Cross-index of a graph

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Abstract. A family of topological invariants of a connected graph associated to all trees is introduced. The member of the family associated to a tree $T$ is called the $T$-cross-index, which takes a non-negative integer or infinity according to whether $T$ is a tree basis of the graph or not. It is shown how this cross-index is independent of the other topological invariants of a connected graph such as the Euler characteristic, the crossing number and the genus.

1. Introduction

A based graph is a pair $(G; T)$ such that $G$ is a connected graph and $T$ is a maximal tree of $G$, called a tree basis of $G$. A based diagram $(D; X)$ of the based graph $(G; T)$ is defined in § 2. Then the crossing number $c(G; T)$ of the based graph $(G; T)$ is defined to be the minimum of the crossing numbers $c(D; X)$ for all based diagrams $(D; X)$ of $(G; T)$. The genus $g(D; X)$, the nullity $\nu(D; X)$ and the cross-index $\varepsilon(D; X)$ are defined by using the $\mathbb{Z}_2$-form

$$\varepsilon : \mathbb{Z}_2[D; X] \times \mathbb{Z}_2[D; X] \to \mathbb{Z}_2$$

on $(D; X)$, which are invariants of non-negative integer values of the based diagram $(D; X)$. The genus $g(G; T)$ and the cross-index $\varepsilon(G; T)$ are defined to be the minimums of the genera $g(D; X)$ and the cross-indexes $\varepsilon(D; X)$ for all based diagrams $(D; X)$ of $(G; T)$, respectively, whereas the nullity $\nu(G; T)$ is defined to be the maximum of the nullities $\nu(D; X)$ for all based diagrams $(D; X)$ of $(G; T)$. Thus, $c(G; T)$, $g(G; T)$, $\nu(G; T)$, $\varepsilon(G; T)$ are topological invariant of the based graph $(G; T)$. The relationships between the topological invariants $c(G; T)$, $g(G; T)$, $\nu(G; T)$, $\varepsilon(G; T)$ of $(G; T)$, the genus $g(G)$ and the Euler characteristic $\chi(G)$ of the graph $G$ are explained in § 2. In particular, the identities

$$c(G; T) = \varepsilon(G; T), \quad g(G; T) = g(G), \quad \nu(G; T) = 1 - \chi(G) - 2g(G)$$

are established. In particular, it turns out that the crossing number $c(G; T)$ of $(G; T)$ is a calculable invariant in principle. The idea of a cross-index is also applied to study complexities of a knitting pattern in [5].

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This invariant \(c(G; T)\) is modified into a topological invariant \(c^T(G)\) of a graph \(G\) associated to a tree \(T\), called the \(T\)-cross-index of \(G\) as follows:

Namely, define \(c^T(G)\) to be the minimum of the invariants \(c(G; T')\) for all tree bases \(T'\) of \(G\) homeomorphic to \(T\). If there is no tree basis of \(G\) homeomorphic to \(T\), then define \(c^T(G) = \infty\).

Let \(c'(G)\) be the family of the invariants \(c^T(G)\) for all trees \(T\). The minimal value \(c^\text{min}(G)\) in the family \(c'(G)\) has been appeared as the crossing number of a \(\Gamma\)-unknotted graph in the paper \([3]\) on spatial graphs. The crossing number \(c(G)\) of a graph \(G\) is defined to be the minimum of the crossing numbers \(c(D)\) of all diagrams \(D\) of \(G\) (in the plane). It is an open question whether \(c^\text{min}(G)\) is equal to the crossing number \(c(G)\) of any connected graph \(G\).

The finite maximal value \(c^\text{max}(G)\) in the family \(c'(G)\) is a well-defined invariant, because there are only finitely many tree bases of \(G\). In the inequalities

\[
c^\text{max}(G) \geq c^\text{min}(G) \geq c(G) \geq g(G)
\]

for every connected graph \(G\) which we establish, the following properties are mutually equivalent:

(i) \(G\) is a planar graph.
(ii) \(c^\text{max}(G) = 0\).
(iii) \(c^\text{min}(G) = 0\).
(iv) \(c(G) = 0\).
(v) \(g(G) = 0\).

In § 3, the case of the \(n\)-complete graph \(K_n\) \((n \geq 5)\) is discussed in a connection to Guy’s conjecture on the crossing number \(c(K_n)\). It is shown that \(c^\text{min}(K_5) = c^\text{max}(K_5)\) and \(c^\text{min}(K_n) < c^\text{max}(K_n)\) for every \(n \geq 6\). Thus, the invariants \(c^\text{min}(G)\) and \(c^\text{max}(G)\) are different invariants for a general connected graph \(G\).

The main purpose of this paper is to show that the family \(c'(G)\) is more or less a new invariant. For this purpose, for a real-valued invariant \(I(G)\) of a connected graph \(G\) which is not bounded when \(G\) goes over the range of all connected graphs, we introduce a virtualized invariant \(\tilde{I}(G)\) of \(G\) which is defined to be \(\tilde{I}(G) = f(I(G))\) for a fixed non-constant real polynomial \(f(x)\) in \(x\). Every time a different non-constant polynomial \(f(t)\) is given, a different virtualized invariant \(\tilde{I}(G)\) is obtained from the invariant \(I(G)\). Then the main result is stated as follows, showing a certain independence between the cross-index \(c'(G)\) and the other invariants \(c(G), g(G), \chi(G)\).

**Theorem 4.1.** Let \(\tilde{c}^\text{max}(G), \tilde{c}(G), \tilde{g}(G), \tilde{\chi}(G)\) be any virtualized invariants of the invariants \(c^\text{max}(G), c(G), g(G), \chi(G)\), respectively. Every linear combination of the invariants \(\tilde{c}^\text{max}(G), \tilde{c}(G), \tilde{g}(G), \tilde{\chi}(G)\) in real coefficients including a non-zero number is not bounded when \(G\) goes over the range of all connected graphs.

