Cross-index of a graph

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Abstract. A family of topological invariants of a connected graph associated to all trees is introduced. The member of the family associated to a tree T is called the T-cross-index, which takes a non-negative integer or infinity according to whether T is a tree basis of the graph or not. It is shown how this cross-index is independent of the other topological invariants of a connected graph such as the Euler characteristic, the crossing number and the genus.

1. Introduction

A *based graph* is a pair $(G;T)$ such that G is a connected graph and T is a maximal tree of G, called a *tree basis* of G. A based diagram $(D;X)$ of the based graph $(G;T)$ is defined in § 2. Then the *crossing number* $c(G;T)$ of the based graph (G,T) is defined to be the minimum of the crossing numbers $c(D;X)$ for all based diagrams $(D;X)$ of $(G;T)$. The genus $g(D;X)$, the nullity $\nu(D;X)$ and the cross-index $\varepsilon(D;X)$ are defined by using the $\mathbb{Z}_2$-form

$$\varepsilon : \mathbb{Z}_2[D;X] \times \mathbb{Z}_2[D;X] \to \mathbb{Z}_2$$

on $(D;X)$, which are invariants of non-negative integer values of the based diagram $(D;X)$. The genus $g(G;T)$ and the cross-index $\varepsilon(G;T)$ are defined to be the minimums of the genera $g(D;X)$ and the cross-indexes $\varepsilon(D;X)$ for all based diagrams $(D;X)$ of $(G;T)$, respectively, whereas the nullity $\nu(G;T)$ is defined to be the maximum of the nullities $\nu(D;X)$ for all based diagrams $(D;X)$ of $(G;T)$. Thus, $c(G;T)$, $g(G;T)$, $\nu(G;T)$, $\varepsilon(G;T)$ are topological invariant of the based graph $(G;T)$. The relationships between the topological invariants $c(G;T)$, $g(G;T)$, $\nu(G;T)$, $\varepsilon(G;T)$ of $(G;T)$, the genus $g(G)$ and the Euler characteristic $\chi(G)$ of the graph G are explained in § 2. In particular, the identities

$$c(G;T) = \varepsilon(G;T), \quad g(G;T) = g(G), \quad \nu(G;T) = 1 - \chi(G) - 2g(G)$$

are established. In particular, it turns out that the crossing number $c(G;T)$ of $(G;T)$ is a calculable invariant in principle. The idea of a cross-index is also applied to study complexities of a knitting pattern in [**5**].

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This invariant $c(G;T)$ is modified into a topological invariant $c^T(G)$ of a graph G associated to a tree T, called the *T-cross-index* of G as follows:

Namely, define $c^T(G)$ to be the minimum of the invariants $c(G;T')$ for all tree bases T' of G homeomorphic to T. If there is no tree basis of G homeomorphic to T, then define $c^T(G)=\infty$.

Let $c^*(G)$ be the family of the invariants $c^T(G)$ for all trees T. The minimal value $c^{\min}(G)$ in the family $c^*(G)$ has been appeared as the crossing number of a Γ-unknotted graph in the paper [**3**] on spatial graphs. The crossing number $c(G)$ of a graph G is defined to be the minimum of the crossing numbers $c(D)$ of all diagrams D of G (in the plane). It is an open question whether $c^{\min}(G)$ is equal to the crossing number $c(G)$ of any connected graph G.

The finite maximal value $c^{\max}(G)$ in the family $c^*(G)$ is a well-defined invariant, because there are only finitely many tree bases of G. In the inequalities

$$c^{\max}(G) \geq c^{\min}(G) \geq c(G) \geq g(G)$$

for every connected graph G which we establish, the following properties are mutually equivalent:

(i) G is a planar graph.

(ii) $c^{\max}(G)=0$.

(iii) $c^{\min}(G)=0$.

(iv) $c(G)=0$.

(v) $g(G)=0$.

In § 3, the case of the n-complete graph K_n ($n \geq 5$) is discussed in a connection to Guy's conjecture on the crossing number $c(K_n)$. It is shown that $c^{\min}(K_5)=c^{\max}(K_5)$ and $c^{\min}(K_n) < c^{\max}(K_n)$ for every $n \geq 6$. Thus, the invariants $c^{\min}(G)$ and $c^{\max}(G)$ are different invariants for a general connected graph G.

The main purpose of this paper is to show that the family $c^*(G)$ is more or less a new invariant. For this purpose, for a real-valued invariant $I(G)$ of a connected graph G which is not bounded when G goes over the range of all connected graphs, we introduce a *virtualized invariant* $\tilde{I}(G)$ of G which is defined to be $\tilde{I}(G)=f(I(G))$ for a fixed *non-constant* real polynomial $f(x)$ in x. Every time a different non-constant polynomial $f(t)$ is given, a different virtualized invariant $\tilde{I}(G)$ is obtained from the invariant $I(G)$. Then the main result is stated as follows, showing a certain independence between the cross-index $c^*(G)$ and the other invariants $c(G), g(G), \chi(G)$.

Theorem 4.1. Let $\tilde{c}^{\max}(G), \tilde{c}(G), \tilde{g}(G), \tilde{\chi}(G)$ be any virtualized invariants of the invariants $c^{\max}(G), c(G), g(G), \chi(G)$, respectively. Every linear combination of the invariants $\tilde{c}^{\max}(G), \tilde{c}(G), \tilde{g}(G), \tilde{\chi}(G)$ in real coefficients including a non-zero number is not bounded when G goes over the range of all connected graphs.

The proof of this theorem is given in § 4.

As an appendix, it is shown how the non-isomorphic tree bases are tabulated in case of the complete graph K_{11}. This tabulation method is important to compute the T-cross index $c^T(K_n)$ for a tree basis T of K_n, because we have $c^T(K_n) = c(K_n;T)$ for every tree basis T (see Lemma 3.1).

2. The cross-index of a graph associated to a tree

By a *graph*, a connected graph G with only topological edges and without vertexes of degrees 0, 1 and 2 is meant. Let G have $n(\geq 1)$ vertexes and $s(\geq 1)$ edges. By definition, if $n = 1$ (that is, G is a bouquet of loops), then every tree basis T of G has one vertex. A *diagram* of a graph G is a representation D of G in the plane so that the vertexes of G are represented by distinct points and the edges of G are represented by arcs joining the vertexes which may have transversely meeting double points avoiding the vertexes. A double point on the edges of a diagram D is called a *crossing* of D. In this paper, to distinguish between a degree 4 vertex and a crossing, a crossing is denoted by a crossing with over-under information except in Figs. 7, 8 representing diagrams of the graphs K_{11} and K_{12} without degree 4 vertexes. A *tree diagram* of a tree T is a diagram X of T without crossings. A *based diagram* of a based graph $(G;T)$ is a pair $(D;X)$ where D is a diagram of G and X is a sub-diagram of D such that X is a tree diagram of the tree basis T without crossings in D. In this case, the diagram X is called a *tree basis diagram.* The following lemma is used without proof in the author's earlier papers [**3, 4**].

Lemma 2.1. Given any based graph $(G;T)$ in $\mathbb{R}^3$, then every spatial graph diagram of G is transformed into a based diagram $(D;X)$ of $(G;T)$ only with crossings with over-under information by the Reidemeister moves I-V (see Fig. 1).

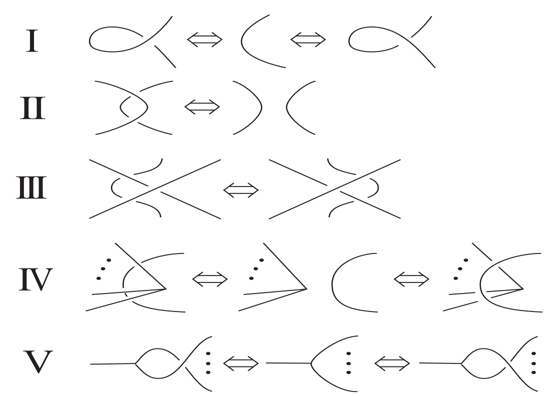


FIGURE 1. The Reidemeister moves

Proof of Lemma 2.1. In any spatial graph diagram D' of G, first transform the sub-diagram $D(T)$ of the tree basis T in D' into a tree diagram X by the Reidemeister moves I-V. Since a regular neighborhood $N(X;\mathbb{R}^2)$ of X in the plane $\mathbb{R}^2$ is a disk, a based diagram is obtained by shrinking this tree diagram into a very small tree diagram within the disk by the Reidemeister moves I-V. See Fig. 2 for this transformation. Thus, we have a based diagram $(D;X)$ of $(G;T)$ only with crossings with over-under information. $\square$

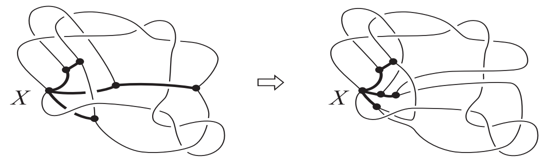


FIGURE 2. Transforming a diagram with a tree graph into a based diagram by shrinking the tree graph

The crossing number $c(D)$ of a based diagram $(D;X)$ is denoted by $c(D;X)$. The *crossing number* $c(G;T)$ of a based graph $(G;T)$ is the minimal number of the crossing numbers $c(D;X)$ of all based diagrams $(D;X)$ of $(G;T)$. For a based diagram $(D;X)$ of $(G;T)$, let $N(X;D) = D \cap N(X;\mathbb{R}^2)$ be a regular neighborhood of X in the diagram D. Then the complement $\mathrm{cl}(D \setminus N(X;D))$ is a *tangle diagram* of m-strings a_i $(i = 1, 2, \ldots, m)$ in the disk $\Delta = S^2 \setminus N(X;\mathbb{R}^2)$ where $S^2 = \mathbb{R}^2 \cup \{\infty\}$ denotes the 2-sphere which is the one-point compactification of the plane $\mathbb{R}^2$.

