \tau_n^*(L_1 \# L_2) - 2\rho_n^*(L_1 \# L_2), \\ \rho_n^*(L_1 \# L_2) &= \sum_{p+q=n} (\rho_q^*(L_1)\tau_q^*(L_2) + \tau_p^*(L*_1)\rho_q^*(L_2)). \end{aligned}$$

The following are obtained from (1.4):

$$(4.2)\qquad \tau_n^*(\overline{L}) = \tau_n^*(L),$$

$$(4.2')\qquad \rho_n^*(\overline{L}) = -(r-1)\tau_n^*(L) - \rho_n^*(L).$$

The following are obtained from (1.5):

$$(4.3)\qquad \sum_{n=0}^{+\infty} 2^{2n}\tau_n^*(L) = (-2)^{r-1},$$

$$(4.3')\qquad \sum_{n=0}^{+\infty} 2^{2n}\rho_n^*(L) = (r-1)(-2)^{r-2}.$$

Let $\epsilon(L)$ be 1 if L is a proper link with trivial Arf invariant and -1 if L is a proper link with non-trivial Arf invariant and 0 if L is not a proper link. For a knot K, it is well-known that $\epsilon(K) = (-1)^{\tau_1(K)}$, so that $\epsilon(K) = (-1)^{\tau_1^*(K)}$. Let $S^3(L)_3$ be the 3-fold cyclic covering space of S^3 branched along L and let $\nu(L) = (1/2)\dim_{Z_2} H_1(S^3(L)_3; Z_2)$. In Appendix B we shall discuss this number $\nu(L)$ (which is in fact an integer) and its generalization. The following (4.4) and (4.5) are obtained from results of H. Murakami [20], [21] and Lickorish/Millett [15]:

$$\sum_{n=0}^{+\infty} 2^n \tau_n^*(L) = \epsilon(L)(-2)^{r-1}. \tag{4.4}$$

$$\sum_{n=0}^{+\infty} \tau_n^*(L) = (-2)^{\nu(L)}. \tag{4.5}$$

We show the following:

THEOREM 4.6. *For any integer $q \geq 0$, we have the following identity*:

$$\begin{aligned}&\sum_{n=0}^{q}(2^{q+2}-2^n)(2^{q+1}-2^n)\tau_n^*(L)\\ &\quad+\sum_{n=q+3}^{+\infty}(2^n-2^{q+2})(2^n-2^{q+1})\tau_n^*(L)\\ &\qquad= 2^{2q+3}(-2)^{\nu(L)} - 3\epsilon(L)2^{q+1}(-2)^{r-1} + (-2)^{r-1}.\end{aligned}$$

PROOF. By (4.4) $\times\, 2^{q+2}$ − (4.3), we have

$$\begin{aligned}&\sum_{n=0}^{q+1} 2^n(2^{q+2}-2^n)\tau_n^*(L) + \sum_{n=q+3}^{+\infty} 2^n(2^{q+2}-2^n)\tau_n^*(L)\\ &\qquad= (\epsilon(L)2^{q+2}-1)(-2)^{r-1}.\end{aligned} \tag{4.7}$$

By (4.5) $\times\, 2^{q+2}$ − (4.4), we have

$$\begin{aligned}&\sum_{n=0}^{q+1}(2^{q+2}-2^n)\tau_n^*(L) + \sum_{n=q+3}^{+\infty}(2^{q+2}-2^n)\tau_n^*(L)\\ &\qquad= (-2)^{\nu(L)}2^{q+2} - \epsilon(L)(-2)^{r-1}.\end{aligned} \tag{4.8}$$

By (4.8) $\times\, 2^{q+1}$ − (4.7), we have

$$\sum_{n=0}^{q}(2^{q+2}-2^n)(2^{q+1}-2^n)\tau_n^*(L)$$

$$+ \sum_{n=q+3}^{+\infty} (2^{q+2} - 2^n)(2^{q+1} - 2^n)\tau_n^*(L)$$

$$= (-2)^{\nu(L)} 2^{2q+3} - \epsilon(L)(-2)^{r-1} 2^{q+1} - (\epsilon(L) 2^{q+2} - 1)(-2)^{r-1}$$

$$= 2^{2q+3}(-2)^{\nu(L)} - 3\epsilon(L) 2^{q+1}(-2)^{r-1} + (-2)^{r-1}.$$

This completes the proof.