The proof of this theorem is given in § 4.

As an appendix, it is shown how the non-isomorphic tree bases are tabulated in case of the complete graph \(K_{11}\). This tabulation method is important to compute
the $T$-cross index $c^T(K_n)$ for a tree basis $T$ of $K_n$, because we have $c^T(K_n) = c(K_n; T)$ for every tree basis $T$ (see Lemma 3.1).

2. The cross-index of a graph associated to a tree

By a graph, a connected graph $G$ with only topological edges and without vertexes of degrees 0, 1 and 2 is meant. Let $G$ have $n(\geq 1)$ vertexes and $s(\geq 1)$ edges. By definition, if $n = 1$ (that is, $G$ is a bouquet of loops), then every tree basis $T$ of $G$ has one vertex. A diagram of a graph $G$ is a representation $D$ of $G$ in the plane so that the vertexes of $G$ are represented by distinct points and the edges of $G$ are represented by arcs joining the vertexes which may have transversely meeting double points avoiding the vertexes. A double point on the edges of a diagram $D$ is called a crossing of $D$. In this paper, to distinguish between a degree 4 vertex and a crossing, a crossing is denoted by a crossing with over-under information except in Figs. 7, 8 representing diagrams of the graphs $K_{11}$ and $K_{12}$ without degree 4 vertexes. A tree diagram of a tree $T$ is a diagram $X$ of $T$ without crossings. A based diagram of a based graph $(G; T)$ is a pair $(D; X)$ where $D$ is a diagram of $G$ and $X$ is a sub-diagram of $D$ such that $X$ is a tree diagram of the tree basis $T$ without crossings in $D$. In this case, the diagram $X$ is called a tree basis diagram. The following lemma is used without proof in the author’s earlier papers [3, 4].

Lemma 2.1. Given any based graph $(G; T)$ in $\mathbb{R}^3$, then every spatial graph diagram of $G$ is transformed into a based diagram $(D; X)$ of $(G; T)$ only with crossings with over-under information by the Reidemeister moves I-V (see Fig. 1).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{reidemeister_moves.png}
\caption{The Reidemeister moves}
\end{figure}

Proof of Lemma 2.1. In any spatial graph diagram $D'$ of $G$, first transform the sub-diagram $D(T)$ of the tree basis $T$ in $D'$ into a tree diagram $X$ by the Reidemeister moves I-V. Since a regular neighborhood $N(X; \mathbb{R}^2)$ of $X$ in the plane $\mathbb{R}^2$ is a disk, a based diagram is obtained by shrinking this tree diagram into a very small tree diagram within the disk by the Reidemeister moves I-V. See Fig. 2 for this transformation. Thus, we have a based diagram $(D; X)$ of $(G; T)$ only with crossings with over-under information. \qed
The crossing number $c(D)$ of a based diagram $(D; X)$ is denoted by $c(D; X)$. The crossing number $c(G; T)$ of a based graph $(G; T)$ is the minimal number of the crossing numbers $c(D; X)$ of all based diagrams $(D; X)$ of $(G; T)$. For a based diagram $(D; X)$ of $(G; T)$, let $N(X; D) = D \cap N(X; \mathbb{R}^2)$ be a regular neighborhood of $X$ in the diagram $D$. Then the complement $\text{cl}(D \setminus N(X; D))$ is a tangle diagram of $m$-strings $a_i$ ($i = 1, 2, \ldots, m$) in the disk $\Delta = S^2 \setminus N(X; \mathbb{R}^2)$ where $S^2 = \mathbb{R}^2 \cup \{\infty\}$ denotes the 2-sphere which is the one-point compactification of the plane $\mathbb{R}^2$.

Let $\mathbb{Z}_2[D; X]$ be the $\mathbb{Z}_2$ vector space with the arcs $a_i$ ($i = 1, 2, \ldots, m$) as a $\mathbb{Z}_2$-basis. For any two arcs $a_i$ and $a_j$ with $i \neq j$, the cross-index $\varepsilon(a_i, a_j)$ is defined to be 0 or 1 according to whether the two boundary points $\partial a_j$ of the arc $a_j$ are in one component of the two open arcs $\partial \Delta \setminus \partial a_i$ or not, respectively. For $i = j$, the identity $\varepsilon(a_i, a_j) = 0$ is taken. Then the cross-index $\varepsilon(a_i, a_j)$ (mod 2) defines the symmetric bilinear $\mathbb{Z}_2$-form

$$\varepsilon : \mathbb{Z}_2[D; X] \times \mathbb{Z}_2[D; X] \rightarrow \mathbb{Z}_2,$$

called the $\mathbb{Z}_2$-form on $(D; X)$. The genus $g(D; X)$ of the based diagram $(D; X)$ is defined to be half of the $\mathbb{Z}_2$-rank of the $\mathbb{Z}_2$-form $\varepsilon : \mathbb{Z}_2[D; X] \times \mathbb{Z}_2[D; X] \rightarrow \mathbb{Z}_2$, which is seen to be even since the $\mathbb{Z}_2$-form $\varepsilon$ is a $\mathbb{Z}_2$-symplectic form.

The genus $g(G; T)$ of a based graph $(G; T)$ is the minimum of the genus $g(D; X)$ for all based diagrams $(D; X)$ of $(G; T)$. The following lemma shows that the genus $g(G)$ of a graph $G$ is calculable from based diagrams $(D; X)$ of any based graph $(G; T)$ of $G$.

