On a cross-section of an immersed sphere-link in 4-space

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ABSTRACT

The torsion Alexander polynomial, the reduced torsion Alexander polynomial and the local signature invariant of a cross-section of an immersed sphere-link are investigated from the viewpoint of how to influence to the immersed sphere-link. It is shown that the torsion Alexander polynomial of a symmetric middle cross-section of a ribbon sphere-link is an invariant of the ribbon sphere-link. A generalization to a symmetric middle cross-section of an immersed ribbon sphere-link is given.

Keywords: Immersed sphere-link, Cross-section, Immersed ribbon sphere-link, Torsion Alexander polynomial, Reduced torsion Alexander polynomial, Local signature.

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1 Introduction

An *immersed sphere-link*, namely an *immersed S^2-link* with r components in the 4-space $\mathbf{R}^4$ is the image L of the disjoint union rS^2 of r copies of the 2-sphere S^2 into the 4-space $\mathbf{R}^4$ by a smooth immersion. When L is connected, it is called an immersed S^2-*knot* in $\mathbf{R}^4$. Assume that the singularity set $S(L)$ of an immersed S^2-link L consists of transverse double points, whose number is called the *double point number* of L and denoted by $c = c(L)$. When $c = 0$, the immersed S^2-link L is just an S^2-*link*. Two immersed S^2-links L and L' in $\mathbf{R}^4$ are *equivalent* if there is an orientation-preserving diffeomorphism $f : \mathbf{R}^4 \to \mathbf{R}^4$ sending L to L' orientation-preservingly. For a subset $F \subset \mathbf{R}^3$ and an interval $I \subset \mathbf{R}^1$, let

$$FI = \{(x,t) \in \mathbf{R}^4 | \, x \in F, t \in I\}.$$

A *cross-section* of an immersed S^2-link L is an oriented link ℓ with s components in the 3-space $\mathbf{R}^3[0]$ obtained by the transverse intersection $\ell = L' \cap \mathbf{R}^3[0]$ for an immersed S^2-link L' equivalent to L with $S(L') \cap \mathbf{R}^3[0] = \emptyset$. For our argument, assume that $L' = L$ and $\mathbf{R}^3[0] = \mathbf{R}^3$. Let $L^- = L \cap \mathbf{R}^3(-\infty, 0]$ and $L^+ = L \cap \mathbf{R}^3[0, +\infty)$. The double point singularities $S(L^-)$ and $S(L^+)$ of L^- and L^+ are called the *lower* and *upper double points* of the immersed S^2-link L separated by the cross-section ℓ, whose numbers $c^- = c(L^-)$ and $c^+ = c(L^+)$ are called the *lower* and *upper double point numbers*, respectively. Then $c = c^- + c^+$. Every double point singularity is constructed by a cone over a Hopf link and the linking number ± 1 of the Hopf link is called the *sign* of the double point. The sums of the signs of the lower and upper double points are called the *signed lower* and *upper double point numbers* and denoted by ξ^- and ξ^+, respectively. Then the sum $\xi = \xi^+ + \xi^-$ is called the *signed double point number* of L. By definition, we have

$$|\xi^-| \leqq c^- \quad \textit{and} \quad |\xi^+| \leqq c^+.$$

A cross-section ℓ of L is *regular* if the natural homomorphism

$$H_1(\mathbf{R}^3 \setminus \ell; \mathbf{Z}) \to H_1(\mathbf{R}^4 \setminus L; \mathbf{Z})$$

induces an isomorphism sending every meridian of ℓ to a meridian of L. For an irregular cross-section ℓ of L, it is assumed that the natural homomorphism

$$H_1(\mathbf{R}^3 \setminus \ell; \mathbf{Z}) \to H_1(\mathbf{R}^4 \setminus L; \mathbf{Z})$$

induces an epimorphism sending every medidian of ℓ to an meridian of L, so that every meridian of L is the image of a meridian of ℓ. It is noted that if ℓ is a (possibly irregular) cross-section of an immersed S^2-link L such that ℓ has at most two components or a regular cross-section of an immersed S^2-link L with any component, then the link ℓ is immersed concordant to a trivial link in the sense of [14]. In § 2, finer notions of a regular cross-section of an immersed S^2-link, namely a *middle cross-section* of an immersed S^2-link and a *symmetric middle cross-section* of an *immersed ribbon S^2-link* are introduced.

The purpose of this paper is to study a relationship between an immersed S^2-link L and a link ℓ which is a cross-section of L. Although it has not been well studied on before, this setting is general and natural because a generic smooth map

$$sS^1 \to \mathbf{R}^3$$

is an embedding and a generic smooth map

$$rS^2 \to \mathbf{R}^4$$

is an immersion. This research is discussed from two viewpoints, namely a viewpoint of the immersed link concordance in [14] where a result of the torsion Alexander polynomial has been given and an earlier viewpoint of the quadratic form of a link in [6, 13] where a result on an irregular cross-sections of an S^2-link (without singularity) has been given. Let $\Lambda = \mathbf{Z}[\mathbf{Z}^r] = \mathbf{Z}[t_1, t_1^{-1}, t_2, t_2^{-1}, \ldots, t_r, t_r^{-1}]$ and $\tilde{\Lambda} = \mathbf{Z}[\mathbf{Z}] = \mathbf{Z}[t, t^{-1}]$ denote the free abelian group $\mathbf{Z}^r$ of rank r and the integral group ring of the infinite cyclic group $\mathbf{Z}$, respectively. The following invariants of an immersed S^2-link L and a cross-section ℓ of L are used to state our main results (Theorems 1.1 and 1.2).

- The *linking sum invariant* $|\mathrm{Link}|(\ell)$ of ℓ which is defined to be the sum of the absolute values of pairwise linking numbers of ℓ.
- The *local signature invariant* $s_J(\ell)$ of ℓ for every subset $J \subset [-1, 1]$.
- The *torsion Alexander polynomials*

$$\Delta^T(L) = \Delta^T(L; t_1, t_2, \ldots, t_r), \quad \Delta^T(\ell) = \Delta^T(\ell; t_1, t_2, \ldots, t_r) \in \Lambda$$

of L and ℓ and the *reduced torsion Alexander polynomials*

$$\tilde{\Delta}^T(L) = \tilde{\Delta}^T(L; t), \quad \tilde{\Delta}^T(\ell) = \tilde{\Delta}^T(\ell; t) \in \tilde{\Lambda}.$$

- The *κ-invariants*

$$\kappa(\ell), \quad \kappa(L^{\pm}), \quad \kappa(L)$$

and the *reduced κ-invariants*

$$\tilde{\kappa}(\ell), \quad \tilde{\kappa}(L^{\pm}), \quad \tilde{\kappa}(L)$$

of $\ell, L^{\pm}$ and L.

Some detailed explanations of these invariants are made in § 3. It is standard to use the notation

$$f \doteq f'$$

for elements $f, f' \in \Lambda$ or $f, f' \in \tilde{\Lambda}$ in the sense that f and f' are equal up to multiplications of units of Λ or $\tilde{\Lambda}$, respectively. The notation

$$f \doteqdot f'$$

for elements $f, f' \in \Lambda$ or $f, f' \in \tilde{\Lambda}$ are used in the sense that f and f' are equal up to multiplications of units of Λ and the elements $t_i - 1 \in \Lambda\, (i = 1, 2, \ldots, r)$ or equal up to multiplications of units of $\tilde{\Lambda}$ and the element $t-1 \in \tilde{\Lambda}$, respectively. For an element $f = f(t_1, t_2, \ldots, t_r) \in \Lambda$ or $f = f(t) \in \tilde{\Lambda}$, let

$$f^* = f(t_1^{-1}, t_2^{-1}, \ldots, t_r^{-1}) \quad \text{or} \quad f^* = f(t^{-1}),$$

respectively. The following theorem is obtained from the viewpoint of the immersed link concordance in [14].

Theorem 1.1. The following (1)-(4) are obtained.
(1) The inequality $2\kappa(L) \leqq c$ holds. If $2\kappa(L) = c$, then for any (regular or irregular) cross-section ℓ of L, the identity

$$\Delta^T(\ell) \doteqdot ff^*$$

holds for an element $f \in \Lambda$.
(2) If ℓ is a regular cross-section of L, then the inequalities

$$\max\{2\kappa(L^+) - c^+, 2\kappa(L^-) - c^-\} \leqq \kappa(\ell) \leqq \min\{c^-, c^+\}$$

holds. Further, if $\kappa(\ell) = \max\{2\kappa(L^+) - c^+, 2\kappa(L^-) - c^-\}$ or $\min\{c^-, c^+\}$, then the identity

$$\Delta^T(\ell) \doteqdot ff^*$$

holds for an element $f \in \Lambda$.
(3) If ℓ is a middle cross-section of L and $\kappa(\ell) = c^- = c^+$, then $2\kappa(L) = c$ and the identities

$$\Delta^T(\ell) \doteqdot \Delta^T(L^+)\Delta^T(L^+)^* \doteq \Delta^T(L^-)\Delta^T(L^-)^* \doteq \Delta^T(L)\Delta^T(L)^* gg^*$$

hold for an element $g \in \Lambda$.
(4) If L is an immersed ribbon S^2-link with $2\kappa(L) = c$ and ℓ is any symmetric middle cross-section of L, then

$$\kappa(\ell) = c^- = c^+ \quad \text{and} \quad \Delta^T(\ell) \doteqdot \Delta^T(L)\Delta^T(L)^*.$$

The following theorem is obtained from the viewpoint of the quadratic form of a link in [13].