Let $\mathbb{Z}_2[D;X]$ be the $\mathbb{Z}_2$ vector space with the arcs a_i $(i = 1, 2, \ldots, m)$ as a $\mathbb{Z}_2$-basis. For any two arcs a_i and a_j with $i \neq j$, the *cross-index* $\varepsilon(a_i, a_j)$ is defined to be 0 or 1 according to whether the two boundary points ∂a_j of the arc a_j are in one component of the two open arcs $\partial\Delta \setminus \partial a_i$ or not, respectively. For $i = j$, the identity $\varepsilon(a_i, a_j) = 0$ is taken. Then the cross-index $\varepsilon(a_i, a_j)$ (mod 2) defines the symmetric bilinear $\mathbb{Z}_2$-form

$$\varepsilon : \mathbb{Z}_2[D;X] \times \mathbb{Z}_2[D;X] \to \mathbb{Z}_2,$$

called the $\mathbb{Z}_2$-*form* on $(D;X)$. The *genus* $g(D;X)$ of the based diagram $(D;X)$ is defined to be half of the $\mathbb{Z}_2$-rank of the $\mathbb{Z}_2$-form $\varepsilon : \mathbb{Z}_2[D;X] \times \mathbb{Z}_2[D;X] \to \mathbb{Z}_2$, which is seen to be even since the $\mathbb{Z}_2$-form ε is a $\mathbb{Z}_2$-symplectic form.

The *genus* $g(G;T)$ of a based graph $(G;T)$ is the minimum of the genus $g(D;X)$ for all based diagrams $(D;X)$ of $(G;T)$. The following lemma shows that the genus $g(G)$ of a graph G is calculable from based diagrams $(D;X)$ of any based graph $(G;T)$ of G.

Lemma 2.2 (Genus Lemma). $g(G) = g(D;X) = g(G;T)$ for any based diagram $(D;X)$ of any based graph (G,T).

Proof of Lemma 2.2. Let $(D;X)$ be a based diagram of a based graph $(G;T)$ with $g(D;X) = g(G;T)$. Constructs a compact connected orientable surface $N(D;X)$ from $(D;X)$ such that

(1) the surface $N(D;X)$ is a union of a disk N in $\mathbb{R}^2$ with the tree basis diagram X as a spine and attaching bands B_i $(i = 1, 2, \ldots, m)$ whose cores are the edges a_i $(i = 1, 2, \ldots, m)$ of D,
(2) the $\mathbb{Z}_2$-form $\varepsilon : \mathbb{Z}_2[D;X] \times \mathbb{Z}_2[D;X] \to \mathbb{Z}_2$ is isomorphic to the $\mathbb{Z}_2$-intersection form on $H_1(N(D;X);\mathbb{Z}_2)$.

Because the nullity of the $\mathbb{Z}_2$-intersection form on $H_1(N(D;X);\mathbb{Z}_2)$ is equal to the number of the boundary components of the bounded surface $N(D;X)$ minus one, the genus $g(N(D;X))$ is equal to the half of the $\mathbb{Z}_2$-rank of the $\mathbb{Z}_2$-form ε. This implies that

$$g(G;T) = g(D;X) = g(N(D;X)) \geq g(G).$$

Conversely, let F be a compact connected orientable surface containing G with genus $g(F) = g(G)$, where F need not be closed. For any based graph $(G;T)$, let $N(G)$ be a regular neighborhood of G in F, which is obtained from a disk N in F with the tree basis T as a spine by attaching bands $B_i\,(i = 1, 2, \ldots, m)$ whose cores are the edges $a_i\,(i = 1, 2, \ldots, m)$ of G. Then the inequality $g(N(G)) \leq g(F)$ holds. Let $(D;X)$ be any based diagram of the based graph $(G;T)$. Identify the disk N with a disk $N(X)$ with the tree basis T as a spine. By construction, the $\mathbb{Z}_2$-form $\varepsilon : \mathbb{Z}_2[D;X] \times \mathbb{Z}_2[D;X] \to \mathbb{Z}_2$ is isomorphic to the $\mathbb{Z}_2$-intersection form on $H_1(N(G);\mathbb{Z}_2)$, which determines the genus $g(N(G))$ as the half of the $\mathbb{Z}_2$-rank of ε. Thus, the inequality

$$g(G;T) \leq g(D;X) = g(N(G)) \leq g(F) = g(G)$$

holds and we have $g(G) = g(D;T) = g(G;T)$ for any based diagram $(D;X)$ of $(G;T)$. $\square$

The *nullity* $\nu(D;X)$ of $(D;X)$ is the nullity of the $\mathbb{Z}_2$-form $\varepsilon : \mathbb{Z}_2[D;X] \times \mathbb{Z}_2[D;X] \to \mathbb{Z}_2$. The *nullity* $\nu(G;T)$ of a based graph $(G;T)$ is the maximum of the nullity $\nu(D;X)$ for all based diagrams $(D;X)$ of $(G;T)$. Then the following corollary is obtained:

Corollary 2.3. The identity $\nu(G;T) = 1-\chi(G)-2g(G)$ holds for any based graph (G,T).

This corollary shows that the nullity $\nu(G;T)$ is independent of a choice of tree bases T of G, and is therefore simply called the *nullity* of G and denoted by $\nu(G)$.

Proof of Corollary 2.3. The graph G is obtained from the tree graph $N(X;D)$ by attaching the mutually disjoint m-strings $a_i\,(i = 1, 2, \ldots, m)$. Since the $\mathbb{Z}_2$-rank of $\mathbb{Z}_2[D;X]$ is m by definition, we see from a calculation of the Euler characteristic $\chi(G)$ that $\chi(G) = 1+m-2m = 1-m$. By the identity $m = 2g(D;X)+\nu(D;X)$ on the rank and the nullity of the $\mathbb{Z}_2$-form ε, the nullity $\nu(D;X)$ of a based diagram $(D;X)$ of $(G;T)$ is given by $\nu(D;X) = m - 2g(D;X) = 1 - \chi(G) - 2g(D;X)$. Hence we have

$$\nu(G;T) = 1 - \chi(G) - 2g(G;T) = 1 - \chi(G) - 2g(G)$$

by Lemma 2.2. $\square$

The *cross-index* of a based diagram $(D;X)$ is the non-negative integer $\varepsilon(D;X)$ defined by

$$\varepsilon(D;X) = \sum_{1\leq i<j\leq m} \varepsilon(a_i, a_j).$$

The following lemma is obtained:

Lemma 2.4. For every based diagram $(D;X)$, the inequality $\varepsilon(D;X) \geq g(D;X)$ holds.