If a knot K has the unknotting number $u(K) = 1$ (or more generally, the weak unknotting number (cf. [13], [9]) $u_w(K) = 1$), then we have $\nu(K) = 0$ or 1 (See Appendix B). The following corollary obtained from Theorem 4.6 by taking $q = 0$ shows that a knot K with $\tau_0^*(K) \neq 1, 5, -7, -3$ constructed in Theorem 3.1 has $\tau_n^*(K) \neq 0$ for some $n \geq 3$.

COROLLARY 4.9. *Assume that $\tau_n^*(K) = 0$ for all $n \geq 3$. If $\nu(K) = 0$, then $\tau_0^*(K) = 3 - 2\epsilon(K)$ and if $\nu(K) = 1$, then $\tau_0^*(K) = -5 - 2\epsilon(K)$.*

Applying this corollary to a double of a knot, we obtain the following:

COROLLARY 4.10. *Let K' be any finitely many iterated (untwisted or twisted) double of a knot K with $\tau_0^*(K) \neq 1$. Then $g(K') = 1$ but $g_c(K') \geq 3$.*

PROOF. Let K' be a double of K. Clearly, $g(K') = 1$. The identity $\tau_0^*(K') = -1 \pm 2\tau_0^*(K)^2$ is easily established. Using $\tau_0^*(K) \equiv 1 \pmod 4$ and $\tau_0^*(K) \neq 1$, we see that $|\tau_0^*(K')| \geq 17$. Since $u(K') = 1$, we have $\tau_n^*(K') \neq 1$ for some $n \geq 3$. By (1.6), $g_c(K') \geq 3$. By induction on the iteration number, we complete the proof.

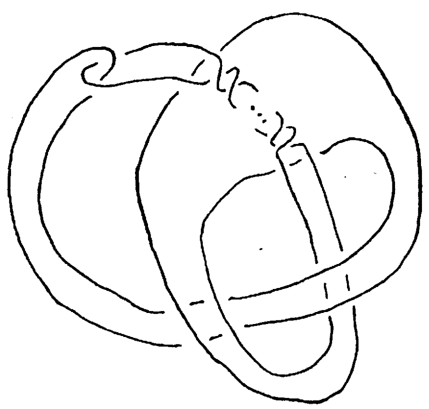

Fig. 4

EXAMPLE 4.11. Let K' be any double of the trefoil knot $K(3_1)$. Directly, we can see that $g_c(K') \le 3$ (See Fig. 4). Clearly, $g(K') = 1$. Since $\tau_0^*(K(3_1)) = -3$, we see from Corollary 4.10 and (1.6) that $\tau_3^*(K) \ne 0$. Hence $g_c(K) = 3$. Let K'_n be the $n(\ge 2)$-fold connected sum of this knot K'. By the additivity of genus, $g(K'_n) = n$. Clearly, $g_c(K_n) \le 3n$. By (4.1), $\tau_{3n}^*(K'_n) = (\tau_3^*(K'))^n \ne 0$, so that $g_c(K'_n) = 3n$. Thus, we have a knot K'_n such that $g_c(K'_n) - g(K'_n) = 2n$ for any positive integer n. In particular, if K' is an untwisted double of $K(3_1)$, then the knot K'_n has the trivial Alexander polynomial and $g_c(K'_n) - g(K'_n) = 2n$.

On the value of $\tau_0^*(K)$, we have the following:

THEOREM 4.12.

(1) $\tau_0^*(K) \equiv 1 \pmod 4$ *for all knots* K,

(2) *For a knot* K, $\tau_0^*(K) \equiv 1 \pmod 8$ *if and only if* $\tau_1(K) \equiv 0 \pmod 2$ (*i.e.*, $\epsilon(K) = 1$),

(3) *If a knot* K *has* $\tau_n(K) \equiv 0 \pmod 2$ *for all* $n \ge 1$, *then* $\tau_0^*(K) \equiv 1 \pmod{16}$,

(4) *If a knot* K *has* $\tau_0^*(K) \ne 1$ *and* $\tau_0^*(K) \equiv 1 \pmod{16}$, *then* $\tau_n^*(K) \ne 0$ *for some* $n \ge 3$.