Theorem 1.2. The following (1)-(4) are obtained.
(1) If ℓ is a regular or irregular cross-section of L, then

$$\max\{|s_{(a,1]}(\ell)| + \tilde{\kappa}(L),\, 2\tilde{\kappa}(L)\} \leqq c$$

for every $a \in (-1,1)$. Further, if $2\tilde{\kappa}(L) = c$, then

$$\xi = 0, \quad s_a(\ell) = 0, \quad |s_1(\ell)| \leqq \frac{c}{2} \quad \text{and} \quad \tilde{\Delta}^T(\ell) \doteqdot ff^*$$

for every $a \in (-1,1)$ and an element $f \in \tilde{\Lambda}$.
(2) If ℓ is a regular cross-section of L, then $|\mathrm{Link}|(\ell) \leqq \min\{c^-, c^+\}$,

$$\max\{2\tilde{\kappa}(L^+) - \tilde{\kappa}(\ell),\, |s_{(a,1]}(\ell) + \xi^+| + \tilde{\kappa}(\ell)\} \leqq c^+,$$
$$\max\{2\tilde{\kappa}(L^-) - \tilde{\kappa}(\ell),\, |s_{(a,1]}(\ell) - \xi^-| + \tilde{\kappa}(\ell)\} \leqq c^-$$

for every $a \in (-1,1)$. Further, if $\tilde{\kappa}(\ell) = 2\tilde{\kappa}(L^+) - c^+$ or $\tilde{\kappa}(L^+) = c^+$ (or $\tilde{\kappa}(\ell) = 2\tilde{\kappa}(L^-) - c^-$ or $\tilde{\kappa}(L^-) = c^-$, respectively), then we have $s_{(a,1]}(\ell) = -\xi^-$ (or $s_{(a,1]}(\ell) = \xi^-$, respectively) for every $a \in (-1,1)$ and

$$\tilde{\Delta}^T(\ell) \doteqdot ff^*$$

for an element $f \in \tilde{\Lambda}$.
(3) If ℓ is a middle cross-section of L and $\tilde{\kappa}(\ell) = c^- = c^+$, then we have $\xi = 0$, $2\tilde{\kappa}(L) = c$ and

$$\tilde{\Delta}^T(\ell) \doteqdot \tilde{\Delta}^T(L^+)\tilde{\Delta}^T(L^+)^* \doteq \tilde{\Delta}^T(L^-)\tilde{\Delta}^T(L^-)^* \doteq \tilde{\Delta}^T(L)\tilde{\Delta}^T(L)^* gg^*$$

for an element $g \in \tilde{\Lambda}$.
(4) If L is an immersed ribbon S^2-link with $2\tilde{\kappa}(L) = c$ and ℓ is any symmetric middle cross-section of L, then

$$\tilde{\kappa}(\ell) = c^- = c^+ \quad \text{and} \quad \tilde{\Delta}^T(\ell) \doteqdot \tilde{\Delta}^T(L)\tilde{\Delta}^T(L)^*.$$

It is noted that the linking sum invariant $|\mathrm{Link}|(\ell)$ can be taken sufficiently large for an irregular cross-section ℓ of an S^2-knot L (see S. Satoh [21]). Also, there are many examples of immersed ribbon S^2-links L with $2\kappa(L) = c(L)$ or $2\tilde{\kappa}(L) = c(L)$, which will be discussed in § 5. As a consequence of Theorem 1.2 (2), we obtain the following corollary.

Corollary 1.3. If ℓ is a regular cross-section of an immersed S^2-link L, then we have

$$|s_{(a,1]}(\ell)| \leqq \min\{c^- + |\xi^-|,\, c^+ + |\xi^+|\} \leqq \min\{2c^-, 2c^+\}$$

for every $a \in (-1,1)$.

For example, let ℓ be the n-fold connected sum of the trefoil knot. If ℓ is a cross-section of an immersed S^2-knot L, then we have $c \geqq 2n$ by Theorem 1.2 (1) and $c^{\pm} \geqq n$ by Corollary 1.3 since $s_{(-1,1]}(\ell) = s(\ell) = \pm 2n$. Further, an immersed S^2-knot L with this link ℓ as a regular cross-section such that $\xi^+ = n, \xi^- = -n$ and $c = 2n$ is easily constructed. From Theorem 1.1 (4), the following corollary is obtained.

Corollary 1.4. Let L be an immersed ribbon S^2-link of r components with $2\kappa(L) = c(L)$, and ℓ a symmetric middle cross-section of L. The torsion Alexander polynomial $\Delta^T(\ell) \in \Lambda$ up to multiplications of units of Λ and the elements $t_i - 1\,(i = 1, 2, \ldots, r)$ is an invariant of L under the immersed S^2-link equivalences.

From Theorem 1.2 (4), the following corollary is obtained.

Corollary 1.5. Let L be an immersed ribbon S^2-link with $2\tilde{\kappa}(L) = c(L)$, and ℓ a symmetric middle cross-section of L. The reduced torsion Alexander polynomial $\tilde{\Delta}^T(\ell) \in \tilde{\Lambda}$ up to multiplications of units of $\tilde{\Lambda}$ and $t-1$ is an invariant of L under the immersed S^2-link equivalences.

Let L be an S^2-link (i.e., an immersed S^2-link with $c = 0$). A (possibly irregular) cross-section ℓ of L is also called a *sphere-slice link*. This cross-sectional link was first considered by R. H. Fox in [3]. It is noted that if ℓ is a (possibly irregular) cross-section of L such that ℓ has at most two components or a regular cross-section of L with any component, then the link ℓ is concordant to a trivial link, which is much investigated until now by many researchers. By Theorems 1.1 (1) and 1.2 (1) and Lemma 3.3 (3), we have

$$\kappa(L) = \tilde{\kappa}(L) = 0 \quad \text{and} \quad \Delta^T(L; 1, 1, \ldots, 1) = \pm 1,\ \tilde{\Delta}^T(L; 1) = \pm 1.$$

If ℓ is a (regular or irregular) cross-section of L, then Theorem 1.2 (1) means that $s_1(\ell) = 0$, so that by [13, Lemma 5.7] $\kappa(\ell) \equiv 0 \pmod 2$ and the multiplicity of $t-1$ in $\tilde{\Delta}^T(\ell)$ is even and hence

$$\tilde{\Delta}^T(\ell) \doteq pp^*$$

for an element $p \in \tilde{\Lambda}$. Furthermore, since the double branched covering space Y_2^+ of $(B^4)^+$ branched along L^+ is embedded in the double branched covering space Y_2 of S^4 branched along L which has the trivial second rational homology $H_2(Y_2; \mathbf{Q}) = 0$, it is seen that the signature $\mathrm{sign}(Y_2^+)$ is 0, which implies that the signature $s(\ell)$ is 0 by [13, Lemma 6.1]. By combining it with Theorem 1.2 (1), we have the local signature $s_a(\ell)$ is 0 for every $a \in [-1, 1]$. The linking matrix $V(\ell)$ of ℓ with Seifert surface framing is an even symmetric matrix with signature $s_V(\ell) = 0$. The evenness and the vanishing signature of the linking matrix $V(\ell)$ are known by [19, 21] and [13, Theorem 8.2], respectively. For completeness, a unified simple topological proof of this fact is given in Lemma 3.4. Then the Arf invariant $\mathrm{Arf}(\ell)$ is well-defined, and taken 0 (mod 2) since ℓ bounds a genus zero surface in the 4-ball (see [12, Corollary 12.3.9]). If ℓ is a regular cross-section of L, then by Theorems 1.1 (2) and 1.2 (2) and Lemma 3.3 (3), we have

$$\kappa(\ell) = \tilde{\kappa}(\ell) = 0 \quad \text{and} \quad \Delta^T(\ell; 1, 1, \ldots, 1) = \pm 1,\ \tilde{\Delta}^T(\ell; 1) = \pm 1.$$

Then the following unique normalizations

$$\Delta^T(\ell) = \Delta^T(\ell)^*,\ \Delta^T(\ell; 1, 1, \ldots, 1) = 1,\ \tilde{\Delta}^T(\ell) = \tilde{\Delta}^T(\ell)^*,\ \tilde{\Delta}^T(\ell; 1) = 1$$

can be made by some unit multiplications of $\tilde{\Lambda}$ and Λ, respectively. By combining the arguments above with Theorems 1.1 and 1.2, we obtain a detailed version of [13, Theorem 8.2] as the following corollary, where the normalizations of $\Delta^T(\ell)$ and $\tilde{\Delta}^T(\ell)$ are used in the assertions (2)-(4).

Corollary 1.6 (Sphere-slice Theorem). Let L be an S^2-link. Then the following (1)-(4) are obtained.

(1) For every (regular or irregular) cross-section ℓ of L, the linking matrix $V(\ell)$ is even and the identities

$$\begin{aligned} &s_V(\ell) = s_a(\ell) = \sigma_a(\ell) = 0, \quad \Delta^T(\ell) \doteqdot ff^*, \\ &\tilde{\Delta}^T(\ell) \doteq pp^*, \quad \tilde{\kappa}(\ell) \equiv \mathrm{Arf}(\ell) \equiv 0 \pmod 2 \end{aligned}$$

hold for every $a \in [-1, 1]$ and elements $f \in \Lambda$ and $p \in \tilde{\Lambda}$.

(2) If ℓ is a regular cross-section of L, then the following identities

$$\Delta^T(\ell) = ff^* \quad \text{and} \quad \tilde{\Delta}^T(\ell) = pp^*$$

hold for an element $f \in \Lambda$ with $f(1, 1, \ldots, 1) = 1$ and an element $p \in \tilde{\Lambda}$ with $p(1) = 1$ without ambiguity of unit multiplications.