Proof of Lemma 2.4. Let V be the $\mathbb{Z}_2$-matrix representing the $\mathbb{Z}_2$-form $\varepsilon : \mathbb{Z}_2[D;X] \times \mathbb{Z}_2[D;X] \to \mathbb{Z}_2$ with respect to the arc basis a_i $(i = 1, 2, \ldots, m)$. Let $\varepsilon_{ij} = \varepsilon(a_i, a_j)$ be the (i, j)-entry of the matrix V. The $\mathbb{Z}_2$-rank r of the matrix V is equal to $2g(D;X)$ by definition. There are r column vectors in V that are $\mathbb{Z}_2$-linearly independent. By changing the indexes of the arc basis a_i, we can find a sequence of integral pairs $(i_k, j_k)\,(k = 1, 2, \ldots, r)$ with $1 \le i_1 < i_2 \cdots < i_r \le m$ and $1 \le j_1 < j_2 < \cdots < j_r \le m$ such that $\varepsilon_{i_k j_k} = 1$ for all $k\,(k = 1, 2, \ldots, r)$. Here, note that this sequence $(i_k, j_k)\,(k = 1, 2, \ldots, r)$ may contain two pairs $(i_k, j_k), (i_{k'}, j_{k'})$ with $k \ne k'$ and $(i_k, j_k) = (j_{k'}, i_{k'})$. By the identities $\varepsilon_{ii} = 0$ and $\varepsilon_{ij} = \varepsilon_{ji}$ for all i, j, we have

$$2\varepsilon(D;X) = \sum_{i \le i,j \le m} \varepsilon_{ij} \ge \sum_{k=1}^{r} \varepsilon_{i_k j_k} = r = 2g(D;X).$$

Thus, the inequality $\varepsilon(D;X) \ge g(D;X)$ is obtained. □

The *cross-index* $\varepsilon(G;T)$ of a based graph $(G;T)$ is the minimum of the cross-index $\varepsilon(D;X)$ for all based diagrams $(D;X)$ of $(G;T)$. It may be used to compute the crossing number $c(G;T)$ of a based graph $(G;T)$ as it is stated in the following lemma:

Lemma 2.5 (Calculation Lemma). $\varepsilon(G;T) = c(G;T)$ for every based graph $(G;T)$.

Proof of Lemma 2.5. Let a_i $(i = 1, 2, \ldots, m)$ be an arc basis of a based diagram $(D;X)$ of $(G;T)$ attaching to the boundary of a regular neighborhood disk N of X in the plane.

By a homotopic deformation of a_i into an embedded arc a'_i keeping the boundary points fixed, we construct a new based diagram $(D';X)$ of $(G;T)$ with a basis a'_i $(i = 1, 2, \ldots, m)$ so that

(1) $a'_i \cap a'_j = \emptyset$ if $\varepsilon(a_i, a_j) = 0$ and $i \ne j$,

(2) a'_i and a'_j meet one point transversely if $\varepsilon(a_i, a_j) = 1$.

Then the cross-index $\varepsilon(D;X)$ is equal to the crossing number $c(D';X)$ of the based diagram $(D';X)$ of $(G;T)$. Hence the inequality $\varepsilon(G;T) \ge c(G;T)$ is obtained. Since $\varepsilon(D;X) \le c(D;X)$ for every based graph $(D;X)$ of $(G;T)$, the inequality $\varepsilon(G;T) \le c(G;T)$ holds. Hence the identity $\varepsilon(G;T) = c(G;T)$ holds. □

Calculation Lemma (Lemma 2.5) gives a computation method of the crossing number $c(G;T)$ of a based graph $(G;T)$ in a finite procedure.

In fact, let $X_i\,(i = 1, 2, \ldots, s)$ be all the tree basis diagrams of T in $\mathbb{R}^2$. For every i, let $(D_{ij}, X_i)\,(j = 1, 2, \ldots, t_i)$ be a finite set of based diagrams of (G, T) such that every based diagram (D, X_i) of (G, T) coincides with a based diagram (D_{ij}, X_i) for some j in a neighborhood of X_i. Then Calculation Lemma implies that the crossing number $c(G;T)$ is equal to the minimum of the cross-indexes $\varepsilon(D_{ij}; X_i)$ for all i, j.

The following corollary is obtained by a combination of Lemmas 2.2, 2.5 and definition and some observation.

Corollary 2.6. The inequality

$$\varepsilon(G;T) = c(G;T) \geq c(G) \geq g(G) = g(G;T)$$

holds for every based graph $(G;T)$.

Proof of Corollary 2.6. The identity $\varepsilon(G;T) = c(G;T)$ is given by Lemma 2.5. By definition, the inequality $c(G;T) \geq c(G)$ is given. To see that $c(G) \geq g(G)$, let D be a diagram of G with over-under information on the sphere S^2 with $c(D) = c(G)$. Put an upper arc around every crossing of D on a tube attaching to S^2 to obtain a closed orientable surface of genus $c(D)$ with G embedded (see Fig. 3). Hence the inequality $c(G) \geq g(G)$ is obtained. The identity $g(G) = g(G;T)$ is given by Lemma 2.2. (Incidentally, the inequality $c(G;T) \geq g(G;T)$ is directly obtained by Lemma 2.4.) $\square$

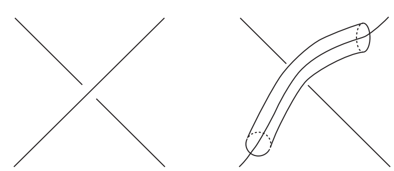


FIGURE 3. Put an upper arc on a tube

For an arbitrary tree T, the T-*cross-index* $c^T(G)$ of a connected graph G is the minimal number of $c(G;T')$ for all tree bases T' of G such that T' is homeomorphic to T if such a tree basis T' of G exists. Otherwise, let $c^T(G) = \infty$. The T-cross-index $c^T(G)$ is a topological invariant of a graph G associated to every tree T, whose computation is in principle simpler than a computation of the crossing number $c(G)$ by Calculation Lemma (Lemma 2.5).

Let $c^*(G)$ be the family of the invariants $c^T(G)$ of a connected graph G for all trees T. The minimal value $c^{\min}(G)$ in the family $c^*(G)$ has appeared as the crossing number of a Γ-unknotted graph in the paper [**3**] on a spatial graph.

The finite maximal value $c^{\max}(G)$ in the family $c^*(G)$ is a well-defined invariant of a connected graph G, because there are only finitely many tree bases T in G. By definition, we have the following inequalities

$$c^{\max}(G) \geq c^{\min}(G) \geq c(G) \geq g(G)$$

for every connected graph G. By definition, the following properties are mutually equivalent:

(i) G is a planar graph.

(ii) $c^{\max}(G) = 0$.

(iii) $c^{\min}(G) = 0$.

(iv) $c(G) = 0$.

(v) $g(G) = 0$.

3. The invariants of a complete graph

Let K_n be the n-complete graph. Let $n \geq 5$, because K_n is planar for $n \leq 4$. To consider a tree basis T of K_n, the following lemma is useful:

Lemma 3.1. For any two isomorphic tree bases T and T' of K_n, there is an automorphism of K_n sending T to T'. In particular, $c^T(K_n) = c(K_n; T)$ for every tree basis T of K_n.

Proof of Lemma 3.1. Let K_n be the 1-skelton of the $(n-1)$-simplex $A = |v_0 v_1 \dots v_{n-1}|$. The isomorphism f from T to T' gives a permutation of the vertexes v_i $(i = 0, 1, 2, \dots, n-1)$ which is induced by a linear automorphism f_A of the $(n-1)$-simplex A. The restriction of f_A to the 1-skelton K_n of A is an automorphism of K_n sending T to T'. $\square$

A *star-tree basis* of K_n is a tree basis T^* of K_n which is homeomorphic to a cone of $n-1$ points to a single point. By Lemma 2.5 (Calculation Lemma), the crossing number $c(K_n; T^*)$ of the based graph $(K_n; T^*)$ is calculated as follows.

Lemma 3.2. $c(K_n; T^*) = \dfrac{(n-1)(n-2)(n-3)(n-4)}{24}$.[1]

Proof of Lemma 3.2. Let T^*_n denote the star-tree basis T^* of K_n in this proof. Since K_5 is non-planar, the computation $c(K_5; T^*_5) = 1$ is easily obtained (see Fig. 4). Suppose the calculation of $c(K_n; T^*_n)$ is done for $n \geq 5$. To consider $c(K_{n+1}; T^*_{n+1})$, let the tree basis T^*_{n+1} be identified with the 1-skelton P^1 of the stellar division of a regular convex n-gon P (in the plane) at the origin v_0. Let v_i $(i = 1, 2, \dots, n)$ be the linearly enumerated vertexes of P^1 in the boundary closed polygon ∂P of P in this order. We count the number of edges of $(K_{n+1}; T^*_{n+1})$ added to $(K_n : T^*_n)$ contributing to the cross-index $\varepsilon(K_{n+1}; T^*_{n+1})$. In the polygonal arcs of ∂P divided by the vertexes v_n, v_2, the vertex v_1 and the vertexes $v_3, \dots, v_{n-1}$ construct pairs of edges contributing to the cross-index 1. In the polygonal arcs of ∂P divided by the vertexes v_n, v_3, the vertexes v_1, v_2 and the vertexes $v_4, \dots, v_{n-1}$ construct pairs of edges contributing to the cross-index 1. Continue this process. As the final step, in the polygonal arcs of ∂P divided by the vertexes v_n, v_{n-2}, the vertexes $v_1, v_2, \dots, v_{n-3}$ and the vertex v_{n-1} construct pairs of edges contributing to the cross-index 1. By Calculation Lemma, we have

$$
\begin{aligned}
c(K_{n+1}; T^*_{n+1}) - c(K_n; T^*_n) &= 1(n-3) + 2(n-4) + \cdots + (n-3)(n-(n-1)) \\
&= \sum_{k=1}^{n-3} k(n-2-k) = \frac{(n-1)(n-2)(n-3)}{6},
\end{aligned}
$$