**Lemma 2.2 (Genus Lemma).** $g(G) = g(D; X) = g(G; T)$ for any based diagram $(D; X)$ of any based graph $(G; T)$.

**Proof of Lemma 2.2.** Let $(D; X)$ be a based diagram of a based graph $(G; T)$ with $g(D; X) = g(G; T)$. Constructs a compact connected orientable surface $N(D; X)$ from $(D; X)$ such that

(1) the surface $N(D; X)$ is a union of a disk $N$ in $\mathbb{R}^2$ with the tree basis diagram $X$ as a spine and attaching bands $B_i$ ($i = 1, 2, \ldots, m$) whose cores are the edges $a_i$ ($i = 1, 2, \ldots, m$) of $D$,

(2) the $\mathbb{Z}_2$-form $\varepsilon : \mathbb{Z}_2[D; X] \times \mathbb{Z}_2[D; X] \rightarrow \mathbb{Z}_2$ is isomorphic to the $\mathbb{Z}_2$-intersection form on $H_1(N(D; X); \mathbb{Z}_2)$.

Because the nullity of the $\mathbb{Z}_2$-intersection form on $H_1(N(D; X); \mathbb{Z}_2)$ is equal to the number of the boundary components of the bounded surface $N(D; X)$ minus one, the genus $g(N(D; X))$ is equal to the half of the $\mathbb{Z}_2$-rank of the $\mathbb{Z}_2$-form $\varepsilon$. This
implies that
\[ g(G; T) = g(D; X) = g(N(D; X)) \geq g(G). \]

Conversely, let \( F \) be a compact connected orientable surface containing \( G \) with genus \( g(F) = g(G) \), where \( F \) need not be closed. For any based graph \((G; T)\), let \( N(G) \) be a regular neighborhood of \( G \) in \( F \), which is obtained from a disk \( N \) in \( F \) with the tree basis \( T \) as a spine by attaching bands \( B_i \) \((i = 1, 2, \ldots, m)\) whose cores are the edges \( a_i \) \((i = 1, 2, \ldots, m)\) of \( G \). Then the inequality \( g(N(G)) \leq g(F) \) holds. Let \((D; X)\) be any based diagram of the based graph \((G; T)\). Identify the disk \( N \) with a disk \( N(X) \) with the tree basis \( T \) as a spine. By construction, the \( \mathbb{Z}_2 \)-form \( \varepsilon : \mathbb{Z}_2[D; X] \times \mathbb{Z}_2[D; X] \rightarrow \mathbb{Z}_2 \) is isomorphic to the \( \mathbb{Z}_2 \)-intersection form on \( H_1(N(G); \mathbb{Z}_2) \), which determines the genus \( g(N(G)) \) as the half of the \( \mathbb{Z}_2 \)-rank of \( \varepsilon \). Thus, the inequalities
\[ g(G; T) \leq g(D; X) = g(N(G)) \leq g(F) = g(G) \]
hold and we have \( g(G) = g(D; T) = g(G; T) \) for any based diagram \((D; X)\) of \((G; T)\). \( \Box \)

The nullity \( \nu(D; X) \) of \((D; X)\) is the nullity of the \( \mathbb{Z}_2 \)-form \( \varepsilon : \mathbb{Z}_2[D; X] \times \mathbb{Z}_2[D; X] \rightarrow \mathbb{Z}_2 \). The nullity \( \nu(G; T) \) of a based graph \((G; T)\) is the maximum of the nullity \( \nu(D; X) \) for all based diagrams \((D; X)\) of \((G; T)\). Then the following corollary is obtained:

**Corollary 2.3.** The identity \( \nu(G; T) = 1 - \chi(G) - 2g(G) \) holds for any based graph \((G; T)\).

This corollary shows that the nullity \( \nu(G; T) \) is independent of a choice of tree bases \( T \) of \( G \), and is therefore simply called the nullity of \( G \) and denoted by \( \nu(G) \).

**Proof of Corollary 2.3.** The graph \( G \) is obtained from the tree graph \( N(X; D) \) by attaching the mutually disjoint \( m \)-strings \( a_i \) \((i = 1, 2, \ldots, m)\). Since the \( \mathbb{Z}_2 \)-rank of \( \mathbb{Z}_2[D; X] \) is \( m \) by definition, we see from a calculation of the Euler characteristic \( \chi(G) \) that \( \chi(G) = 1 + m - 2m = 1 - m \). By the identity \( m = 2g(D; X) + \nu(D; X) \) on the rank and the nullity of the \( \mathbb{Z}_2 \)-form \( \varepsilon \), the nullity \( \nu(D; X) \) of a based diagram \((D; X)\) of \((G; T)\) is given by
\[ \nu(D; X) = m - 2g(D; X) = 1 - \chi(G) - 2g(D; X). \]
Hence we have
\[ \nu(G; T) = 1 - \chi(G) - 2g(G; T) = 1 - \chi(G) - 2g(G) \]
by Lemma 2.2. \( \Box \)

The cross-index of a based diagram \((D; X)\) is the non-negative integer \( \varepsilon(D; X) \) defined by
\[ \varepsilon(D; X) = \sum_{1 \leq i < j \leq m} \varepsilon(a_i, a_j). \]
The following lemma is obtained:

**Lemma 2.4.** For every based diagram \((D; X)\), the inequality \( \varepsilon(D; X) \geq g(D; X) \) holds.
Proof of Lemma 2.4. Let $V$ be the $\mathbb{Z}_2$-matrix representing the $\mathbb{Z}_2$-form $\varepsilon : \mathbb{Z}_2[D;X] \times \mathbb{Z}_2[D;X] \to \mathbb{Z}_2$ with respect to the arc basis $a_i$ ($i = 1, 2, \ldots, m$). Let $\varepsilon_{ij} = \varepsilon(a_i, a_j)$ be the $(i,j)$-entry of the matrix $V$. The $\mathbb{Z}_2$-rank $r$ of the matrix $V$ is equal to $2g(D;X)$ by definition. There are $r$ column vectors in $V$ that are $\mathbb{Z}_2$-linearly independent. By changing the indexes of the arc basis $a_i$, we can find a sequence of integral pairs $(i_k, j_k)$ ($k = 1, 2, \ldots, r$) with $1 \leq i_1 < i_2 < \cdots < i_r \leq m$ and $1 \leq j_1 < j_2 < \cdots < j_r \leq m$ such that $\varepsilon_{i_k,j_k} = 1$ for all $k (k = 1, 2, \ldots, r)$. Here, note that this sequence $(i_k, j_k)$ is obtained by combining two pairs $(i_k, j_k), (i_k', j_k')$ with $k \neq k'$ and $(i_k, j_k) = (j_k, i_k')$. By the identities $\varepsilon_{ii} = 0$ and $\varepsilon_{ij} = \varepsilon_{ji}$ for all $i, j$, we have

$$2\varepsilon(D;X) = \sum_{1 \leq i,j \leq m} \varepsilon_{ij} \geq \sum_{k=1}^{r} \varepsilon_{i_k,j_k} = r = 2g(D;X).$$

Thus, the inequality $\varepsilon(D;X) \geq g(D;X)$ is obtained. $\square$

The cross-index $\varepsilon(G;T)$ of a based graph $(G;T)$ is the minimum of the cross-index $\varepsilon(D;X)$ for all based diagrams $(D;X)$ of $(G;T)$. It may be used to compute the crossing number $c(G;T)$ of a based graph $(G;T)$ as it is stated in the following lemma:

Lemma 2.5 (Calculation Lemma). $\varepsilon(G;T) = c(G;T)$ for every based graph $(G;T)$.

Proof of Lemma 2.5. Let $a_i$ ($i = 1, 2, \ldots, m$) be an arc basis of a based diagram $(D;X)$ of $(G;T)$ attaching to the boundary of a regular neighborhood disk $N$ of $X$ in the plane.

By a homotopic deformation of $a_i$ into an embedded arc $a_i'$ keeping the boundary points fixed, we construct a new based diagram $(D';X)$ of $(G;T)$ with a basis $a_i'$ ($i = 1, 2, \ldots, m$) so that

1. $a_i' \cap a_j' = \emptyset$ if $\varepsilon(a_i, a_j) = 0$ and $i \neq j$.
2. $a_i'$ and $a_j'$ meet one point transversely if $\varepsilon(a_i, a_j) = 1$.

Then the cross-index $\varepsilon(D;X)$ is equal to the crossing number $c(D';X)$ of the based diagram $(D';X)$ of $(G;T)$. Hence the inequality $\varepsilon(G;T) \geq c(G;T)$ is obtained. Since $\varepsilon(D;X) \leq c(D;X)$ for every based graph $(D;X)$ of $(G;T)$, the inequality $\varepsilon(G;T) \leq c(G;T)$ holds. Hence the identity $\varepsilon(G;T) = c(G;T)$ holds. $\square$

Calculation Lemma (Lemma 2.5) gives a computation method of the crossing number $c(G;T)$ of a based graph $(G;T)$ in a finite procedure.

In fact, let $X_i$ ($i = 1, 2, \ldots, s$) be all the tree basis diagrams of $T$ in $\mathbb{R}^2$. For every $i$, let $(D_{ij}, X_i)$ ($j = 1, 2, \ldots, t_i$) be a finite set of basic diagrams of $(G;T)$ such that every based diagram $(D;X_i)$ of $(G;T)$ coincides with a based diagram $(D_{ij}, X_i)$ for some $j$ in a neighborhood of $X_i$. Then Calculation Lemma implies that the crossing number $c(G;T)$ is equal to the minimum of the cross-indexes $\varepsilon(D_{ij};X_i)$ for all $i, j$.

The following corollary is obtained by a combination of Lemmas 2.2, 2.5 and definition and some observation.
**Corollary 2.6.** The inequalities
\[ \varepsilon(G; T) = c(G; T) \geq c(G) \geq g(G) = g(G; T) \]
hold for every based graph \((G; T)\).

**Proof of Corollary 2.6.** The identity \(\varepsilon(G; T) = c(G; T)\) is given by Lemma 2.5. By definition, the inequality \(c(G; T) \geq c(G)\) is given. To see that \(c(G) \geq g(G)\), let \(D\) be a diagram of \(G\) with over-under information on the sphere \(S^2\) with \(c(D) = c(G)\). Put an upper arc around every crossing of \(D\) on a tube attaching to \(S^2\) to obtain a closed orientable surface of genus \(c(D)\) with \(G\) embedded (see Fig. 3). Hence the inequality \(c(G) \geq g(G)\) is obtained. The identity \(g(G) = g(G; T)\) is given by Lemma 2.2. (Incidentally, the inequality \(c(G; T) \geq g(G; T)\) is directly obtained by Lemma 2.4.) \(\square\)

**Figure 3.** Put an upper arc on a tube

For an arbitrary tree \(T\), the \(T\)-cross-index \(c^T(G)\) of a connected graph \(G\) is the minimal number of \(c(G; T')\) for all tree bases \(T'\) of \(G\) such that \(T'\) is homeomorphic to \(T\) if such a tree basis \(T'\) of \(G\) exists. Otherwise, let \(c^T(G) = \infty\). The \(T\)-cross-index \(c^T(G)\) is a topological invariant of a graph \(G\) associated to every tree \(T\), whose computation is in principle simpler than a computation of the crossing number \(c(G)\) by Calculation Lemma (Lemma 2.5).