(3) If ℓ is a middle cross-section of L, then the following identities

$$\begin{aligned} \Delta^T(\ell) &= \Delta^T(L^+)\Delta^T(L^+)^* = \Delta^T(L^-)\Delta^T(L^-)^* = \Delta^T(L)\Delta^T(L)^* gg^*, \\ \tilde{\Delta}^T(\ell) &= \tilde{\Delta}^T(L^+)\tilde{\Delta}^T(L^+)^* = \tilde{\Delta}^T(L^-)\tilde{\Delta}^T(L^-)^* = \tilde{\Delta}^T(L)\tilde{\Delta}^T(L)^* qq^* \end{aligned}$$

hold for an element $g \in \Lambda$ with $g(1, 1, \ldots, 1) = 1$ and an element $q \in \tilde{\Lambda}$ with $q(1) = 1$ without ambiguity of unit multiplications.

(4) If L is a ribbon S^2-link and ℓ is a symmetric middle cross-section of L, then the following identities

$$\Delta^T(\ell) = \Delta^T(L)\Delta^T(L)^* \quad \text{and} \quad \tilde{\Delta}^T(\ell) = \tilde{\Delta}^T(L)\tilde{\Delta}^T(L)^*$$

hold without ambiguity of unit multiplications. In particular, the torsion Alexander polynomial $\Delta^T(\ell)$ and the reduced torsion Alexander polynomial $\tilde{\Delta}^T(\ell)$ are invariants of L under the equivalences of S^2-links.

For example, if a possibly irregular cross-section ℓ of an immersed S^2-link L is the Hopf link, then we have $c(L) \geqq 1$ and by a direct construction the equality is realized by an irregular cross-section. If a possibly irregular cross-section ℓ of an immersed S^2-link L is the Whitehead link or the Borromean rings, then we have $c(L) \geqq 2$ and by a direct construction the equality is realized by a regular cross-section or an irregular cross-section, respectively. This inequality is shown from the fact that $\mathrm{Arf}(\ell) \neq 0 \pmod 2$, implying that ℓ cannot bound any genus zero surface in the 4-ball by [12, Corollary 12.3.9]. The following corollary is direct from Corollary 1.6.

Corollary 1.7. For every ribbon link ℓ, only finitely many elements of Λ and $\tilde{\Lambda}$ up to unit multiplications which are factors of $\Delta^T(\ell)$ and $\tilde{\Delta}^T(\ell)$ can be the torsion Alexander polynomials $\Delta^T(L)$ and $\tilde{\Delta}^T(L)$ for all S^2-links L with ℓ as a middle cross-section, respectively.

For $r = 1$, the torsion Alexander polynomial $\Delta^T(\ell)$ and the reduced torsion Alexander polynomial $\tilde{\Delta}^T(\ell)$ of ℓ are just the Alexander polynomial $\Delta(\ell; t)$ of the knot ℓ which is equivalent to the Conway polynomial $\nabla(\ell; z)$. Thus, the following corollary is also direct from Corollary 1.6:

Corollary 1.8. The Conway polynomial $\nabla(\ell; z)$ of every symmetric middle cross-section ℓ of every ribbon S^2-knot L is an invariant of L under the equivalences of S^2-knots.

Corollary 1.8 can be also derived from the Fox calculus (see [2]) on a ribbon band calculation done by H. Terasaka [24] and the author's recent result in [15] on the ribbon moves of a ribbon S^2-knot. This result has been applied to a recent joint paper with Y. Joung, S. Kamada and S. Y. Lee in [4].

In § 2, an immersed S^2-link is explained from the viewpoint of a motion picture. In § 3, some invariants of a link and an immersed S^2-link are explained. In § 4, the proofs of Theorems 1.1 and 1.2 are done. In § 5, a symmetric construction of immersed ribbon S^2-links is considered to explain some examples.

2 A motion picture of an immersed S^2-link

A *band surgery* on an oriented link ℓ in the 3-space $\mathbf{R}^3$ is a transformation of ℓ into an oriented link ℓ' by a finite family of mutually disjoint bands $\beta_i\,(i = 1, 2, \ldots, m)$ spanning ℓ such that

$$\ell' = \mathrm{cl}(\ell \setminus \cup_{i=1}^m \ell \cap \beta_i) \cup_{i=1}^m \mathrm{cl}(\partial\beta_i \setminus \ell \cap \beta_i).$$

In particular, the band surgery $\ell \to \ell'$ is called a *fusion* or *fission* respectively according to whether $|\ell| - m = |\ell'|$ or $|\ell| + m = |\ell'|$, where $|\ell|$ and $|\ell'|$ denote the component numbers of the links ℓ and ℓ', respectively. The *realizing surface* in $\mathbf{R}^3[a, b]$ for a band surgery $\ell \to \ell'$ on finitely many mutually disjoint bands β_j $(j = 1, 2, \ldots, s)$ is a surface F_a^b in $\mathbf{R}^3[a, b]$ defined by the following identity:

$$F_a^b \cap \mathbf{R}^3[t] = \begin{cases} \ell'[t] & (\frac{a+b}{2} < t \leqq b), \\ (\ell \cup \beta_1 \cup \beta_2 \cup \cdots \cup \beta_s)[t] & (t = \frac{a+b}{2}), \\ \ell[t] & (a \leqq t < \frac{a+b}{2}). \end{cases}$$

For a division $a = a_0 < a_1 < \cdots < a_m = b$ of the interval $[a, b]$ and a band surgery sequence $\ell_0 \to \ell_1 \to \cdots \to \ell_m$, the realizing surface F_a^b in $\mathbf{R}^3[a, b]$ is constructed by

$$F_a^b = F_{a_0}^{a_1} \cup F_{a_1}^{a_2} \cup \cdots \cup F_{a_{m-1}}^{a_m}.$$

An H-*trivial link* is a split union of a finite number of trivial knots and Hopf links.

For non-negative integers c^-, c^+, a *normalized surface* with c^- *lower Hopf links* and c^+ *upper Hopf links* is the realizing surface $F = F_{-1}^1$ in $\mathbf{R}^3[-1, 1]$ for a division $-1 < 0 < 1$ of the interval $[-1, 1]$ and a band surgery sequence

$$\theta^- \to \ell \to \theta^+$$

which has the following additional conditions:

(1) The link θ^- and θ^+ are H-trivial links such that θ^- and θ^+ have c^- and c^+ Hopf links, respectively.

(2) The band surgery $\theta^- \to \ell$ is a fusion and the band surgery $\ell \to \theta^+$ is a fission.

Let θ be an H-trivial link with trivial knot components $o_i\,(i = 1, 2, \ldots, s_o)$ and Hopf link components $H_j\,(j = 1, 2, \ldots, s_H)$. For an interval $[a, b]$, let $\theta_\vee[a, b]$ be the disjoint union of a disjoint disk system $d_i\,(i = 1, 2, \ldots, s_o)$ for the trivial knots $o_i\,(i = 1, 2, \ldots, s_o)$ in $\mathbf{R}^3[b]$ and a disjoint Hopf link cone system $C_j\,(j = 1, 2, \ldots, s_H)$ in $\mathbf{R}^3[a, b]$ such that C_j is a cone with Hopf link base $H_j[b]$ and vertex $v_j \in \mathbf{R}^3[a]$

for every j. Similarly, let $\theta_\wedge[a,b]$ be the disjoint union of a disjoint disk system $d_i\,(i=1,2,\dots,s_o)$ for the trivial knots $o_i\,(i=1,2,\dots,s_o)$ in $\mathbf{R}^3[a]$ and a disjoint Hopf link cone system $C_j\,(j=1,2,\dots,s_H)$ in $\mathbf{R}^3[a,b]$ such that C_j is a cone with Hopf link base $H_j[a]$ and vertex $v_j \in \mathbf{R}^3[b]$ for every j. Then the union

$$L = \theta^-_\vee[-2,-1] \cup F^1_{-1} \cup \theta^+_\wedge[1,2]$$

is an immersed S^2-link in $\mathbf{R}^4$ with a regular cross-section ℓ. This immersed S^2-link L is said to be *in a normal form* and the regular cross-section ℓ of L is called a *middle cross-section* of L. It is noted that the vertices of $\theta^-_\vee[-2,-1]$ and $\theta^+_\wedge[1,2]$ correspond to the lower and upper double points of L separated by the link ℓ, respectively, so that L has the lower and upper double point numbers c^- and c^+, respectively. It is shown by an argument on a normal form of a cobordism surface in [16, 17] that every immersed S^2-link is equivalent to an immersed S^2-link L in a normal form whose lower and upper double points are given by any previously given division of the double points and the equivalence of L is independent of any choices of the disk systems used to construct $\theta^-_\vee[-2,-1]$ and $\theta^+_\wedge[1,2]$. It is noted that a more general immersed ribbon surface-link is studied by Kamada and Kawamura in [5]. A main difference between a regular cross-section and a middle cross-section of an immersed S^2-link L is that the natural homomorphisms

$$\begin{aligned}\pi_1(\mathbf{R}^3\setminus\ell,x_0) &\to \pi_1(\mathbf{R}^3[-\infty,0]\setminus L^-,x_0),\\ \pi_1(\mathbf{R}^3\setminus\ell,x_0) &\to \pi_1(\mathbf{R}^3[0,+\infty)\setminus L^+,x_0)\end{aligned}$$

for any base point $x_0 \in \mathbf{R}^3\setminus\ell$ are always onto for a middle cross-section ℓ of L, but it is not true for a general regular cross-section ℓ of L. For example, for a middle cross-section ℓ of an immersed S^2-link L in a normal form, take the connected sum $L_K = L\#K$ for an S^2-knot K with non-abelian fundamental group so that the operation is done in the upper open half 4-space $\mathbf{R}^3(0,+\infty)$. Then the link ℓ is a regular cross-section of the immersed S^2-link L_K such that the natural homomorphism $\pi_1(\mathbf{R}^3\setminus\ell,x_0)\to\pi_1(\mathbf{R}^3[-\infty,0]\setminus L^+_K,x_0)$ is not onto. If the fission $\ell\to\theta^+$ is just the inverse of the fusion $\theta^-\to\ell$, then the immersed S^2-link L in $\mathbf{R}^4$ is called an *immersed ribbon S^2-link* and the link ℓ is a *symmetric middle cross-section* of L. By construction, we have $\xi=\xi^+-\xi^-=0$. Unless confusion might occur, an immersed S^2-link equivalent to this immersed ribbon S^2-link is also called an immersed ribbon S^2-link. We observe the following lemma.