PROOF of THEOREM 4.12. Since $\tau_0(K) = 1$, $\rho_0(K) = 0$, there is a Laurent polynomial $f(x)$ with $c_0(K;x) = 1+(x-1)^2 f(x)$, so that (1) is obtained. Further, by Lemma 2.4, $\tau_1(K) = -c_0''(K;1)/2 = -f(1)$. Then $\tau_0^*(K) - 1 = 4f(-1) \equiv -4f(1) = 4\tau_1(K) \pmod 8$ and (2) is proved. For (3), (4), we use Theorem 4.6. Taking $q = 0$ in Theorem 4.6, we have

$$3\tau_0^*(K) + \sum_{n=3}^{+\infty}(2^n - 2^2)(2^n - 2)\tau_n^*(K) = 8(-2)^{\nu(K)} - 6\epsilon(K) + 1.$$

Since $\tau_n(K) \equiv 0 \pmod 2$ for all $n \ge 1$, we see from Appendix B that $\nu(K) = 0$. We have also $\epsilon(K) = (-1)^{\tau_1(K)} = 1$. Using that $(2^n - 2^2)(2^n - 2) \equiv 0 \pmod 8$ and $\tau_n^*(K) \equiv \tau_n(K) \equiv 0 \pmod 2$ for all $n \ge 3$, it follows that

$$3\tau_0^*(K) \equiv 3 \pmod{16}.$$

Hence $\tau_0^*(K) \equiv 1 \pmod{16}$. For (4), suppose that $\tau_n^*(K) = 0$ for all $n \ge 3$. Then we have $3\tau_0^*(K) = 8(-2)^{\nu(K)} - 6\epsilon(K) + 1$. If $\nu(K) \ge 1$, then $3\tau_0^*(K) \equiv \pm 6 + 1 \pmod{16}$, contradicting that $\tau_0^*(K) \equiv 1 \pmod{16}$. Hence $\nu(K) = 0$ and $\tau_0^*(K) = 1$ or 5, which is impossible. Thus, $\tau_n^*(K) \ne 0$ for some $n \ge 3$. This completes the proof.

REMARK 4.13. (1) Let K be the $4m$-fold connected sum of the trefoil knot $K(3_1)$ with m a positive integer. Then $\tau_0^*(K) = \tau_0^*(K(3_1))^{4m} = (-3)^{4m} \equiv 1$

(mod 16). But, by (1.3), $c_{4m}(K;x) = c_1(K(3_1);x)^{4m} = 1$, since $c_n(K(3,1);x) = 1$ (if $n = 1$) or 0 (if $n \geq 2$). This means that the converse of Theorem 4.12(3) is not true.

(2) Let K_{KT} be the Kinoshita-Terasaka knot illustrated in Fig. 5. Then we have $\nabla(K_{KT};z) = 1$ and $c_0(K_{KT};x) = 1 + 2\overline{x}(x-1)^3$ and hence $\tau_0^*(K_{KT}) = 1 + 16$. Let K be the knot illustrated in Fig. 6. Then we have $\nabla(K;z) = 1$ and $c_0(K;x) = 1 + 2(x-1)^3$ and hence $\tau_0^*(K) = 1 - 16$. However, we do not know *whether* $1+16m$ *for every integer* m *is realizable by* $\tau_0^*(K)$ *of a knot* K *with trivial Alexander polynomial*. This is related to a question: *Is every Laurent polynomial* $f(x)$ *with* $f(1) = 1$, $f'(1) = f''(1) = 0$, $f(-1) \equiv 1$ (mod 16) *realizable by* $c_0(K;x)$ *of a knot* K *with trivial Alexander polynomial*?

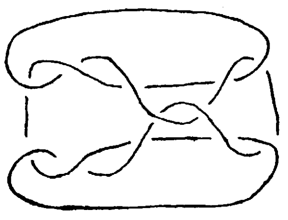

Fig. 5

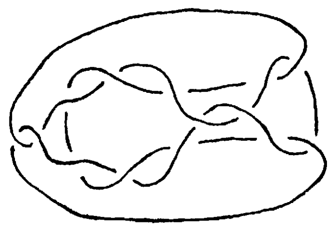

Fig. 6

Appendix A. In this section, we show the following.