[1] Thanks to Y. Matsumoto for suggesting this calculation result.

so that

$$\begin{aligned} c(K_{n+1}; T^*_{n+1}) &= c(K_n; T^*_n) + \frac{(n-1)(n-2)(n-3)}{6} \\ &= \frac{(n-1)(n-2)(n-3)(n-4)}{24} + \frac{(n-1)(n-2)(n-3)}{6} \\ &= \frac{n(n-1)(n-2)(n-3)(n-4)}{24}. \end{aligned}$$

Thus, the desired identity on $c(K_n; T^*) = c(K_n; T^*_n)$ is obtained. □

For the crossing number $c(K_n)$, R. K. Guy's conjecture is known (see [**2**]):

Guy's conjecture. $c(K_n) = Z(n)$ where

$$Z(n) = \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor,$$

where $\lfloor\ \rfloor$ denotes the floor function.

Until now, this conjecture was confirmed to be true for $n \le 12$. In fact, Guy confirmed that it is true for $n \le 10$, and if it is true for any odd n, then it is also true for $n+1$. S. Pan and P. B. Richter in [**7**] confirmed that it is true for $n = 11$, so that it is also true for $n = 12$. Thus,

$$\begin{aligned} c(K_n) &= 1\ (n=5),\ \ 3\ (n=6),\ \ 9\ (n=7),\ \ 18\ (n=8),\ \ 36\ (n=9), \\ &\qquad 60\ (n=10),\ \ 100\ (n=11),\ \ 150\ (n=12). \end{aligned}$$

It is further known by D. McQuillana, S. Panb, R. B. Richterc in [**6**] that $c(K_{13})$ belongs to the set $\{219, 221, 223, 225\}$ where 225 is the Guy's conjecture.

Given a tree basis diagram X of a tree basis T of K_n, we can construct a based diagram $(D; X)$ of $(K_n; T)$ by Lemma 3.1, so that $c(K_n; T) \le c(D; X)$.

To investigate $c^{\min}(K_5)$ and $c^{\max}(K_5)$, observe that the graph K_5 has just 3 non-isomorphic tree bases, namely a linear-tree basis T^L, a T-shaped-tree basis T^s and a star-tree basis T^*, where the T-shaped-tree basis T^s is a graph constructed by two linear three-vertex graphs ℓ and ℓ' by identifying the degree 2 vertex of ℓ with a degree one vertex of ℓ'. Since any of T^L, T^s, T^* is embedded in the planar diagram obtained from the diagram of K_5 in Fig. 4 by removing the two crossing edges, we have $c(K_5; T) \le 1$ for every tree basis T of K_5. Since $c(K_5; T) \ge c(K_5) = 1$,

$$c(K_5) = c^{\min}(K_5) = c^{\max}(K_5) = 1.$$

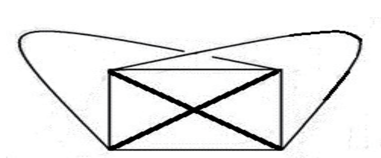


FIGURE 4. A based diagram of K_5 with a star-tree basis $T^* = T^*_5$

To investigate $c^{\min}(K_6)$ and $c^{\max}(K_6)$, observe that K_6 has just 6 non-isomorphic tree bases (see Fig. 5). In Appendix, it is shown how the non-isomorphic tree bases are tabulated in case of the complete graph K_{11}. For every tree basis T in Fig. 5, we can construct a based diagram $(D; X)$ of (K_6, T) with $c(D; X) \leq 5$ by Lemma 3.1. Thus, by $c(K_6) = 3$ and $c(K_6; T^*) = 5$ and $c(K_6; T^L) \leq 3$ for a linear-tree basis T^L of K_6 (see Fig. 6), we have

$$c(K_6) = c^{\min}(K_6) = c(K_6; T^L) = 3 < c(K_6; T^*) = c^{\max}(K_6) = 5.$$

In particular, this means that $c^{\max}(G)$ is different from $c(G)$ for a general connected graph G. It is observed in [**2**] that

$$c(K_n) \leq \frac{1}{4} \cdot \frac{n}{2} \cdot \frac{n-1}{2} \cdot \frac{n-2}{2} \cdot \frac{n-3}{2} = \frac{n(n-1)(n-2)(n-3)}{64}.$$

More precisely, it is observed in [**7**] that

$$0.8594 Z(n) \leq c(K_n) \leq Z(n).$$

By Lemma 3.2, we have

$$c^{\max}(K_n) \geq c(K_n; T^*) = \frac{(n-1)(n-2)(n-3)(n-4)}{24}.$$

Hence the difference $c^{\max}(K_n) - c(K_n)$ is estimated as follows:

$$\begin{aligned} c^{\max}(K_n) - c(K_n) \quad &\geq \quad \frac{(n-1)(n-2)(n-3)(n-4)}{24} - \frac{n(n-1)(n-2)(n-3)}{64} \\ &= \quad \frac{(n-1)(n-2)(n-3)(5n-32)}{192}. \end{aligned}$$

Hence we have

$$\lim_{n \to +\infty} (c^{\max}(K_n) - c(K_n)) = \lim_{n \to +\infty} c^{\max}(K_n) = +\infty.$$

As another estimation, we have

$$0 < \frac{c(K_n)}{c^{\max}(K_n)} \leq \frac{24}{(n-1)(n-2)(n-3)(n-4)} \cdot \frac{n(n-1)(n-2)(n-3)}{64} = \frac{3}{8} \cdot \frac{n}{n-4},$$

so that for $n \geq 16$

$$0 < \frac{c(K_n)}{c^{\max}(K_n)} \leq \frac{1}{2}.$$

Thus, we have the following lemma, which is used in § 4:

Lemma 3.3.

$$\begin{aligned} \lim_{n \to +\infty} (c^{\max}(K_n) - c(K_n)) \quad &= \quad \lim_{n \to +\infty} c^{\max}(K_n) = +\infty, \\ 0 < \frac{c(K_n)}{c^{\max}(K_n)} \quad &\leq \quad \frac{1}{2} \quad (n \geq 16). \end{aligned}$$

Here is a question on a relationship between the crossing number and the minimally based crossing number.

Question (open). $c(G) = c^{\min}(G)$ for every connected graph G ?

The authors confirmed that

$$c(K_n) = c(K_n; T^L) = c^{\min}(K_n)$$

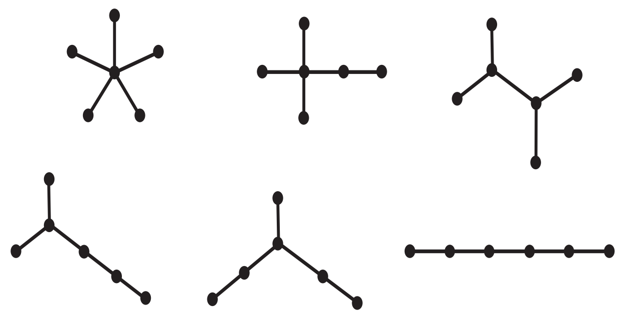
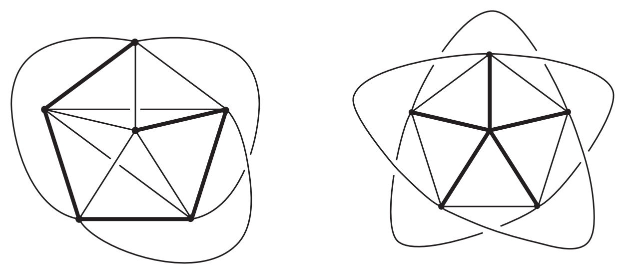


FIGURE 5. The tree bases of K_6



$c(K_6; T^L) = 3$ $\qquad$ $c(K_6; T^*) = 5$

FIGURE 6. Based diagrams of K_6 with a linear-tree basis T^L and a star-tree basis T^*

for $n \leq 12$, where T^L is a linear-tree basis of K_n. The diagrams with minimal cross-index for K_{11} and K_{12} are given in Fig. 7 and Fig. 8, respectively. It is noted that if this question is yes for K_{13}, then the crossing number $c(K_{13})$ would be computable with use of a computer. If this question is no, then the T-cross-index $c^T(G)$ would be more or less a new invariant for every tree T. Some related questions on the cross-index of K_n remain also open. *Is there a linear-tree basis T^L in K_n with $c(K_n; T^L) = c^{\min}(K_n)$ for every $n \geq 13$? Furthermore, is the linear-tree basis T^L extendable to a Hamiltonian loop without crossing* ?