Lemma 2.1. (1) Every immersed S^2-link admits infinitely many (up to equivalences of links) middle cross-sections.

(2) Every immersed ribbon S^2-link admits infinitely many (up to equivalences of links) symmetric middle cross-sections.

(3) For every previously given link ℓ, there is an immersed ribbon S^2-link with ℓ as a symmetric middle cross-section.

Proof. For (1), let $k_n\,(n \geqq 1)$ be the n-fold connected sum of the connected sum knot $k\#(-k^*)$ for any non-trivial knot k and the inversed mirror image $-k^*$ of k. This knot k_n is a regular cross-section of a trivial S^2-knot O by Zeeman's 1-twist spinning in [26] (see also [23]) which is in fact shown to be a middle cross-section of a trivial S^2-knot O (see [16]). Let L be any given immersed S^2-link, and ℓ any middle cross-section of L. A connected sum $L\#O$ is equivalent to L and admits a connected sum $\ell\#k_n$ as a middle cross-section of $L\#O$, making an infinite family for $n = 0, 1, 2, \dots$ of mutually inequivalent middle cross-sections of L, showing (1).

For (2), let k_n be the n-fold connected sum of the Kinoshita-Terasaka knot k_{KT} for any $n \geqq 1$ which is a symmetric middle cross-section of a trivial S^2-knot O (see for example [11]). For any given immersed ribbon S^2-link L and any symmetric middle cross-section ℓ of L, a connected sum $L\#O$ is an immersed ribbon S^2-link equivalent to L and admits a connected sum $\ell\#k_n$ as a symmetric middle cross-section of $L\#O$, making an infinite family for $n = 0, 1, 2, \dots$ of mutually inequivalent symmetric middle cross-sections, showing (2).

For (3), let ℓ be any link in $\mathbf{R}^3[0]$, and γ a singular disk system with only simple clasp singularities in $\mathbf{R}^3[0]$ bounded by ℓ. Push the interior of γ into an immersed disk system L^+ in $\mathbf{R}^3[0, +\infty)$ and then take the image L^- of L^+ in $\mathbf{R}^3(-\infty, 0]$ under the homeomorphism $\mathbf{R}^3[0, +\infty) \to \mathbf{R}^3(-\infty, 0]$ sending (x, t) to $(x, -t)$. The union $L = L^- \cup L^+$ is an immersed ribbon S^2-link with ℓ a symmetric middle cross-section, showing (3). $\square$

3 Some invariants of links and immersed S^2-links

Let L be an immersed S^2-link with r components, and ℓ a cross-section of L. The pair (L, ℓ) in the $(4,3)$-space pair $(\mathbf{R}^4, \mathbf{R}^3)$ is often regarded as it is in the $(4,3)$-sphere pair (S^4, S^3) obtained from the $(4,3)$-space pair $(\mathbf{R}^4, \mathbf{R}^3)$ by the one-point compactification. Let $X = \mathrm{cl}(S^4 \setminus N(L))$ be the compact exterior of L, where $N(L)$ is a regular neighborhood of L in S^4. Let $X^0 = \mathrm{cl}(S^3 \setminus N(\ell))$ be the compact exterior of the link ℓ in S^3 with $N(\ell) = S^3 \cap N(L)$ a tubular neighborhood of ℓ in S^3. The inclusion $X^0 \subset X$ induces an epimorphism $H_1(X^0; \mathbf{Z}) \to H_1(X; \mathbf{Z})$ on the free abelian groups sending every meridian of the link ℓ to an meridian of the S^2-link L, which is an isomorphism when ℓ is a regular cross-section of L. Let X^+ and X^- be the compact connected oriented 4-manifolds obtained from X by splitting along X^0. In other words, for the 4-balls $(B^4)^\pm$ obtained S^4 by splitting along S^3, let $X^\pm$ be the compact exteriors of the immersed 2-disk knots $L^{\pm} = L \cap (B^4)^{\pm}$ in the 4-balls $(B^4)^{\pm}$. By the assumption of the cross-section ℓ of L, the inclusions $X^0 \subset X^{\pm} \subset X$ induce epimorphisms

$$H_1(X^0; \mathbf{Z}) \to H_1(X^{\pm}; \mathbf{Z}) \to H_1(X; \mathbf{Z})$$

sending the meridians to the meridians. Let $(\bar{X}, \bar{X}^+, \bar{X}^0)$ and $(\tilde{X}, \tilde{X}^+, \tilde{X}^0)$ be the maximal abelian covering and the infinite cyclic covering of (X, X^+, X^0) belonging to the isomorphism $H_1(X; \mathbf{Z}) \to \mathbf{Z}^r$ sending every meridian of L to the standard basis and the epimorphism $H_1(X; \mathbf{Z}) \to \mathbf{Z}$ sending every meridian of L to 1, respectively. The homology groups

$$H_d(\bar{X}; \mathbf{Z}), H_d(\bar{X}^+; \mathbf{Z}), H_d(\bar{X}^0; \mathbf{Z}) \quad \text{and} \quad H_d(\tilde{X}; \mathbf{Z}), H_d(\tilde{X}^+; \mathbf{Z}), H_d(\tilde{X}^0; \mathbf{Z})$$

form finitely generated Λ-modules and $\tilde{\Lambda}$-modules, respectively. By using that Λ is a unique factorization domain, the characteristic polynomial invariants for a finitely generated Λ-module are defined as follows (cf. [7, 12, 14]). For a finitely generated Λ-module H, let A be a finite Λ-presentation matrix of H, namely a matrix A representing the Λ-homomorphism λ in a Λ-exact sequence

$$\Lambda^{m'} \xrightarrow{\lambda} \Lambda^m \to H \to 0 \quad (m' \geqq m).$$

This Λ-exact sequence is always constructed for every H since Λ has the Noetherian property. For a non-negative integer d, the d^{th} *characteristic polynomial* $\Delta^{(d)}(H) \in \Lambda$ of H is defined to be the g.c.d. of all $(m-d)$-minors of the matrix A if $d \leqq m$ and 1 if $d > m$. The characteristic polynomial invariants are unique up to multiplications of the units of Λ. The 0^{th} characteristic polynomial $\Delta^{(0)}(H)$ is called the *Alexander polynomial* of H and denoted by $\Delta(H)$. For the Λ-torsion part TH of H, which is also finitely generated over Λ by the Noetharian property, let $\Delta^T(H) = \Delta(TH)$ which we call the *torsion Alexander polynomial* of H. The *null Λ-submodule* of H is the Λ-submodule DH of TH consisting of all elements x such that $f_i x = 0$ for coprime elements $f_i \in \Lambda$ $(i = 1, 2, \dots, s)$ for some $s \geqq 2$. Let $BH = H/TH$ be the Λ-torsion-free part of H. Let $\beta(H)$ denote the Λ-rank of H, namely the $Q(\Lambda)$-dimension of the $Q(\Lambda)$-vector space $H \bigotimes_{\Lambda} Q(\Lambda)$ for the quotient field $Q(\Lambda)$ of Λ. Then $\beta(H) = \beta(BH)$. The following known properties are very often used (see [7] for the proof):

Lemma 3.1 (Properties on the characteristic polynomials).

(1) For every short Λ-exact sequence $0 \to T' \to T \to T'' \to 0$ of finitely generated torsion Λ-modules T', T, T'', we have $\Delta(T) \doteq \Delta(T')\Delta(T'')$.

(2) For every finitely generated Λ-module H, we have $\Delta^{(d)}(H) = 0$ for all $d < \beta(H)$ and $\Delta^{(d)}(TH) \doteq \Delta^{(d+\beta(H))}(H) \neq 0$ for all $d \geq 0$.

(3) For the null Λ-submodule DH of every finitely generated Λ-module H, we have $\Delta(DH) \doteq \pm 1$.
(4) Let $(\bar{P}, \bar{P}')$ be a regular $\mathbf{Z}^r$-covering of a compact polyhedral pair (P, P'). If $H_d(P, P'; \mathbf{Z}) = 0$, then the Λ-module $H = H_d(\bar{P}, \bar{P}'; \mathbf{Z})$ is a finitely generated torsion Λ-module with $\Delta(H)(1, 1, \ldots, 1) = \pm 1$.

By Property (2), we have the identity

$$\Delta^T(H) \doteq \Delta^{(\beta(H))}(H),$$

whose non-zero property is useful in our argument. Let $(\bar{P}, \bar{P}')$ be a regular $\mathbf{Z}^r$-covering of a compact polyhedral pair (P, P') (possibly with $P' = \emptyset$). Then let

$$\Delta^T(\bar{P}, \bar{P}') = \Delta^T(H_1(\bar{P}, \bar{P}'; \mathbf{Z})) \in \Lambda, \quad \beta_d(\bar{P}, \bar{P}') = \beta(H_d(\bar{P}, \bar{P}'; \mathbf{Z})).$$

In particular, the torsion Alexander polynomials

$$\Delta^T(\ell) = \Delta^T(\bar{X}^0), \quad \Delta^T(L^\pm) = \Delta^T(\bar{X}^\pm), \quad \Delta^T(L) = \Delta^T(\bar{X})$$

are called the *torsion Alexander polynomials* of ℓ, $L^\pm$ and L, respectively. Also, let

$$\beta_d(\ell) = \beta_d(\bar{X}^0), \quad \beta_d(L^\pm) = \beta_d(\bar{X}^\pm), \quad \beta_d(L) = \beta_d(\bar{X}).$$

For an immersed S^2-link L with r components and its a cross-section ℓ with s components of L, the *κ-invariants* $\kappa(\ell)$ and $\kappa(L)$ are defined by the following identities

$$\kappa(\ell) = s - 1 - \beta_1(\ell) \quad \text{and} \quad \kappa(L) = r - 1 - \beta_1(L).$$

It is noted that the properties in Lemma 3.1 are still true for a finitely generated $\tilde{\Lambda}$-module by taking $r = 1$.