THEOREM A.1. *If there is a* $Z[Z]$*-homology equivalence* $f : (S^3, L^*) \to (S^3, L)$, *then any Seifert matrices* A^*, A *associated with any connected Seifert surfaces for* L^*, L *are* S*-equivalent.*

PROOF. Let F be a connected Seifert surface for L. It is known that any two Seifert matrices associated with any two connected Seifert surfaces for L are S-equivalent (cf. [10]). By t-regularity, $F^* = f^{-1}F$ can be assumed to be a (possibly disconnected) surface with boundary L^*. The restriction $f \mid F^* : F^* \to F$ is a ∂-diffeomorphic degree one map and induces an epimorphism $H_1(F^*, \partial F^*; Z) \to H_1(F, \partial F; Z)$ and an isomorphism $H_1(\partial F^*; Z) \cong H_1(\partial F; Z)$. Using that F is connected, we see that F^* has just one bounded component. Hence a Seifert matrix associated with F^* is a Seifert matrix associated with a connected Seifert surface obtained from F^* by piping the components of F^*. Let $E(F^*) = cl(S^3 - N(F^*))$, $E(F) = cl(S^3 - N(F))$ for regular neighborhoods $N(F^*)$, $N(F)$ of F^*, F in S^3, respectively, with $N(F^*) = f^{-1}N(F)$. Let $f^! : H_1(F;Z) \to H(F^*, Z)$ be the composite

$$H_1(F;Z) \xrightarrow{\partial^{-1}} H_2(S^3,F;Z) \cong H_2(S^3,N(F);Z) \cong H_2(E(F),\partial E(F);Z)$$
$$\xrightarrow{D^{-1}} H^1(E(F);Z) \xrightarrow{f^*} H^1(E(F^*);Z) \xrightarrow{D} H_2(E(F^*),\partial E(F^*);Z)$$
$$\cong H_2(S^3,N(F^*);Z) \cong H_2(S^3,F^*;Z) \xrightarrow{\partial} H_1(F^*;Z),$$

where ∂ denotes the boundary isomorphism and D denotes the Poincaré duality isomorphism and $\cong$ denotes the inclusion isomorphism or its inverse. Then we see that the composite of $f^!$ with $f_* : H_1(F^*;Z) \to H_1(F;Z)$ is the identity on $H_1(F;Z)$. Let $S_1(F^*;Z) = f^!(H_1(F;Z))$. Then the restriction $f_* \mid S_1(F^*;Z) : S_1(F^*;Z) \to H_1(F;Z)$ is an isomorphism. Let $K_1(F^*;Z) = \operatorname{Ker} f_*$. Then $H_1(F^*;Z) = S_1(F^*;Z) \oplus K_1(F^*;Z)$. Let $F^*_+, F_+ \subset S^3$ be slight translations of F^*, F in a positive normal direction, respectively, and F^*_-, $F_- \subset S^3$, in a negative normal direction. Let $L^*_\pm : H_1(F^*;Z) \times H_1(F^*;Z) \to Z$ be the Seifert linking form on F^* such that $L^*_\pm(x,y) = \operatorname{Link}(x,y_\pm)$ for $x,y \in H_1(F^*;Z)$ where $y_\pm$ denotes a copy of y in $H_1(F^*_\pm;Z)$. Similarly, let $L_\pm : H_1(F;Z) \times H_1(F;Z) \to Z$ be the Seifert linking form on F. Then we have that $L_\pm(f_*(x),f_*(y)) = L^*_\pm(x,y)$ for $x \in S_1(F^*;Z)$, $y \in H_1(F^*;Z)$. In fact, let $u \in H^1(E(F);Z)$ correspond to the element $f_*(x) \in H_1(F;Z)$ for $x \in S_1(F^*;Z)$ in the above diagram. Then $f^*(u) \in H^1(E(F^*);Z)$ corresponds to x in the above diagram. Hence $L^*_\pm(x,y) = \langle f^*(u), y_\pm\rangle = \langle u, f_*(y_\pm)\rangle = L_\pm(f_*(x),f_*(y))$, as desired, where we regard that $y_\pm \in H_1(E(F);Z)$. In particular, $L^*_\pm(S_1(F^*;Z),K_1(F^*;Z)) = 0$. Let $A^*_\pm$, $A_\pm$ be matrices representing $L^*_\pm \mid S_1(F^*;Z) \times S_1(F^*;Z)$, $L_\pm$ associated with a basis for $S_1(F^*;Z)$ and a basis for $H_1(F;Z)$ obtained from the basis for $S_1(F^*;Z)$ by applying f_*, respectively. Then we have $A^*_\pm = A_\pm$. Let $A^K_\pm$ be a matrix representing $L^*_\pm \mid K_1(F^*;Z) \times K_1(F^*;Z)$ associated with a basis for $K_1(F^*;Z)$. Note that $A^K_+ \oplus A^*_+$ and A_+ are Seifert matrices of L^*, L and the transpose matrices $(A^K_+)', (A^*_+)' = (A_+)'$ are equal to $A^K_-, A^*_- = A_-$, respectively. Let Λ be the integral group ring of $\langle t\rangle$. Since $(tA^K_+ - A^K_-) \oplus (tA^*_+ - A^*_-)$, and $tA_+ - A_-(= tA^*_+ - A^*_-)$ are Λ-presentation matrices of the Λ-modules $H_1(\widetilde{E}(L^*);Z)$, $H_1(\widetilde{E}(L);Z)$ which are Λ-isomorphic, it follows from the Noetherian property of Λ that $\det(tA^K_+ - A^K_-)$ is a unit $\pm t^m$ ($m \in Z$) of Λ. By Trotter [24], A^K_+ is S-equivalent to the zero matrix. Hence $A^K_+ \oplus A^*_+$ and A_+ are S-equivalent. This completes the proof.