Let \(c^*(G)\) be the family of the invariants \(c^T(G)\) of a connected graph \(G\) for all trees \(T\). The minimal value \(c^\text{min}(G)\) in the family \(c^*(G)\) has appeared as the crossing number of a \(\Gamma\)-unknotted graph in the paper [3] on a spatial graph.

The finite maximal value \(c^\text{max}(G)\) in the family \(c^*(G)\) is a well-defined invariant of a connected graph \(G\), because there are only finitely many tree bases \(T\) in \(G\). By definition, we have the following inequalities
\[ c^\text{max}(G) \geq c^\text{min}(G) \geq c(G) \geq g(G) \]
for every connected graph \(G\). By definition, the following properties are mutually equivalent:

(i) \(G\) is a planar graph.
(ii) \(c^\text{max}(G) = 0\).
(iii) \(c^\text{min}(G) = 0\).
(iv) \(c(G) = 0\).
(v) \(g(G) = 0\).
3. The invariants of a complete graph

Let $K_n$ be the $n$-complete graph. Let $n \geq 5$, because $K_n$ is planar for $n \leq 4$. To consider a tree basis $T$ of $K_n$, the following lemma is useful:

**Lemma 3.1.** For any two isomorphic tree bases $T$ and $T'$ of $K_n$, there is an automorphism of $K_n$ sending $T$ to $T'$. In particular, $c^T(K_n) = c(K_n; T)$ for every tree basis $T$ of $K_n$.

**Proof of Lemma 3.1.** Let $K_n$ be the 1-skelton of the $(n-1)$-simplex $A = \{v_0v_1 \ldots v_{n-1}\}$. The isomorphism $f$ from $T$ to $T'$ gives a permutation of the vertexes $v_i$ ($i = 0, 1, 2, \ldots, n-1$) which is induced by a linear automorphism $f_A$ of the $(n-1)$-simplex $A$. The restriction of $f_A$ to the 1-skelton $K_n$ of $A$ is an automorphism of $K_n$ sending $T$ to $T'$. □

A star-tree basis of $K_n$ is a tree basis $T^*$ of $K_n$ which is homeomorphic to a cone of $n-1$ points to a single point. By Lemma 2.5 (Calculation Lemma), the crossing number $c(K_n; T^*)$ of the based graph $(K_n; T^*)$ is calculated as follows.

**Lemma 3.2.** $c(K_n; T^*) = \frac{(n - 1)(n - 2)(n - 3)(n - 4)}{24}$

**Proof of Lemma 3.2.** Let $T_n^*$ denote the star-tree basis $T^*$ of $K_n$ in this proof. Since $K_5$ is non-planar, the computation $c(K_5; T_5^*) = 1$ is easily obtained (see Fig. 4). Suppose the calculation of $c(K_n; T_n^*)$ is done for $n \geq 5$. To consider $c(K_{n+1}; T_{n+1}^*)$, let the tree basis $T_{n+1}^*$ be identified with the 1-skelton $P^1$ of the stellar division of a regular convex $n$-gon $P$ in the plane at the origin $v_0$. Let $v_i$ ($i = 1, 2, \ldots, n$) be the linearly enumerated vertexes of $P^1$ in the boundary closed polygon $\partial P$ of $P$ in this order. We count the number of edges of $(K_n; T_n^*)$ contributing to the cross-index $\varepsilon(K_{n+1}; T_{n+1}^*)$. In the polygonal arcs of $\partial P$ divided by the vertexes $v_n, v_2$, the vertex $v_1$ and the vertexes $v_3, \ldots, v_{n-1}$ construct pairs of edges contributing to the cross-index 1. In the polygonal arcs of $\partial P$ divided by the vertexes $v_n, v_3$, the vertexes $v_1, v_2$ and the vertexes $v_4, \ldots, v_{n-1}$ construct pairs of edges contributing to the cross-index 1. Continue this process. As the final step, in the polygonal arcs of $\partial P$ divided by the vertexes $v_n, v_{n-2}$, the vertexes $v_1, v_2, \ldots, v_{n-3}$ and the vertex $v_{n-1}$ construct pairs of edges contributing to the cross-index 1. By Calculation Lemma, we have

\[
c(K_{n+1}; T_{n+1}^*) - c(K_n; T_n^*) = 1(n - 3) + 2(n - 4) + \cdots + (n - 3)(n - (n - 1)) = \sum_{k=1}^{n-3} k(n - 2 - k) = \frac{(n - 1)(n - 2)(n - 3)}{6},
\]

1Thanks to Y. Matsumoto for suggesting this calculation result.
so that
\[
c(K_{n+1}; T_{n+1}^*) = c(K_n; T_n^*) + \frac{(n-1)(n-2)(n-3)}{6}
\]
\[
= \frac{(n-1)(n-2)(n-3)(n-4)}{24} + \frac{(n-1)(n-2)(n-3)}{6}
\]
\[
= \frac{n(n-1)(n-2)(n-3)(n-4)}{24}
\]
Thus, the desired identity on \( c(K_n; T^*) = c(K_n; T_n^*) \) is obtained. □

For the crossing number \( c(K_n) \), R. K. Guy’s conjecture is known (see [2]):

Guy’s conjecture. \( c(K_n) = Z(n) \) where

\[
Z(n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor,
\]
where \( \left\lfloor \cdot \right\rfloor \) denotes the floor function.

Until now, this conjecture was confirmed to be true for \( n \leq 12 \). In fact, Guy confirmed that it is true for \( n \leq 10 \), and if it is true for any odd \( n \), then it is also true for \( n + 1 \). S. Pan and P. B. Richter in [7] confirmed that it is true for \( n = 11 \), so that it is also true for \( n = 12 \). Thus,
\[
c(K_n) = 1 \quad (n = 5), \quad 3 \quad (n = 6), \quad 9 \quad (n = 7), \quad 18 \quad (n = 8), \quad 36 \quad (n = 9),
\]
\[
60 \quad (n = 10), \quad 100 \quad (n = 11), \quad 150 \quad (n = 12).
\]
It is further known by D. McQuillana, S. Panb, R. B. Richterc in [6] that \( c(K_{13}) \) belongs to the set \( \{219, 221, 223, 225\} \) where 225 is the Guy’s conjecture.