Quite recently, a research group of the second and third authors confirmed in [**1**] that

$$c(K_n; T^L) = Z(n)$$

for all n.

The genus $g(K_n)$ of K_n is known by G. Ringel and J. W. T. Youngs in [**8**] to be

$$\begin{aligned} g(K_n) &= \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \\ &= 1\,(n = 5, 6, 7),\ 2(n = 8),\ 3\,(n = 9),\ 4\,(n = 10),\ 5\,(n = 11),\ 6\,(n = 12),\ \ldots, \end{aligned}$$

where $\lceil\ \rceil$ denotes the ceiling function. Then the nullity $\nu(K_n)$ of K_n is computed as follows:

$$\begin{aligned}\nu(K_n) &= 1-\chi(K_n)-2g(K_n)\\ &= (n-1)(n-2)/2-2\left\lceil\frac{(n-3)(n-4)}{12}\right\rceil\\ &= 4\,(n=5),\ 8\,(n=6),\ 13\,(n=7),\ 17\,(n=8),\ 22\,(n=9),\ 28\,(n=10),\\ &\qquad 35\,(n=11),\ 43\,(n=12),\ \ldots.\end{aligned}$$

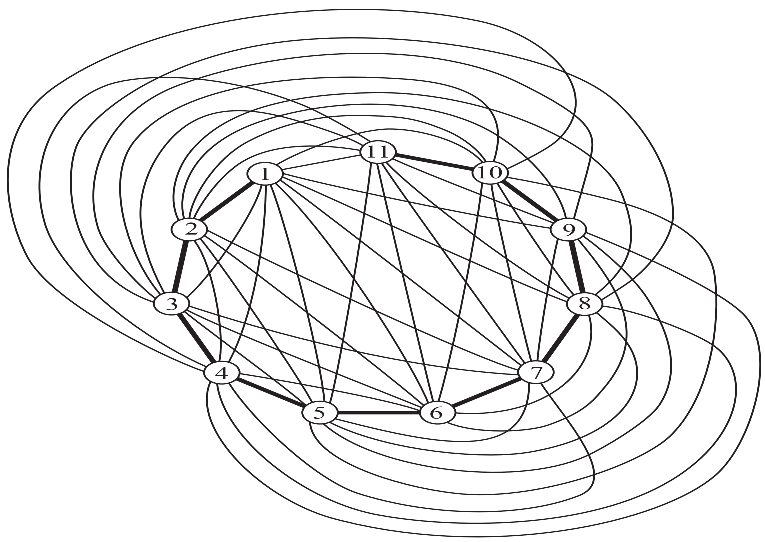


FIGURE 7. A diagram of K_{11} with minimal cross-index 100

4. Independence of the cross-index

In this section, we show that the invariant $c^*(G)$ is more or less a new invariant. For this purpose, for a real-valued invariant $I(G)$ of a connected graph G which is not bounded when G goes over the range of all connected graphs, a *virtualized invariant* $\tilde{I}(G)$ of G is defined to be $\tilde{I}(G) = f(I(G))$ for a fixed *non-constant* real polynomial $f(x)$ in x. Every time a different non-constant polynomial $f(t)$ is given, a different virtualized invariant $\tilde{I}(G)$ is obtained from the invariant $I(G)$. The following theorem is the main result of this paper showing a certain independence between the cross-index $c^*(G)$ and the other invariants $c(G), g(G), \chi(G)$.

Theorem 4.1. Let $\tilde{c}^{\max}(G), \tilde{c}(G), \tilde{g}(G), \tilde{\chi}(G)$ be any virtualized invariants of the invariants $c^{\max}(G), c(G), g(G), \chi(G)$, respectively. Every linear combination of the invariants $\tilde{c}^{\max}(G), \tilde{c}(G), \tilde{g}(G), \tilde{\chi}(G)$ in real coefficients including a non-zero number is not bounded when G goes over the range of all connected graphs.

Let $N(v_G)$ be the regular neighborhood of the vertex set v_G in G. A connected graph G is *vertex-congruent* to a connected graph G' if there is a homeomorphism $N(v_G) \cong N(v_{G'})$. Then we have the same Euler characteristic: $\chi(G) = \chi(G')$.

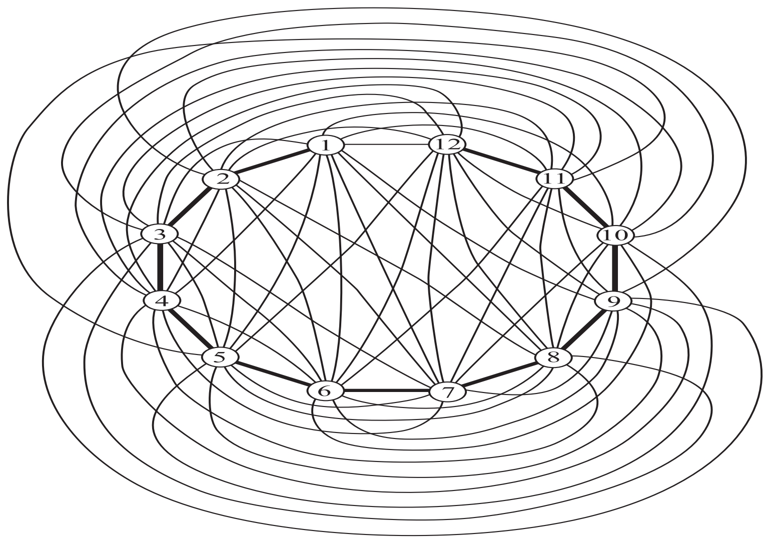


Figure 8. A diagram of K_{12} with minimal cross-index 150

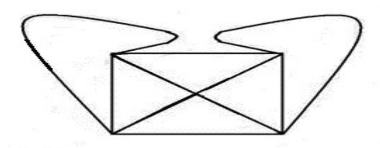


Figure 9. The planar diagram D_5^0

To show this theorem, the following lemma is used.

Lemma 4.2
(1) For every $n > 1$, there are vertex-congruent connected graphs G^i $(i = 0, 1, 2, \dots, n)$ such that

$$c^{\max}(G^i) = c(G^i) = g(G^i) = i$$

for all i.
(2) For every $n > 1$, there are connected graphs H^i $(i = 1, 2, \dots, n)$ such that

$$c^{\max}(H^i) = c(H^i) = i \quad \text{and} \quad g(H^i) = 1$$

for all i.

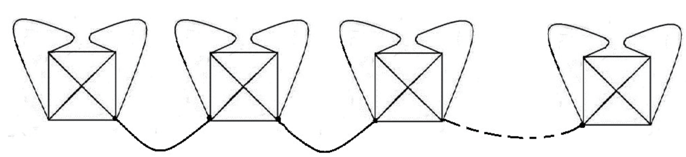


Figure 10. The planar diagram G^0

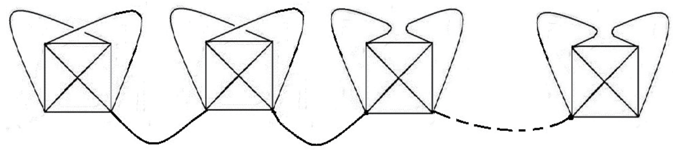


FIGURE 11. The graph G^2

Proof of Lemma 4.2. Use the based diagram $(D_5; X)$ of K_5 in Fig. 4 with $c(D_5; X) = c(K_5; T^*) = g(K_5) = 1$. Let D_5^0 be the planar diagram without crossing by obtained from D_5 by smoothing the crossing, illustrated in Fig. 9. Let K_5^0 be the planar graph given by D_5^0. For the proof of (1), let G^0 be the connected graph obtained from the n copies of K_5^0 by joining $n-1$ edges one after another linearly by introducing them (see Fig. 10).