Appendix B. Let $S^3(L)_d$ be the $d(\geq 2)$-fold cyclic covering space of S^3 branched along a link L. Let L_d be the lift of L. Let E_d be the lift of $E = S^3 - L$ to $S^3(L)_d$. Let $p_d : E_d \to E$ be the covering projection. The infinite cyclic covering $p : \widetilde{E} \to E$ factors into the infinite cyclic covering $p^d : \widetilde{E} \to E_d$ and the covering $p_d : E_d \to E$. Let $f : \widetilde{E} \to S^3(L)_d$ be the composite of p^d and the inclusion $E_d \subset S^3(L)_d$. Let $\zeta_d(t) = (1-t^d)/(1-t)$.

THEOREM B.1. *The map* $f : \widetilde{E} \to S^3(L)_d$ *induces an isomorphism*

$$f_* : H_1(\widetilde{E}; Z)/\zeta_d(t)H_1(\widetilde{E}; Z) \cong H_1(S^3(L)_d; Z).$$

Sakuma [23] has established an isomorphism from the inverse direction in a more general setting. It is a merit of our proof that this identification is seen to be natural.

PROOF of THEOREM B.1. Note that the homomorphism $f_* : H_1(\widetilde{E}; Z) \to H_1(S^3(L)_d; Z)$ is t-invariant. By a transfer argument [2, p.119], we have $\zeta_d(t) = 0$ on $H_1(S^3(L)_d; Z)$. Hence $\operatorname{Ker} f_* \supset \zeta_d(t)H_1(\widetilde{E}; Z)$. Let the covering $p^d : \widetilde{E} \to E_d$ be associated with an element $\gamma^d \in H^1(E_d; Z) = \operatorname{Hom}(H_1(E_d; Z), Z)$. Let $\gamma = \gamma^1$. Let L_i, $i = 1, 2, \ldots, r$, be the components of L and L_i^d be the corresponding component of L^d. Let m_i, m_i^d be fixed meridians of L_i, L_i^d in E, E_d, respectively. Let $x \in H_1(S^3(L)_d; Z)$. Since $\gamma^d\{m_i^d\} = 1$, we see that x is represented by a cycle c in E_d such that $\gamma^d\{c\} = 0$. Clearly, c is homologous to a simple closed curve c' in E_d. For any component c_0' of $(p^d)^{-1}c'$, we have that $p^d \mid c_0' : c_0' \to c'$ is a homeomorphism. Hence $f_*\{c_0'\} = x$ and f_* is surjective. For $y \in H_1(\widetilde{E}; Z)$, assume that $f_*(y) = 0$. Then $(p^d)_*(y) = \sum_{i=1}^r a_i\{m_i^d\}$ for some integers a_i, $i = 1, 2, \ldots, r$, with $\sum_{i=1}^r a_i = 0$. Note that $(p_d)_*\{m_i^d\} = d\{m_i\}$. Let $z = \sum_{i=1}^r a_i\{m_i\} \in H_1(E; Z)$. Since $\gamma(z) = 0$, we have $z = p_*(\widetilde{z})$ for some $\widetilde{z} \in H_1(\widetilde{E}; Z)$ by the Wang exact sequence