Given a tree basis diagram \( X \) of a tree basis \( T \) of \( K_n \), we can construct a based diagram \( (D; X) \) of \( (K_n; T) \) by Lemma 3.1, so that \( c(K_n; T) \leq c(D; X) \).

To investigate \( c^\min(K_5) \) and \( c^\max(K_5) \), observe that the graph \( K_5 \) has just 3 non-isomorphic tree bases, namely a linear-tree basis \( T^L \), a \( T \)-shaped-tree basis \( T^* \) and a star-tree basis \( T^s \), where the \( T \)-shaped-tree basis \( T^* \) is a graph constructed by two linear three-vertex graphs \( \ell \) and \( \ell' \) by identifying the degree 2 vertex of \( \ell \) with a degree one vertex of \( \ell' \). Since any of \( T^L, T^s, T^* \) is embedded in the planar diagram obtained from the diagram of \( K_5 \) in Fig. 4 by removing the two crossing edges, we have \( c(K_5; T) \leq 1 \) for every tree basis \( T \) of \( K_5 \). Since \( c(K_5; T) \geq c(K_5) = 1 \),
\[
c(K_5) = c^\min(K_5) = c^\max(K_5) = 1.
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{k5star}
\caption{A based diagram of \( K_5 \) with a star-tree basis \( T^* = T_5^s \)}
\end{figure}

To investigate \( c^\min(K_6) \) and \( c^\max(K_6) \), observe that \( K_6 \) has just 6 non-isomorphic tree bases (see Fig. 5). In Appendix, it is shown how the non-isomorphic
tree bases are tabulated in case of the complete graph $K_{11}$. For every tree basis $T$ in Fig. 5, we can construct a based diagram $(D; X)$ of $(K_6, T)$ with $c(D; X) \leq 5$ by Lemma 3.1. Thus, by $c(K_6) = 3$ and $c(K_6; T^*) = 5$ and $c(K_6; T_L) \leq 3$ for a linear-tree basis $T_L$ of $K_6$ (see Fig. 6), we have

$$c(K_6) = c^{\min}(K_6) = c(K_6; T_L) = 3 < c(K_6; T^*) = c^{\max}(K_6) = 5.$$ 

In particular, this means that $c^{\max}(G)$ is different from $c(G)$ for a general connected graph $G$. It is observed in [2] that

$$c(K_n) \geq \frac{n}{4} \cdot \frac{n-1}{2} \cdot \frac{n-2}{2} \cdot \frac{n-3}{2} = \frac{n(n-1)(n-2)(n-3)}{64}.$$ 

More precisely, it is observed in [7] that

$$0.8594 Z(n) \leq c(K_n) \leq Z(n).$$ 

By Lemma 3.2, we have

$$c^{\max}(K_n) \geq c(K_n; T^*) = \frac{(n-1)(n-2)(n-3)(n-4)}{24}.$$ 

Hence the difference $c^{\max}(K_n) - c(K_n)$ is estimated as follows:

$$c^{\max}(K_n) - c(K_n) \geq \frac{(n-1)(n-2)(n-3)(n-4)}{24} - \frac{n(n-1)(n-2)(n-3)}{64} = \frac{(n-1)(n-2)(n-3)(5n-32)}{192}.$$ 

Hence we have

$$\lim_{n \to +\infty} (c^{\max}(K_n) - c(K_n)) = \lim_{n \to +\infty} c^{\max}(K_n) = +\infty.$$ 

As another estimation, we have

$$0 < \frac{c(K_n)}{c^{\max}(K_n)} \leq \frac{24}{n-1}(n-2)(n-3)(n-4) \cdot \frac{n(n-1)(n-2)(n-3)}{64} = \frac{3}{8} \cdot \frac{n}{n-4},$$

so that for $n \geq 16$

$$0 < \frac{c(K_n)}{c^{\max}(K_n)} \leq \frac{1}{2}.$$ 

Thus, we have the following lemma, which is used in § 4:

**Lemma 3.3.**

$$\lim_{n \to +\infty} (c^{\max}(K_n) - c(K_n)) = \lim_{n \to +\infty} c^{\max}(K_n) = +\infty,$$

$$0 < \frac{c(K_n)}{c^{\max}(K_n)} \leq \frac{1}{2} \quad (n \geq 16).$$

Here is a question on a relationship between the crossing number and the minimally based crossing number.

**Question (open).** $c(G) = c^{\min}(G)$ for every connected graph $G$?

The authors confirmed that

$$c(K_n) = c(K_n; T_L) = c^{\min}(K_n)$$
Figure 5. The tree bases of $K_6$

Figure 6. Based diagrams of $K_6$ with a linear-tree basis $T^L$ and a star-tree basis $T^*$

$c(K_6; T^L) = 3$  
$c(K_6; T^*) = 5$

for $n \leq 12$, where $T^L$ is a linear-tree basis of $K_n$. The diagrams with minimal cross-index for $K_{11}$ and $K_{12}$ are given in Fig. 7 and Fig. 8, respectively. It is noted that if this question is yes for $K_{13}$, then the crossing number $c(K_{13})$ would be computable with use of a computer. If this question is no, then the $T$-cross-index $c^T(G)$ would be more or less a new invariant for every tree $T$. Some related questions on the cross-index of $K_n$ remain also open. Is there a linear-tree basis $T^L$ in $K_n$ with $c(K_n; T^L) = c_{\text{min}}(K_n)$ for every $n \geq 13$? Furthermore, is the linear-tree basis $T^L$ extendable to a Hamiltonian loop without crossing?