Let G^i $(i = 1, 2, \dots, n)$ be the connected graphs obtained from G^0 by replacing the first i copies of K_5^0 with the i copies of K_5 (see Fig. 11 for $i = 2$). Since $c(K_5; T) = 1$ for every tree basis T and every tree basis T^i of G^i is obtained from the i tree bases of K_5 and the $n-i$ tree bases of K_5^0 by joining the $n-1$ edges one after another linearly. Then $g(G^i) \leq c^{T^i}(G^i) \leq i$ for every i. By Genus Lemma and Calculation Lemma, we obtain $g(G^i) = g(G^i; T^i) \geq i$ so that

$$g(G^i) = c(G^i) = c^{\max}(G^i) = i$$

for all i, showing (1). For (2), let H^i be the graph obtained from K_5 by replacing every edge except one edge by i multiple edges with $|v_{H^i}| = |v_{K_5}| = 5$. Then $g(H^i) = g(K_5) = 1$. Note that every tree basis T of H^i is homeomorphic to a tree basis of K_5. Then the identity $c^{\max}(K_5) = 1$ implies $c^{\max}(H^i) \leq i$. Since H^i contains i distinct K_5-graphs with completely distinct edges except common one edge. Then we have $c(H^i) \geq i$ and hence

$$c^{\max}(H^i) = c(H^i) = i \quad \text{and} \quad g(H^i) = 1$$

for all i. $\square$

By using Lemma 4.2, the proof of Theorem 4.1 is given as follows:

Proof of Theorem 4.1. Let

$$\begin{aligned} \tilde{c}^{\max}(G) &= f_1(c^{\max}(G)), \\ \tilde{c}(G) &= f_2(c(G)), \\ \tilde{g}(G) &= f_3(g(G)), \\ \tilde{\chi}(G) &= f_4(\chi(G)) \end{aligned}$$

for non-constant real polynomials $f_i(x)$ $(i = 1, 2, 3, 4)$. Suppose that the absolute value of a linear combination

$$a_1\tilde{c}^{\max}(G) + a_2\tilde{c}(G) + a_3\tilde{g}(G) + a_4\tilde{\chi}(G)$$

with real coefficients a_i $(i = 1, 2, 3, 4)$ is smaller than or equal to a positive constant a for all connected graphs G. Then it is sufficient to show that $a_1 = a_2 = a_3 = a_4 = 0$. If G is taken to be a planar graph, then $c^{\max}(G) = c(G) = g(G) = 0$. There is an infinite family of connected planar graphs whose Euler characteristic family is not bounded. Hence the polynomial $a_4 f_4(x)$ is a constant polynomial in x. Since $f_4(x)$ is a non-constant polynomial in x, we must have $a_4 = 0$. Then the inequality

$$|a_1\tilde{c}^{\max}(G) + a_2\tilde{c}(G) + a_3\tilde{g}(G)| \leq a$$

holds. By Lemma 4.2 (1), the polynomial $a_1f_1(x) + a_2f_2(x) + a_3f_3(x)$ in x must be a constant polynomial. By Lemma 4.2 (2), the polynomial $a_1f_1(x) + a_2f_2(x)$ in x must be a constant polynomial. These two claims mean that the polynomial $a_3f_3(x)$ is a constant polynomial in x, so that $a_3 = 0$ since $f_3(x)$ is a non-constant polynomial. Let $a' = a_1f_1(x) + a_2f_2(x)$ which is a constant polynomial in x. Then

$$a_1\tilde{c}^{\max}(G) + a_2\tilde{c}(G) = a_1(f_1(c^{\max}(G)) - f_1(c(G)) + a',$$

so that

$$|a_1(f_1(c^{\max}(G)) - f_1(c(G)) + a'| \leq a$$

for all connected graphs G. By Lemma 3.3, we have

$$\begin{aligned} \lim_{n\to+\infty}(c^{\max}(K_n) - c(K_n)) &= \lim_{n\to+\infty} c^{\max}(K_n) = +\infty, \\ 0 < \frac{c(K_n)}{c^{\max}(K_n)} &\leq \frac{1}{2} \quad (n \geq 16). \end{aligned}$$

Let d and e be the highest degree and the highest degree coefficient of the polynomial $f_1(t)$. Then we have

$$\begin{aligned} \lim_{n\to+\infty} |f_1(c^{\max}(K_n)) &- f_1(c(K_n))| \\ &= \lim_{n\to+\infty} \left| ec^{\max}(K_n)^d \left(1 - (\frac{c(K_n)}{c^{\max}(K_n)})^d\right)\right| = +\infty. \end{aligned}$$

Thus, we must have $a_1 = 0$, so that $a_1 = a_2 = a_3 = a_4 = 0$. □

5. Appendix: Tabulation of the tree bases of K_{11}

In this appendix, it is shown how the non-isomorphic tree bases are tabulated in case of the complete graph K_{11}. This tabulation method is important to compute the T-cross index $c^T(K_n)$ for a tree basis T of K_n, which is equal to the cross-index $c(K_n;T)$ by Lemma 3.1.

Our tabulation method is based on a formula on the numbers of vertexes with respect to degrees. Let T be a tree on the 2-sphere, and v_i the number of vertexes of T of degree i. Then the number V of the vertexes of T is the sum of all v_is for $i = 1, 2, \ldots$;

$$V = v_1 + v_2 + \cdots + v_i + \ldots.$$

Since there are i edges around every vertex of degree i and each edge has two end points, the total number E of edges of T is as follows:

$$E = \frac{1}{2}\left(v_1 + 2v_2 + 3v_3 + \cdots + iv_i + \ldots\right).$$

Since T is a tree, the number F of faces of T is 1. Then the following formula is obtained from the Euler characteristic of the 2-sphere $V - E + F = 2$:

$$v_1 = 2 + v_3 + 2v_4 + \cdots + (i-2)v_i + \ldots. \tag{1}$$

Let $V = 11$, i.e., let T be a tree basis of K_{11}. Since $E = 10$ by the Euler characteristic, the following equality holds:

$$\frac{1}{2}(v_1 + 2v_2 + 3v_3 + \cdots + 10v_{10}) = 10. \tag{2}$$

From the equalities (1) and (2), the following formula is obtained:

$$v_2 + 2v_3 + 3v_4 + \cdots + (i-1)v_i + \cdots + 9v_{10} = 9. \tag{3}$$

In Table 1, all the possible combinations of v_is which satisfy $V = 11$ and the formula (3) are listed. In Fig. 12, all the graphs in Table 1 are shown, where degree-two vertexes are omitted for simplicity. By giving vertexes with degree two to each graph in Fig. 12, all the tree bases of K_{11} are obtained as shown in Figs. 13, 14, 15 and 16.

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case	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8	v_9	v_{10}
A	10	0	0	0	0	0	0	0	0	1
B	9	1	0	0	0	0	0	0	1	0
C	8	2	0	0	0	0	0	1	0	0
D	9	0	1	0	0	0	0	1	0	0
E	7	3	0	0	0	0	1	0	0	0
F	8	1	1	0	0	0	1	0	0	0
G	9	0	0	1	0	0	1	0	0	0
H	9	0	0	0	1	1	0	0	0	0
I	8	1	0	1	0	1	0	0	0	0
J	8	0	2	0	0	1	0	0	0	0
K	7	2	1	0	0	1	0	0	0	0
L	6	4	0	0	0	1	0	0	0	0
M	8	1	0	0	2	0	0	0	0	0
N	8	0	1	1	1	0	0	0	0	0
O	7	2	0	1	1	0	0	0	0	0
P	7	1	2	0	1	0	0	0	0	0
Q	6	3	1	0	1	0	0	0	0	0
R	5	5	0	0	1	0	0	0	0	0
S	8	0	0	3	0	0	0	0	0	0
T	7	1	1	2	0	0	0	0	0	0
U	6	3	0	2	0	0	0	0	0	0
V	7	0	3	1	0	0	0	0	0	0
W	6	2	2	1	0	0	0	0	0	0
X	5	4	1	1	0	0	0	0	0	0
Y	4	6	0	1	0	0	0	0	0	0
Z	6	1	4	0	0	0	0	0	0	0
α	5	3	3	0	0	0	0	0	0	0
β	4	5	2	0	0	0	0	0	0	0
γ	3	7	1	0	0	0	0	0	0	0
δ	2	9	0	0	0	0	0	0	0	0