For an infinite cyclic covering $(\tilde{P}, \tilde{P}')$ of a compact polyhedral pair (P, P') (possibly with $P' = \emptyset$), let

$$\Delta^T(\tilde{P}, \tilde{P}') = \Delta^T(H_1(\tilde{P}, \tilde{P}'; \mathbf{Z})) \in \tilde{\Lambda}, \quad \beta_d(\tilde{P}, \tilde{P}') = \beta(H_d(\tilde{P}, \tilde{P}'; \mathbf{Z})).$$

The following lemma is a consequence of the Fox calculus in [2].

Lemma 3.2 For a compact connected polyhedron P, assume that there is an epimorphism $\mathbf{Z}^r \to \mathbf{Z}$ for the covering transformation group $\mathbf{Z}^r$ of $\bar{P}$ and the covering transformation group $\mathbf{Z}$ of $\tilde{P}$. Then

$$\beta_1(\bar{P}) \leqq \beta_1(\tilde{P}).$$

Proof of Lemma 3.2. Let $\varphi : \Lambda \to \tilde{\Lambda}$ be the ring epimorphism sending t_i to t for every i. By the Fox calculus in [2] (see also [12, Chapter 7]), there is a finite Λ-presentation matrix $A = (a_{ij})$ of the Λ-module $H_1(\bar{P}, \bar{x}_0; \mathbf{Z})$ and $\varphi(A) = (\varphi(a_{ij}))$ is a finite $\tilde{\Lambda}$-presentation matrix of the $\tilde{\Lambda}$-module $H_1(\tilde{P}, \tilde{x}_0; \mathbf{Z})$, where $\bar{x}_0$ and $\tilde{x}_0$ denote the preimages of a base point $x_0 \in P$ under the coverings $\bar{P} \to P$ and $\tilde{P} \to P$, respectively. Then we obtain that

$$\beta_1(\tilde{P}) + 1 = \beta(H_1(\tilde{P}, \tilde{x}_0; \mathbf{Z})) \geqq \beta(H_1(\bar{P}, \bar{x}_0; \mathbf{Z})) = \beta_1(\bar{P}) + 1.$$

Thus, the desired inequality holds. □

For an immersed S^2-link L with ℓ as a cross-section, the reduced torsion Alexander polynomials

$$\tilde{\Delta}^T(\ell) = \Delta^T(\tilde{X}^0), \quad \tilde{\Delta}^T(L^{\pm}) = \Delta^T(\tilde{X}^{\pm}), \quad \tilde{\Delta}^T(L) = \Delta^T(\tilde{X})$$

are called the *reduced torsion Alexander polynomials* of ℓ, $L^{\pm}$ and L, respectively. Also, let

$$\tilde{\beta}_d(\ell) = \beta_d(\tilde{X}^0), \quad \tilde{\beta}_d(L^{\pm}) = \beta_d(\tilde{X}^{\pm}), \quad \tilde{\beta}_d(L) = \beta_d(\tilde{X}).$$

For an immersed S^2-link L with r components and its a cross-section ℓ with s components of L, the *reduced κ-invariants* $\tilde{\kappa}(\ell)$ and $\tilde{\kappa}(L)$ are defined by the following identities

$$\tilde{\kappa}(\ell) = s - 1 - \tilde{\beta}_1(\ell) \quad \text{and} \quad \tilde{\kappa}(L) = r - 1 - \tilde{\beta}_1(L).$$

Then we have the following lemma.

Lemma 3.3. For a regular cross-section ℓ of an immersed S^2-link L with r components, the following (1)-(3) are obtained.

(1)
$$\begin{aligned}
&0 \leqq \beta_1(\ell) \leqq \tilde{\beta}_1(\ell) \leqq r - 1,\\
&0 \leqq \beta_1(L^{\pm}) \leqq \tilde{\beta}_1(L^{\pm}) \leqq r - 1,\\
&0 \leqq \beta_1(L) \leqq \tilde{\beta}_1(L) \leqq r - 1,\\
&\beta_1(L) \leqq \beta_1(L^{\pm}) \leqq \beta_1(\ell),\\
&\tilde{\beta}_1(L) \leqq \tilde{\beta}_1(L^{\pm}) \leqq \tilde{\beta}_1(\ell).
\end{aligned}$$

(2)
$$\begin{aligned}
&0 \leqq \tilde{\kappa}(\ell) \leqq \kappa(\ell) \leqq r - 1,\\
&0 \leqq \tilde{\kappa}(L^{\pm}) \leqq \kappa(L^{\pm}) \leqq r - 1,\\
&0 \leqq \tilde{\kappa}(L) \leqq \kappa(L) \leqq r - 1,\\
&\kappa(\ell) \leqq \kappa(L^{\pm}) \leqq \kappa(L),\\
&\tilde{\kappa}(\ell) \leqq \tilde{\kappa}(L^{\pm}) \leqq \tilde{\kappa}(L).
\end{aligned}$$

(3) If $\kappa(\ell)=0$, $\kappa(L^{\pm})=0$ or $\kappa(L)=0$, then

$$\Delta^T(\ell)(1,1,\dots,1)=\pm1,\ \Delta^T(L^{\pm})(1,1,\dots,1)=\pm1 \,\text{or}\, \Delta^T(L)(1,1,\dots,1)=\pm1,$$

respectively. If $\tilde{\kappa}(\ell)=0$, $\tilde{\kappa}(L^{\pm})=0$ or $\tilde{\kappa}(L)=0$, then

$$\tilde{\Delta}^T(\ell)(1)=\pm1,\quad \tilde{\Delta}^T(L^{\pm})(1)=\pm1 \quad\text{or}\quad \tilde{\Delta}^T(L)(1)=\pm1,$$

respectively.

Proof of Lemma 3.3. The reduced κ-invariant $\tilde{\kappa}(\ell)=r-1-\tilde{\beta}_1(\ell)$ coincides with the κ_1-invariant in [13] which is the $\mathbf{Z}$-rank of the kernel of the homomorphism $1-t:TH_1(\tilde{X}^0;\mathbf{Z})\to TH_1(\tilde{X}^0;\mathbf{Z})$. Hence $\tilde{\beta}_1(\ell)\leqq r-1$. By Lemma 3.2, $0\leqq\beta_1(\ell)\leqq\tilde{\beta}_1(\ell)\leqq r-1$ is shown. By the same method, the second and third inequalities are obtained. Since

$$H_1(X,X^{\pm};\mathbf{Z})=H_1(X^{\pm},X^0;\mathbf{Z})=0,$$

it follows from Lemma 3.1 (4) that the fourth and fifth inequalities are obtained, and thus (1) is shown. Then (2) is direct from (1) by definition.

For (3), first assume that $\kappa(\ell)=0$, i.e., $\beta(\ell)=r-1$. Let $G=\cup_{i=1}^r(m_i\cup\alpha_i)$ be a bouquet in X^0 such that m_i is a meridian of the i^{th} component ℓ_i of ℓ and α_i is a simple arc in X^0 joining a based vertex v and a point of m_i for every i. Then the homology group $H_1(X^0,G;\mathbf{Z})$ of the pair (X^0,G) is 0. Let $(\bar{G},\bar{v})$ be the preimage of (G,v) under the maximal free abelian covering $\bar{X}^0\to X^0$. By Lemma 3.1 (4), the Λ-module $T=H_1(\bar{X}^0,\bar{G};\mathbf{Z})$ is a finitely generated torsion Λ-module with $\Delta(T)(1,1,\dots,1)=\pm1$. The pair $(\bar{G},\bar{v})$ induces an exact sequence

$$0\to H_1(\bar{G};\mathbf{Z})\to H_1(\bar{G},\bar{v};\mathbf{Z})\to \mathrm{Ker}\varphi_*\to 0,$$

where $\mathrm{Ker}\varphi_*$ denotes the kernel of the epimorphism $\varphi_*:\Lambda\to\mathbf{Z}$ sending t_i to 1 for every i. Since $H_1(\bar{G},\bar{v})\cong\Lambda^r$ and $\mathrm{Ker}\varphi_*$ is a torsion-free Λ-module of rank one, the Λ-module $H_1(\bar{G};\mathbf{Z})$ is a torsion-free Λ-module of Λ-rank $r-1$. By combining it with $\beta(\ell)=r-1$, the pair $(\bar{X}^0,\bar{G})$ must induce an exact sequence:

$$0\to H_1(\bar{G};\mathbf{Z})\to H_1(\bar{X}^0;\mathbf{Z})\to T\to 0,$$

so that the Λ-torsion part $TH_1(\bar{X}^0;\mathbf{Z})$ of the Λ-module $H_1(\bar{X}^0;\mathbf{Z})$ is embedded in T and hence we have $\Delta^T(\ell;1,1,\dots,1)=\pm1$ by Lemma 3.1 (1). By using similar bouquets G in $X^{\pm}$ and X in place of X^0, the identities $\Delta^T(L^{\pm};1,1,\dots,1)=\pm1$ and $\Delta^T(L;1,1,\dots,1)=\pm1$ are shown similarly. Next, assume that $\tilde{\kappa}(\ell)=0$, i.e., $\tilde{\beta}(\ell)=r-1$. Let $\tilde{H}=H_1(\tilde{X}^0;\mathbf{Z})$ be a finitely generated $\tilde{\Lambda}$-module. Then $\Delta^{(d)}(\tilde{H})=0$ for $d<r-1$ and $\Delta^{(r-1)}(\tilde{H})=\Delta^T(\tilde{H})\neq0$ by Lemma 3.1 (2). Let $\eta:\tilde{\Lambda}\to\mathbf{Z}$ be the

ring epimorphism sending t to 1. Let $\tilde{A}$ be a finite $\tilde{\Lambda}$-presentation matrix of $\tilde{H}$. Then $\eta(\tilde{A})$ is a $\mathbf{Z}$-presentation matrix of the $\mathbf{Z}$-module $\tilde{H} \otimes_\eta \mathbf{Z} \cong \mathbf{Z}^{r-1}$. This means that

$$\eta(\Delta^T(\tilde{H})) = \Delta^T(\tilde{H}; 1) = \tilde{\Delta}^T(\ell; 1) = \pm 1.$$