Quite recently, a research group of the second and third authors confirmed in [1] that

$$c(K_n; T^L) = Z(n)$$

for all $n$.

The genus $g(K_n)$ of $K_n$ is known by G. Ringel and J. W. T. Youngs in [8] to be

$$g(K_n) = \left\lfloor \frac{(n - 3)(n - 4)}{12} \right\rfloor$$

$$= 1 \ (n = 5, 6, 7), 2 \ (n = 8), 3 \ (n = 9), 4 \ (n = 10), 5 \ (n = 11), 6 \ (n = 12), \ldots,$$
where \([ \cdot ]\) denotes the ceiling function. Then the nullity \(\nu(K_n)\) of \(K_n\) is computed as follows:

\[
\nu(K_n) = 1 - \chi(K_n) - 2g(K_n) \\
= (n - 1)(n - 2)/2 - 2 \left\lceil \frac{(n - 3)(n - 4)}{12} \right\rceil \\
= 4(n = 5), 8(n = 6), 13(n = 7), 17(n = 8), 22(n = 9), 28(n = 10), \\
35(n = 11), 43(n = 12), \ldots
\]

![Figure 7. A diagram of \(K_{11}\) with minimal cross-index 100](image)

### 4. Independence of the cross-index

In this section, we show that the invariant \(c^*(G)\) is more or less a new invariant. For this purpose, for a real-valued invariant \(I(G)\) of a connected graph \(G\) which is not bounded when \(G\) goes over the range of all connected graphs, a virtualized invariant \(\hat{I}(G)\) of \(G\) is defined to be \(\hat{I}(G) = f(I(G))\) for a fixed non-constant real polynomial \(f(x)\) in \(x\). Every time a different non-constant polynomial \(f(t)\) is given, a different virtualized invariant \(\hat{I}(G)\) is obtained from the invariant \(I(G)\). The following theorem is the main result of this paper showing a certain independence between the cross-index \(c^*(G)\) and the other invariants \(c(G), g(G), \chi(G)\).

**Theorem 4.1.** Let \(c_{\text{max}}(G), \tilde{c}(G), \tilde{g}(G), \tilde{\chi}(G)\) be any virtualized invariants of the invariants \(c_{\text{max}}(G), c(G), g(G), \chi(G)\), respectively. Every linear combination of the invariants \(c_{\text{max}}(G), \tilde{c}(G), \tilde{g}(G), \tilde{\chi}(G)\) in real coefficients including a non-zero number is not bounded when \(G\) goes over the range of all connected graphs.

Let \(N(v_G)\) be the regular neighborhood of the vertex set \(v_G\) in \(G\). A connected graph \(G\) is vertex-congruent to a connected graph \(G'\) if there is a homeomorphism \(N(v_G) \cong N(v_{G'})\). Then we have the same Euler characteristic: \(\chi(G) = \chi(G')\).
To show this theorem, the following lemma is used.

**Lemma 4.2**

1. For every $n > 1$, there are vertex-congruent connected graphs $G^i (i = 0, 1, 2, \ldots, n)$ such that
   \[ c_{\text{max}}(G^i) = c(G^i) = g(G^i) = i \]
   for all $i$.

2. For every $n > 1$, there are connected graphs $H^i (i = 1, 2, \ldots, n)$ such that
   \[ c_{\text{max}}(H^i) = c(H^i) = i \quad \text{and} \quad g(H^i) = 1 \]
   for all $i$. 

Proof of Lemma 4.2. Use the based diagram \((D_5; X)\) of \(K_5\) in Fig. 4 with \(c(D_5; X) = c(K_5; T^*) = g(K_5) = 1\). Let \(D_5^0\) be the planar diagram without crossing obtained from \(D_5\) by smoothing the crossing, illustrated in Fig. 9. Let \(K_5^0\) be the planar graph given by \(D_5^0\). For the proof of (1), let \(G^0\) be the connected graph obtained from the \(n\) copies of \(K_5^0\) by joining \(n - 1\) edges one after another linearly by introducing them (see Fig. 10).

Let \(G^i (i = 1, 2, \ldots, n)\) be the connected graphs obtained from \(G^0\) by replacing the first \(i\) copies of \(K_5^0\) with the \(i\) copies of \(K_5\) (see Fig. 11 for \(i = 2\)). Since \(c(K_5; T) = 1\) for every tree basis \(T\) and every tree basis \(T^i\) of \(G^i\) is obtained from the \(i\) tree bases of \(K_5\) and the \(n - i\) tree bases of \(K_5^0\) by joining the \(n - 1\) edges one after another linearly. Then \(g(G^i) \leq c^\text{max}(G^i) \leq i\) for every \(i\). By Genus Lemma and Calculation Lemma, we obtain \(g(G^i) = g(G^i; T^i) \geq i\) so that
\[
g(G^i) = c(G^i) = c^\text{max}(G^i) = i
\]
for all \(i\), showing (1). For (2), let \(H^i\) be the graph obtained from \(K_5\) by replacing every edge except one edge by \(i\) multiple edges with \(|v_{H^i}| = |v_{K_5}| = 5\). Then \(g(H^i) = g(K_5) = 1\). Note that every tree basis \(T\) of \(H^i\) is homeomorphic to a tree basis of \(K_5\). Then the identity \(c^\text{max}(K_5) = 1\) implies \(c^\text{max}(H^i) \leq i\). Since \(H^i\) contains \(i\) distinct \(K_5\)-graphs with completely distinct edges except common one edge. Then we have \(c(H^i) \geq i\) and hence
\[
c^\text{max}(H^i) = c(H^i) = i\quad\text{and}\quad g(H^i) = 1
\]
for all \(i\). \(\square\)