TABLE 1

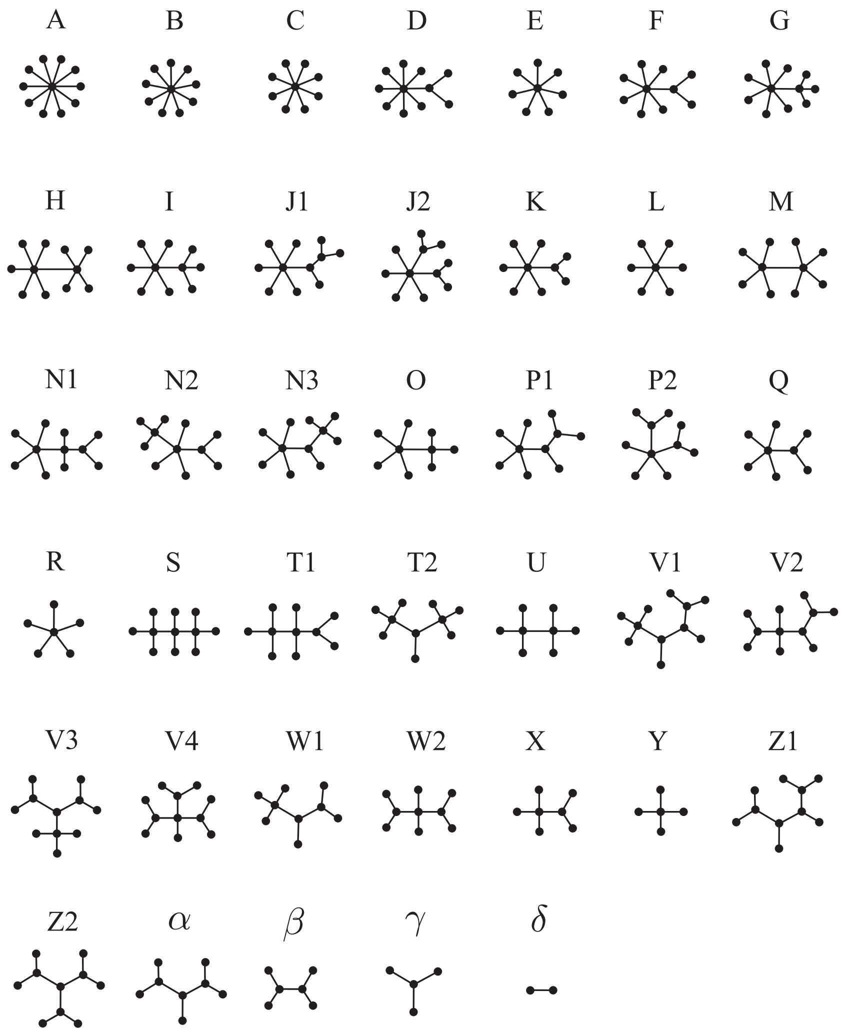

FIGURE 12. The tree bases of K_{11} without degree-two vertexes.

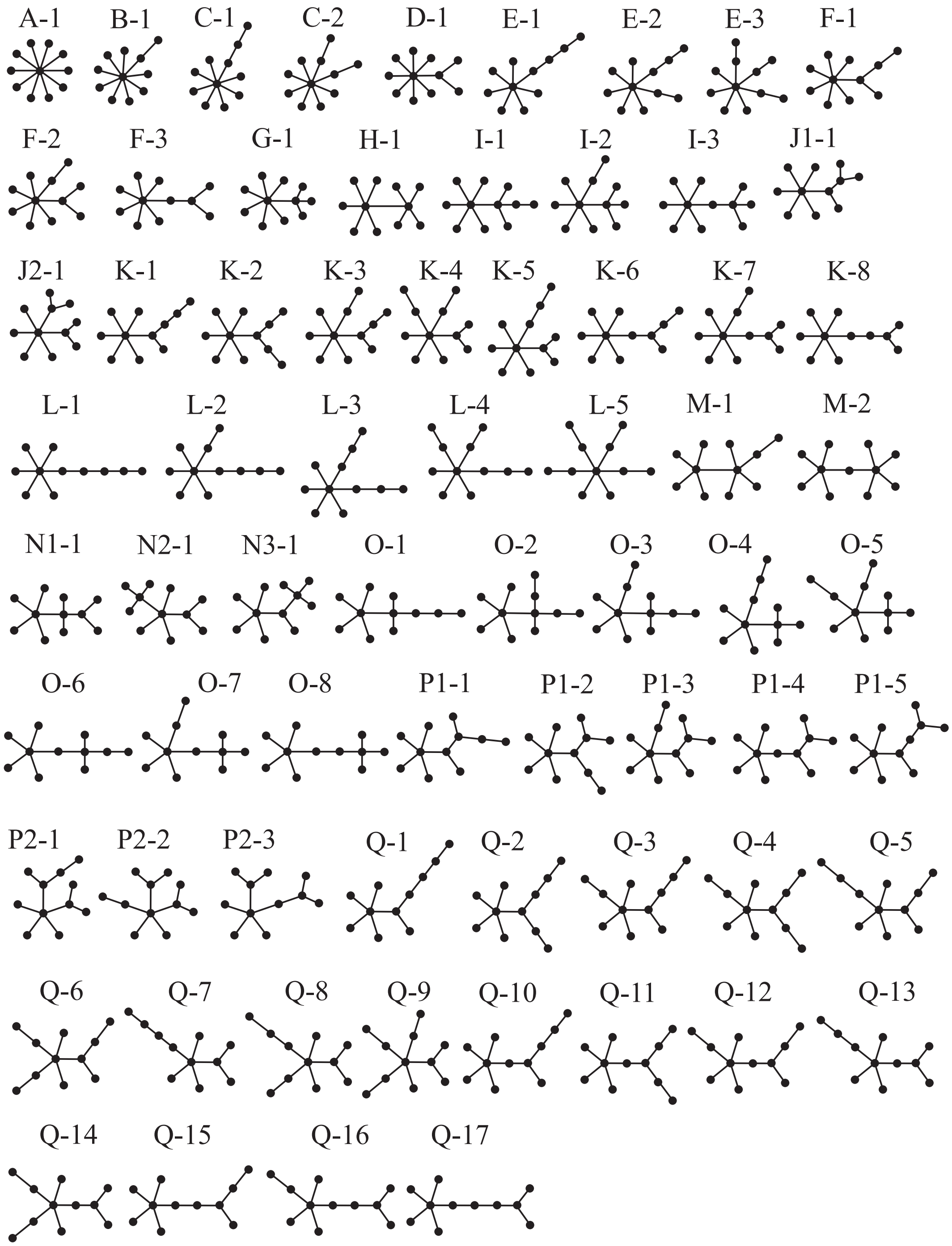

FIGURE 13. The tree bases of type A to Q.

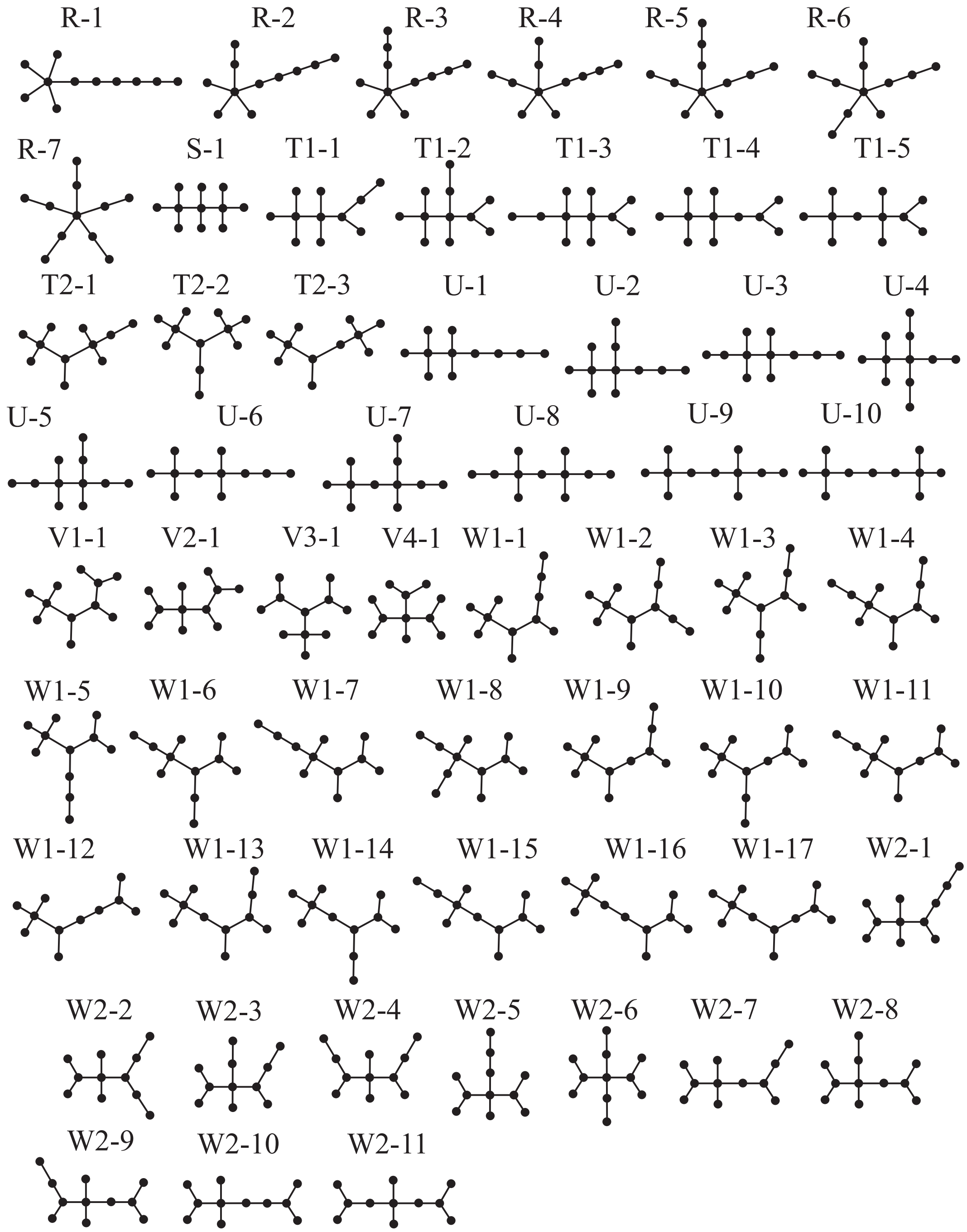

Figure 14. The tree bases of type R to W.