for the infinite cyclic covering p. Let $y' = y - \zeta_d(t)\widetilde{z} \in H_1(\widetilde{E}; Z)$. Then $f_*(y') = 0$ and $p_*(y') = (p_d)_*(p^d)_*(y') = 0$. Then we have $(p^d)_*(y') = 0$ in $H_1(E_d; Z)$. By the Wang exact sequence for the infinite cyclic covering p^d, there is an element $y'' \in H_1(\widetilde{E}; Z)$ such that $y' = (t^d - 1)y''$. Thus, $y = \zeta_d(t)((t-1)y'' + \widetilde{z}) \in \zeta_d(t)H_1(\widetilde{E}; Z)$ and we have $\operatorname{Ker} f_* = \zeta_d(t)H_1(\widetilde{E}; Z)$. This completes the proof.

For an integer $s \geq 0$, let $Z_s = Z/sZ$ (thus, $Z_0 = Z$). Let $m_d(L; Z_s)$ be the minimal number of abelian generators of $H_1(S^3(L)_d; Z_s)$. Note that the number $\nu(L)$ introduced in §4 is equal to $m_3(L; Z_2)/2$. Let $m(L; Z_s)$ be the minimal number of Λ-generators of the Λ-module $H_1(\widetilde{E}; Z_s)$. Let $m(L) = m(L; Z_0)$. It is known in [9] that the weak unlinking number $u_w(L)$ of any r-component link L has $u_w(L) + r - 1 \geq m(L)$. We have the following.

COROLLARY B.2. $m(L) \geq m(L; Z_s) \geq m_d(L; Z_s)/(d-1)$.

PROOF. The left-hand inequality is obvious. Let $m = m(L; Z_s)$. Then there is a Λ-epimorphism $\Lambda^m \to H_1(\widetilde{E}; Z_s)$, inducing a Λ-epimorphism $(\Lambda/\zeta_d(t)\Lambda)^m \to H_1(\widetilde{E}; Z_s)/\zeta_d(t)H_1(\widetilde{E}; Z_s)$. By Theorem B.1 and the universal coefficient theorem, we have an isomorphism

$$H_1(\widetilde{E}; Z_s)/\zeta_d(t)H_1(\widetilde{E}; Z_s) \cong H_1(S^3(L)_d; Z_s).$$

Since $\Lambda/\zeta_d(t)\Lambda$ has $d-1$ abelian generators, it follows that $m_d(L;Z_s) \leq (d-1)m$. Hence the right-hand inequality is obtained. This completes the proof.

Since $\zeta_3(t) = 1+t+t^2 = 0$ in $H_1(S^3(L)_3;Z_2)$, we see that $t+1 = t^2 = t^{-1}$ is an automorphism of $H_1(S^3(L)_3;Z_2)$. Assume that $x_1, tx_1, \ldots, x_{i-1}, tx_{i-1}, x_i$ are Z_2-linearly independent elements of $H_1(S^3(L)_3;Z_2)$. Then we can see easily that $x_1, tx_1, \ldots, x_{i-1}, tx_{i-1}, x_i, tx_i$ are Z_2-linearly independent. By induction, $H_1(S^3(L)_3;Z_2)$ has a Z_2-basis of the type: $x_1, tx_1, \ldots, x_m, tx_m$. Hence $\nu(L) = m_3(L;Z_2)/2$ is an integer.

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