By applying the same method for $X^\pm$ and X in place of X^0, the identities $\tilde{\Delta}^T(L^\pm; 1) = \pm 1$ and $\tilde{\Delta}^T(L; 1) = \pm 1$ are shown similarly. □

Next, the local signature invariant $s_J(\ell)$ for every subset J of $[-1, 1]$ of an oriented link ℓ in S^3 is explained. For $x \in (-1, 1)$, let $\omega_x = x + \sqrt{1 - x^2}\sqrt{-1}$, which is a complex number of norm one. For a Seifert matrix S of a link ℓ, we can consider the signature $\mathrm{sign}S(\omega_x)$ of the Hermitian matrix

$$S(\omega_x) = (1 - \omega_x)S + (1 - \bar{\omega}_x)S^T$$

which is a step function on x (see [8, 9]). This signature invariant is called the *Tristram signature* of ℓ (see [25]). Let $s(\ell)$ be the ordinary signature $\mathrm{sign}(S + S^T)$ of ℓ. By using the Tristram signature and the ordinary signature, the *signature invariants* $s_{[a,1]}(\ell)$, $s_{(a,1]}(\ell)$ of ℓ for $a \in [-1, 1]$ are defined as follows:

$$\begin{aligned} s_{[a,1]}(\ell) &= \lim_{x \to a-0} S(\omega_x) \quad \text{for} \quad a \in (-1, 1], \\ s_{(a,1]}(\ell) &= \lim_{x \to a+0} S(\omega_x) \quad \text{for} \quad a \in [-1, 1), \\ s_{[-1,1]}(\ell) &= s(\ell). \end{aligned}$$

Then the integer invariant $s_a(\ell)\,(a \in [-1, 1])$ of ℓ is defined by the difference $s_a(\ell) = s_{[a,1]}(\ell) - s_{(a,1]}(\ell)$ for $a \in [-1, 1)$ and $s_1(\ell) = s_{[1,1]}(\ell)$, which is 0 except for a finite number of a. For every subset J of $[-1, 1]$, we define

$$s_J(\ell) = \sum_{a \in J} s_a(\ell).$$

Let $\Lambda_{\mathbf{R}}$ be the real polynomial ring $\mathbf{R}[t, t^{-1}]$. In [13], the local signature invariant $\sigma_a(\ell)\,(a \in [-1, 1])$ of a link ℓ is defined from the quadratic form

$$q : TH_1(\tilde{X}^0; \mathbf{R}) \times TH_1(\tilde{X}^0; \mathbf{R}) \to \mathbf{R}$$

on the $\Lambda_{\mathbf{R}}$-torsion part $TH_1(\tilde{X}^0; \mathbf{R})$ of the $\Lambda_{\mathbf{R}}$-module $H_1(\tilde{X}^0; \mathbf{R})$ by restricting to the $p_a(t)$-primary component for every $a \in [-1, 1]$, where $p_a(t)$ is defined by $p_a(t) = 2a - (t + t^{-1})$ for $a \in (-1, 1)$ and $p_{\pm 1}(t) = 1 - (\pm t)$. For a link ℓ with components $\ell_i\,(i = 1, 2, \ldots, s)$, let $V(\ell) = (v_{ij})$ be the symmetric matrix of size s such that $v_{ij} = \mathrm{Link}(\ell_i, \ell_j)$ for every pair (i, j) with $i \neq j$ and $\sum_{j=1}^{s} v_{ij} = 0$ for every i. Since the diagonal entry v_{ii} is the self-linking number of the component ℓ_i with Seifert surface framing given the link ℓ, we call the matrix $V(\ell)$ the *linking matrix of* ℓ *(with Seifert framing)* (see [13]). Let $s_V(\ell)$ be the signature of $V(\ell)$ which is called the *linking signature invariant* of ℓ. When ℓ is a knot, we understand that $V(\ell)$ is the zero matrix with $s_V(\ell) = 0$. Then the identities

$$\sigma_a(\ell) = s_a(\ell)\,(a \in (-1,1)) \quad \text{and} \quad \sigma_1(\ell) = s_1(\ell) - s_V(\ell)$$

are established in [13, Theorem 5.3]. For example, if o^H be a positive Hopf link, then

$$\sigma_a(o^H) = 0\,(a \in [-1,1]) \quad \text{and} \quad s_1(o^H) = s_V(o^H) = -1.$$

A unified simple topological proof of the following lemma is promised in the introduction.

Lemma 3.4. The linking matrix $V(\ell)$ of every (regular or irregular) cross-section ℓ of every S^2-link L is a symmetric even matrix with signature $s_V(\ell) = 0$.

Proof of Lemma 3.4. Let $L^{\pm}$ be the planar surfaces in the 4-balls $B^{\pm}$ with $\partial L^{\pm} = \ell$ obtained from L by splitting along the 3-sphere S^3. Let $k_i\,(i = 1, 2, \ldots, m)$ be mutually disjoint simple loops in the interior of L^+ representing a $\mathbf{Z}$-basis for $H_1(L^+; \mathbf{Z})$. Give the framings determined by collars in L to $k_i\,(i = 1, 2, \ldots, m)$. The 4-manifold obtained from S^4 by the 2-handle surgeries along the framed loops $k_i\,(i = 1, 2, \ldots, m)$ is homeomorphic to the 4-manifold $M = \#m(S^2 \times S^2)$ and contains, as a submanifold, the 4-manifold Y obtained from the 4-ball B^- by attaching 2-handles to S^3 along the link ℓ with Seifert surface framing. Since the 4-manifold M is spin, the 4-submanifold Y is spin. By construction, we may consider that the image $\mathrm{Im}(i_*)$ of the natural homomorphism $i_* : H_2(Y; Q) \to H_2(M; Q)$ contains a half basis $y_i\,(i = 1, 2, \ldots, m)$ of $H_2(M; Q)$ such that $y_i = [S^2 \times p_i]$ for the connected summands $S^2 \times S^2_i\,(i = 1, 2, \ldots, m)$ of M and a point $p \in S^2$. Since the Q-intersection number $\mathrm{Int}(y_i, y_j) = 0$ in M for all i, j, there is a re-indexed Q-basis $y_i\,(i = 1, 2, \ldots, m), y'_j\,(j = 1, 2, \ldots, s(\leqq m))$ for the image $\mathrm{Im}(i_*)$ such that $\mathrm{Int}(y_i, y'_j) = \delta_{ij}$ for all i, j. Since the kernel $\mathrm{Ker} i_*$ of the natural homomorphism $i_* : H_2(Y; Q) \to H_2(M; Q)$ contributes the Q-intersection number 0 in M, it is seen that the signature $\mathrm{sign}(Y) = 0$. Hence the matrix $V(\ell)$ is a symmetric even form with signature $s_V(\ell) = 0$ since it is an intersection matrix of this spin 4-manifold Y with $\mathrm{sign}(Y) = 0$. $\square$

The following lemma is useful to compute this signature invariant for a link ℓ:

Lemma 3.5. (1) If the reduced torsion Alexander polynomial $\tilde{\Delta}^T(\ell)$ does not contain the polynomial $p_a(t)$ for a number $a \in [-1,1]$ as a real polynomial factor or contains $p_1(t) = 1-t$ without multiplicity, then $s_a(\ell) = \sigma_a(\ell) = 0$ or $\sigma_1(\ell) = 0$, respectively.
(2) If the reduced torsion Alexander polynomial $\tilde{\Delta}^T(\ell)$ contains the polynomial $p_a(t)$ for $a \in (-1,1)$ as a real polynomial factor without multiplicity, then $s_a(\ell) = \sigma_a(\ell) = \pm 2$.

Proof of Lemma 3.5. The assertion of (1) for $\sigma_a(\ell)$ is direct from the definition of the local signature in [13]. Then by [13, Corollary 5.4], $s_a(\ell) = 0$ for $a \in [0,1)$. To see that $s_1(\ell) = 0$, it is noted that the assumption $\tilde{\Delta}^T(\ell;1) \neq 0$ means that the homomorphism $1-t : TH_1(\tilde{X}^0;\mathbf{Z}) \to TH_1(\tilde{X}^0;\mathbf{Z})$ is injective and $\tilde{\kappa}(\ell) = 0$ by the proof of Lemma 3.3, so that $\tilde{\Delta}^T(\ell)(1) = \pm 1$ by Lemma 3.3 (3). Then, by [13, Lemma 3.3], all the pairwise linking numbers of ℓ are 0 and hence $s_V(\ell) = 0$. By [13, Corollary 5.4], we have $s(\ell) = \sigma_1(\ell) = 0$. The assertion of (2) is direct from from the definition of $\sigma_a(\ell)$ in [13] (see J. W. Milnor [18]) since $s_a(\ell) = \sigma_a(\ell)$ for $a \in (-1,1)$. □

4 Proof of Theorems 1.1 and 1.2

Throughout this section, the proofs of Theorems 1.1 and 1.2 will be done. The proof of Theorem 1.1 is done as follows.