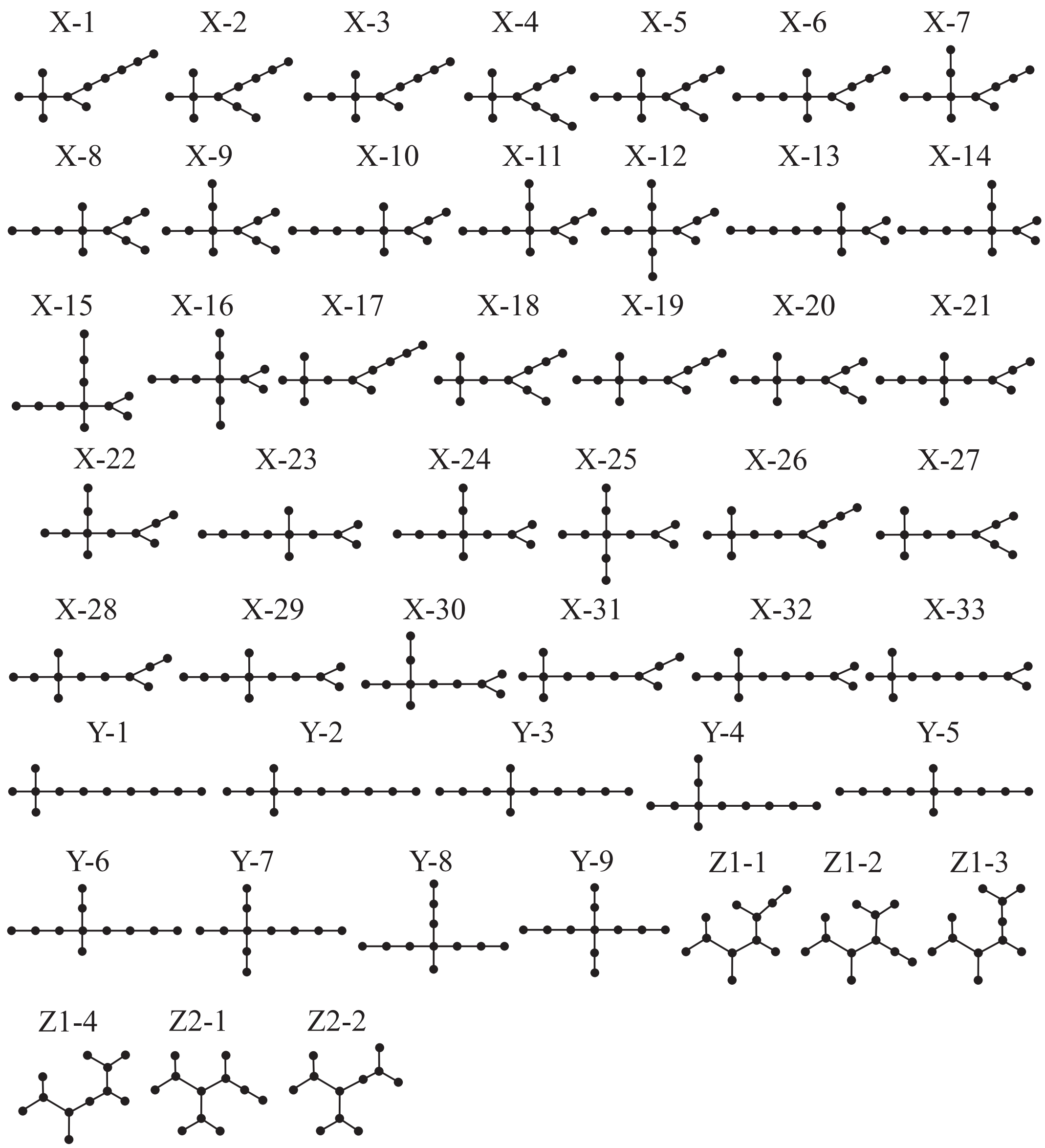

FIGURE 15. The tree bases of type X to Z.

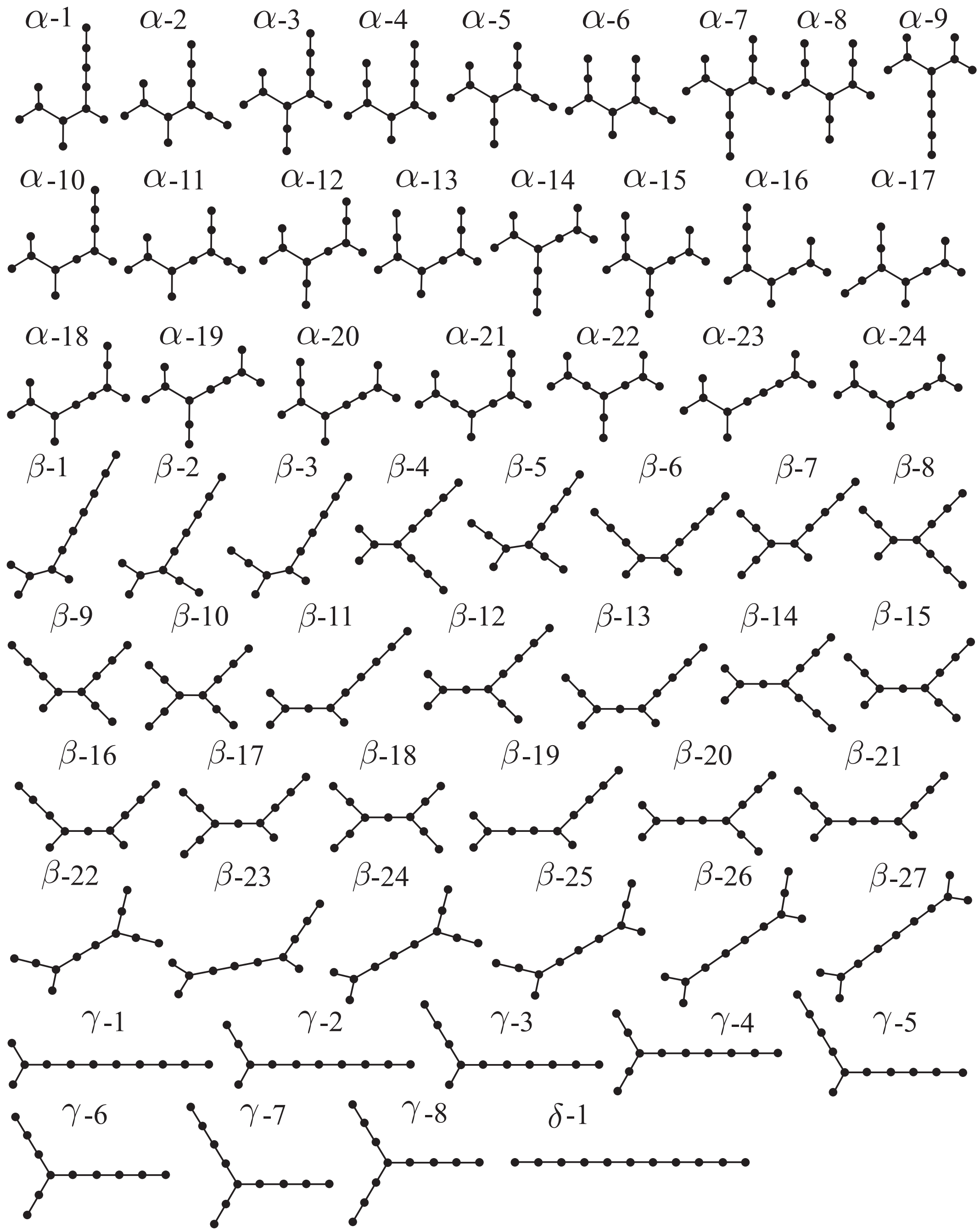

Figure 16. The tree bases of type α to δ.