By using Lemma 4.2, the proof of Theorem 4.1 is given as follows:

Proof of Theorem 4.1. Let
\[
c^\text{max}(G) = f_1(c^\text{max}(G)),
\]
\[
c(G) = f_2(c(G)),
\]
\[
g(G) = f_3(g(G)),
\]
\[
\chi(G) = f_4(\chi(G))
\]
for non-constant real polynomials \(f_i(x) (i = 1, 2, 3, 4)\). Suppose that the absolute value of a linear combination
\[
a_1c^\text{max}(G) + a_2c(G) + a_3g(G) + a_4\chi(G)
\]
with real coefficients \(a_i (i = 1, 2, 3, 4)\) is smaller than or equal to a positive constant \(a\) for all connected graphs \(G\). Then it is sufficient to show that \(a_1 = a_2 = a_3 = a_4 = 0\).

If \(G\) is taken to be a planar graph, then \(c^\text{max}(G) = c(G) = g(G) = 0\). There is an infinite family of connected planar graphs whose Euler characteristic family is not
bounded. Hence the polynomial $a_4f_4(x)$ is a constant polynomial in $x$. Since $f_4(x)$ is a non-constant polynomial in $x$, we must have $a_4 = 0$. Then the inequality

$$|a_1c^{\max}(G) + a_2\tilde{c}(G) + a_3\tilde{g}(G)| \leq a$$

holds. By Lemma 4.2 (1), the polynomial $a_1f_1(x) + a_2f_2(x) + a_3f_3(x)$ in $x$ must be a constant polynomial. By Lemma 4.2 (2), the polynomial $a_1f_1(x) + a_2f_2(x)$ in $x$ must be a constant polynomial. These two claims mean that the polynomial $a_3f_3(x)$ is a constant polynomial in $x$, so that $a_3 = 0$ since $f_3(x)$ is a non-constant polynomial. Let $a' = a_1f_1(x) + a_2f_2(x)$ which is a constant polynomial in $x$. Then

$$a_1c^{\max}(G) + a_2\tilde{c}(G) = a_1(f_1(c^{\max}(G)) - f_1(c(G))) + a'$$

so that

$$|a_1(f_1(c^{\max}(G)) - f_1(c(G))) + a'| \leq a$$

for all connected graphs $G$. By Lemma 3.3, we have

$$\lim_{n \to +\infty} (c^{\max}(K_n) - c(K_n)) = \lim_{n \to +\infty} c^{\max}(K_n) = +\infty,$$

$$0 < \frac{c(K_n)}{c^{\max}(K_n)} \leq \frac{1}{2} \ (n \geq 16).$$

Let $d$ and $e$ be the highest degree and the highest degree coefficient of the polynomial $f_1(t)$. Then we have

$$\lim_{n \to +\infty} |f_1(c^{\max}(K_n)) - f_1(c(K_n))|$$

$$= \lim_{n \to +\infty} \left| c^{\max}(K_n)^d \left(1 - \left(\frac{c(K_n)}{c^{\max}(K_n)}\right)^d\right)\right| = +\infty.$$ 

Thus, we must have $a_1 = 0$, so that $a_1 = a_2 = a_3 = a_4 = 0$. □

5. Appendix: Tabulation of the tree bases of $K_{11}$

In this appendix, it is shown how the non-isomorphic tree bases are tabulated in case of the complete graph $K_{11}$. This tabulation method is important to compute the $T$-cross index $c^T(K_n)$ for a tree basis $T$ of $K_n$, which is equal to the cross-index $c(K_n; T)$ by Lemma 3.1.

Our tabulation method is based on a formula on the numbers of vertexes with respect to degrees. Let $T$ be a tree on the 2-sphere, and $v_i$ the number of vertexes of $T$ of degree $i$. Then the number $V$ of the vertexes of $T$ is the sum of all $v_i$s for $i = 1, 2, \ldots$;

$$V = v_1 + v_2 + \cdots + v_i + \ldots.$$ 

Since there are $i$ edges around every vertex of degree $i$ and each edge has two end points, the total number $E$ of edges of $T$ is as follows:

$$E = \frac{1}{2} (v_1 + 2v_2 + 3v_3 + \cdots + iv_i + \ldots).$$

Since $T$ is a tree, the number $F$ of faces of $T$ is 1. Then the following formula is obtained from the Euler characteristic of the 2-sphere $V - E + F = 2$:

$$v_1 = 2 + v_3 + 2v_4 + \cdots + (i-2)v_i + \ldots.$$ (1)
Let $V = 11$, i.e., let $T$ be a tree basis of $K_{11}$. Since $E = 10$ by the Euler characteristic, the following equality holds:

$$\frac{1}{2} (v_1 + 2v_2 + 3v_3 + \cdots + 10v_{10}) = 10.$$  

From the equalities (1) and (2), the following formula is obtained:

$$v_2 + 2v_3 + 3v_4 + \cdots + (i-1)v_i + \cdots + 9v_{10} = 9.$$  

In Table 1, all the possible combinations of $v_i$s which satisfy $V = 11$ and the formula (3) are listed. In Fig. 12, all the graphs in Table 1 are shown, where degree-two vertexes are omitted for simplicity. By giving vertexes with degree two to each graph in Fig. 12, all the tree bases of $K_{11}$ are obtained as shown in Figs. 13, 14, 15 and 16.

References


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Table 1
Figure 12. The tree bases of $K_{11}$ without degree-two vertexes.
Figure 13. The tree bases of type A to Q.
Figure 14. The tree bases of type R to W.
Figure 15. The tree bases of type X to Z.
Figure 16. The tree bases of type $\alpha$ to $\delta$. 