Proof of Theorem 1.1. To see (1), the Euler characteristic $\chi(X) = 2+c-2r$ is used, which is obtained from the computations that $\chi(L) = 2r - c$ and $\chi(X) + \chi(L) = \chi(S^4) = 2$. It is noted that the boundary ∂X of X is the union of the product 3-manifold $F \times S^1$ for an r-component compact planar surface F and c Hopf link exteriors pasting along tori, so that the homology group $H_d(\partial\bar{X};\mathbf{Z})$ forms a finitely generated torsion Λ-module. Since

$$\begin{aligned}\beta_4(\bar{X}) &= \beta_0(\bar{X},\partial\bar{X}) = \beta_0(\bar{X}) = 0,\\ \beta_3(\bar{X}) &= \beta_1(\bar{X},\partial\bar{X}) = \beta_1(\bar{X})\end{aligned}$$

by Blanchfield duality in [1], the definition of the $Q(\Lambda)$-Euler characteristic $\chi(\bar{X};Q(\Lambda))$ means that

$$\beta_2(\bar{X}) - 2\beta_1(\bar{X}) = \chi(\bar{X};Q(\Lambda)) = \chi(X) = 2+c-2r.$$

Hence the following identities

$$\beta_2(\bar{X}) = c - 2(r-1-\beta_1(\bar{X})) = c - 2(r-1-\beta_1(L)) = c - 2\kappa(L)$$

are obtained. Thus, $2\kappa(L) \leqq c$. Assume that $2\kappa(L) = c$. Then $\beta_2(\bar{X}) = 0$. For any cross-section ℓ of L, this means that the Λ-intersection form Int_Λ on $H_2(\bar{X}^+; \mathbf{Z})$ vanishes, so that the natural homomorphism

$$BH_2(\bar{X}^+; \mathbf{Z}) \to BH_2(\bar{X}^+, \partial\bar{X}^+; \mathbf{Z})$$

must be trivial by the Poincaré duality on $\bar{X}^+$. Using this triviality, we see that the semi-exact sequence

$$(\#) \qquad TH_2(\bar{X}^+, \partial\bar{X}^+; \mathbf{Z}) \xrightarrow{\partial_*} TH_1(\partial\bar{X}^+; \mathbf{Z}) \xrightarrow{i_*} TH_1(\bar{X}^+; \mathbf{Z})$$

is exact. Thus, we have a short exact sequence

$$0 \to \mathrm{Im}\partial_* \hookrightarrow TH_1(\partial\bar{X}^+; \mathbf{Z}) \xrightarrow{i_*} \mathrm{Im}i_* \to 0.$$

By Lemma 3.1 (1) and [7, Theorem 3.1], we have

$$\Delta(TH_1(\partial\bar{X}^+; \mathbf{Z})) \doteq \Delta(\mathrm{Im}\partial_*)\Delta(\mathrm{Im}i_*) \quad \text{and} \quad \Delta(\mathrm{Im}\partial_*)^* \doteq \Delta(\mathrm{Im}i_*).$$

By an argument of [14, Lemma 3.3], it is seen that

$$\Delta^T(\ell) = \Delta^T(H_1(\bar{X}^0; \mathbf{Z})) \doteqdot \Delta^T(H_1(\partial\bar{X}^+; \mathbf{Z})) \doteq \Delta(\mathrm{Im}i_*)\Delta(\mathrm{Im}i_*)^*,$$

showing (1) by taking $f = \Delta(\mathrm{Im}i_*)$.

To see (2), the homology $H_*(X^+; \mathbf{Z})$ is computed as follows:

$$H_d(X^+; \mathbf{Z}) = \begin{cases} \mathbf{Z}^{c^+} & (d = 2), \\ \mathbf{Z}^r & (d = 1), \\ \mathbf{Z} & (d = 0), \\ 0 & (\text{others}). \end{cases}$$

Hence $\chi(X^+) = c^+ + 1 - r$. By Blanchfield duality in [1], we have $\beta_3(\bar{X}^+) = \beta_1(\bar{X}^+, \partial\bar{X}^+) = 0$, so that

$$\beta_d(\bar{X}^+) = 0\,(d = 3, 4), \quad \beta_2(\bar{X}^+) - \beta_1(\bar{X}^+) = c^+ + 1 - r.$$

Note that $\beta_1(\ell) = \beta_1(\partial\bar{X}^+)$ by the argument of (1). Since $\beta_2(\bar{X}^+) = \beta_2(\bar{X}^+, \partial\bar{X}^+)$ and $\beta_1(\partial\bar{X}^+) \leqq \beta_2(\bar{X}^+, \partial\bar{X}^+) + \beta_1(\bar{X}^+)$, we have

$$\beta_1(\ell) \leqq \beta_2(\bar{X}^+) + \beta_1(\bar{X}^+) = c^+ + 1 - r + 2\beta_1(\bar{X}^+) = c^+ - \kappa(L^+) + \beta_1(L^+).$$

Hence $2\kappa(L^+) - \kappa(\ell) \leqq c^+$. On the other hand,

$$c^+ - \kappa(\ell) = c^+ + 1 - r + \beta_1(\partial\bar{X}^+) = \beta_2(\bar{X}^+) + \beta_1(\partial\bar{X}^+) - \beta_1(\bar{X}^+) \geqq 0,$$

because the vanishing $\beta_1(\bar{X}^+, \partial\bar{X}^+) = 0$ means that $\beta_1(\partial\bar{X}^+) - \beta_1(\bar{X}^+) \geqq 0$. Hence the inequality $\kappa(\ell) \leqq c^+$ is obtained. By a similar argument on X^-, the inequalities $2\kappa(L^-) - \kappa(\ell) \leqq c^-$ and $\kappa(\ell) \leqq c^-$ are obtained. Thus, the first half of (2) is shown.

Assume that $\kappa(L^+) - c^+ = \kappa(\ell)$. Then we have

$$\beta_1(\partial\bar{X}^+) = \beta_2(\bar{X}^+, \partial\bar{X}^+) + \beta_1(\bar{X}^+).$$

This means that the sequence (#) in (1) is exact. Assume that $\kappa(\ell) = c^+$. Then $\beta_2(\bar{X}^+) = 0$ and $\beta_1(\partial\bar{X}^+) = \beta_1(\bar{X}^+)$. This also means that the sequence (#) in (1) is exact. The conclusion of (2) is obtained from the argument of (1).

For (3), assume that ℓ is a middle cross-section of L and $\kappa(\ell) = c^- = c^+$. By Lemma 3.3 and (1), we have

$$c = 2\kappa(\ell) \leqq 2\kappa(L^\pm) \leqq 2\kappa(L) \leqq c$$

meaning that

$$\kappa(\ell) = \kappa(L^\pm) = \kappa(L) = c^- = c^+.$$

Hence we obtain

$$\beta_1(\ell) = \beta_1(L^\pm) = \beta_1(L).$$

The natural homomorphism $H_1(\bar{X}^0; \mathbf{Z}) \to H_1(\bar{X}^+; \mathbf{Z})$ is onto since ℓ is a middle cross-section of L. By the identity $\beta_1(\ell) = \beta_1(L^+)$, the natural homomorphism $BH_1(\bar{X}^0; \mathbf{Z}) \to BH_1(\bar{X}^+; \mathbf{Z})$ induces a Λ-isomorphism. Hence the natural homomorphism $TH_1(\bar{X}^0; \mathbf{Z}) \to TH_1(\bar{X}^+; \mathbf{Z})$ is onto, so that $\mathrm{Im} i_* = TH_1(\bar{X}^+; \mathbf{Z})$ and hence

$$\Delta(\mathrm{Im} i_*) \doteq \Delta(TH_1(\bar{X}^+; \mathbf{Z})) \doteq \Delta^T(L^+).$$

Also using X^- instead of X^+, we can conclude that

$$\Delta^T(\ell) \doteq \Delta^T(L^+)\Delta^T(L^+)^* \doteq \Delta^T(L^-)\Delta^T(L^-)^*.$$

The natural homomorphism $H_1(\bar{X}^+; \mathbf{Z}) \to H_1(\bar{X}; \mathbf{Z})$ is also onto. By the identity $\beta_1(L^+) = \beta_1(L)$, the natural homomorphism

$$BH_1(\bar{X}^+; \mathbf{Z}) \to BH_1(\bar{X}; \mathbf{Z})$$

induces an isomorphism. Hence the natural homomorphism $TH_1(\bar{X}^+; \mathbf{Z}) \to TH_1(\bar{X}; \mathbf{Z})$ is onto, so that $\Delta^T(L)$ is a factor of $\Delta^T(L^+)$ and hence $\Delta^T(L^+) = \Delta^T(L)g$ for an element $g \in \Lambda$. Thus, the conclusion of (3) is shown.

To see (4), use the fact that the 4-manifold X is the double of X^+ pasting along the 3-manifold X^0. Let $r : X \to X^+$ be the retraction, i.e., the composite $ri : X^+ \to X^+$ is the identity for the inclusion $i : X^+ \subset X$. Then the induced homomorphism

$i_\# : \pi_1(X^+, x_0) \to \pi_1(X, x_0)$ for a base point $x_0 \in X^0$ is a monomorphism. On the other hand, by the definition of a symmetric middle cross-section, the inclusion $i' : X^0 \subset X^+$ induces an epimorphism $i'_\# : \pi_1(X^0, x_0) \to \pi_1(X^+, x_0)$ so that the inclusion $\bar{i} : X^0 \subset X$ induces an epimorphism $\bar{i}_\# : \pi_1(X^0, x_0) \to \pi_1(X, x_0)$. Thus, the induced homomorphism $i_\# : \pi_1(X^+, x_0) \to \pi_1(X, x_0)$ is an isomorphism, giving a Λ-isomorphism $H_1(\bar{X}^+; \mathbf{Z}) \cong H_1(\bar{X}; \mathbf{Z})$. Similarly, there is a Λ-isomorphism $H_1(\bar{X}^-; \mathbf{Z}) \cong H_1(\bar{X}; \mathbf{Z})$. If $2\kappa(L) = c$, then we have $\beta_2(\bar{X}) = 0$, because $\beta_2(\bar{X}) = c - 2\kappa(L)$. On the other hand, since $\beta_1(\bar{X}^\pm) = \beta_1(\bar{X})$, the Mayer-Vietoris exact sequence

$$H_2(\bar{X}; \mathbf{Z}) \to H_1(\bar{X}^0; \mathbf{Z}) \to H_1(\bar{X}^+; \mathbf{Z}) \oplus H_1(\bar{X}^-; \mathbf{Z}) \to H_1(\bar{X}; \mathbf{Z}) \to 0$$

implies that

$$\beta_1(\ell) = \beta_1(\bar{X}^0) = \beta_1(\bar{X}) = \beta_1(L),$$

so that $2\kappa(\ell) = c$ and $\kappa(\ell) = c^- = c^+$. By the Λ-isomorphism $H_1(\bar{X}^+; \mathbf{Z}) \cong H_1(\bar{X}; \mathbf{Z})$, we have $\Delta(L^+) \doteq \Delta(L)$ and the identity on $\Delta^T(\ell)$ in (4) is obtained from (3). $\square$

The proof of Theorem 1.2 is done as follows.

Proof of Theorem 1.2. To see (1), use the Euler characteristic $\chi(X) = 2 + c - 2r$ given by the argument in the proof of Theorem 1.2 (1). Moreover, the homology group $H_d(\partial\tilde{X}; \mathbf{Z})$ forms a finitely generated torsion $\tilde{\Lambda}$-module and by using the zeroth duality of [10] instead of the Blanchfield duality and the $Q(\tilde{\Lambda})$-Euler characteristic $\chi(\tilde{X}; Q(\tilde{\Lambda}))$ instead of $\chi(\bar{X}; Q(\Lambda))$, we have

$$\begin{aligned}
\beta_4(\tilde{X}) &= \beta_0(\tilde{X}, \partial\tilde{X}) = \beta_0(\tilde{X}) = 0, \\
\beta_3(\tilde{X}) &= \beta_1(\tilde{X}, \partial\tilde{X}) = \beta_1(\tilde{X}), \\
\beta_2(\tilde{X}) &- 2\beta_1(\tilde{X}) = \chi(\tilde{X}; Q(\tilde{\Lambda})) = \chi(X) = 2 + c - 2r.
\end{aligned}$$

Hence the following identities

$$\beta_2(\tilde{X}) = c - 2(r - 1 - \beta_1(\tilde{X})) = c - 2\tilde{\kappa}(L)$$

are obtained. In particular, the inequality $2\tilde{\kappa}(L) \leqq c$ is obtained. Let F be a proper planar surface in a 4-disk B^4 obtained from the pair $((B^4)^+, L^+)$ by removing regular neighborhoods of α_i $(i = 1, 2, \ldots, c)$. It is noted that the boundary $\ell^{H_+} = \partial F$ of the surface F is a connected sum of the link ℓ and c^+ Hopf links o_i^H $(i = 1, 2, \ldots, c^+)$ in S^3 and the exterior $\mathrm{cl}(B^4 \setminus N(F))$ is homeomorphic to X^+. Let $\hat{\beta}_2(\tilde{X}^\pm)$ be the $\tilde{\Lambda}$-rank of the $\tilde{\Lambda}$-intersection form

$$\mathrm{Int}_{\tilde{\Lambda}} : BH_2(\tilde{X}^\pm; \mathbf{Z}) \times BH_2(\tilde{X}^\pm; \mathbf{Z}) \to \tilde{\Lambda}$$

(where $\pm$ is taken with the same sign). Then

$$\hat{\beta}_2(\tilde{X}^+) + \hat{\beta}_2(\tilde{X}^-) \leqq \beta_2(\tilde{X}).$$

By [13, Lemma 6.1], we have the inequality

$$|s_{(a,1]}(\ell^{H_+})| \leqq \hat{\beta}_2(\tilde{X}^+)$$

for every $a \in (-1, 1)$. Here, it is noted that

$$s_{(a,1]}(\ell^{H_+}) = s_{(a,1]}(\ell) + \xi^+.$$

By a similar consideration for X^-, we can conclude that

$$|s_{(a,1]}(\ell) + \xi^+| + |-s_{(a,1]}(\ell) + \xi^-| \leqq \beta_2(\tilde{X}) = c - 2\tilde{\kappa}(L),$$

so that

$$|s_{(a,1]}(\ell)| \leqq \frac{c}{2} - \tilde{\kappa}(L) + \frac{|\xi^+| + |\xi^-|}{2} \leqq c - \tilde{\kappa}(L)$$

for every $a \in (-1, 1)$, showing the first half of (1). For the second half of (1), assume that $2\tilde{\kappa}(L) = c$. Then $\beta_2(\tilde{X}) = 0$ and $s_{(a,1]}(\ell) = -\xi^+ = \xi^-$ for every $a \in (-1, 1)$. In particular, $\xi = 0$, $s_a(\ell) = 0$ for $a \in (-1, 1)$ and $|s_1(\ell)| \leqq \frac{c}{2}$. Since $\beta_2(\tilde{X}) = 0$, by an analogous proof of Theorem 1.1 (1) the identity $\tilde{\Delta}^T(\ell) \doteqdot ff^*$ for an element $f \in \tilde{\Lambda}$ is obtained, showing the second half of (1).

For (2), since the identity

$$\mathrm{Link}(\ell_1, \ell_2) = \pm \mathrm{Int}(C_1, C_2)$$

holds for any disjoint knots $\ell_i\,(i = 1, 2)$ in $\mathbf{R}^3[0]$ bounding compact oriented surfaces $C_i\,(i = 1, 2)$ in $\mathbf{R}^3[0, +\infty)$, the inequality $|\mathrm{Link}|(\ell) \leqq \min\{c^-, c^+\}$ holds. By the $Q(\tilde{\Lambda})$-version $\chi(\tilde{X}^+; Q(\tilde{\Lambda}))$ of the Euler characteristic $\chi(X^+) = c^+ + 1 - r$, we have

$$\beta_d(\tilde{X}^+) = 0\,(d = 3, 4), \quad \beta_2(\tilde{X}^+) - \beta_1(\tilde{X}^+) = c^+ + 1 - r.$$

Hence, $\beta_2(\tilde{X}^+) = c^+ - \tilde{\kappa}(L^+)$. As an analogy of the proof of Theorem 1.1 (2), we have $2\tilde{\kappa}(L^+) - \tilde{\kappa}(\ell) \leqq c^+$ and $2\tilde{\kappa}(L^-) - \tilde{\kappa}(\ell) \leqq c^-$. On the other hand, as they are shown in (1), we have

$$|s_{(a,1]}(\ell) + \xi^+| \leqq \hat{\beta}_2(\tilde{X}^+) \leqq c^+ - \tilde{\kappa}(L^+)$$

meaning that

$$|s_{(a,1]}(\ell) + \xi^+| + \tilde{\kappa}(L^+) \leqq c^+$$

for every $a \in (-1, 1)$. Similarly,

$$|-s_{(a,1]}(\ell) + \xi^-| + \tilde{\kappa}(L^-) \leqq c^-$$

for every $a \in (-1, 1)$. Since $\tilde{\kappa}(\ell) \leqq \tilde{\kappa}(L^\pm)$, the first half of (2) is shown. Assume that $2\tilde{\kappa}(L^+) - \tilde{\kappa}(\ell) = c^+$. Since

$$\beta_1(\partial\tilde{X}^+) = \beta_2(\tilde{X}^+, \partial\tilde{X}^+) + \beta_1(\tilde{X}^+),$$

we have $\hat{\beta}_2(\tilde{X}^+) = 0$. Hence $s_{(a,1]}(\ell^{H_+}) = s_{(a,1]}(\ell) + \xi^+ = 0$ for every $a \in (-1, 1)$. In this case, the semi-exact sequence

$$(\tilde{\#}) \qquad TH_2(\tilde{X}^+, \partial\tilde{X}^+; \mathbf{Z}) \overset{\partial_*}{\to} TH_1(\partial\tilde{X}^+; \mathbf{Z}) \overset{i_*}{\to} TH_1(\tilde{X}^+; \mathbf{Z})$$

is exact, to which a similar argument of Theorem 1.1 (2) using the first duality for an infinite cyclic covering (see [10]) is applied to see $\tilde{\Delta}^T(\ell) \doteqdot ff^*$ for an element $f \in \tilde{\Lambda}$. Assume that $\tilde{\kappa}(\ell) = c^+$. Then it is direct to see that $s_{(a,1]}(\ell) = -\xi^+$ for every $a \in (-1, 1)$. Further, we have $\beta_2(\tilde{X}^+) = 0$ which implies also that the sequence $(\tilde{\#})$ is exact and we have a desired splitting of $\tilde{\Delta}^T(\ell)$. Thus, (2) is shown.

For (3), if $\tilde{\kappa}(\ell) = c^- = c^+$, then we have $s_{(a,1]}(\ell) = -\xi^+ = \xi^-$ by (2), so that $\xi = 0$. Furhter, the identity $2\tilde{\kappa}(L) = c$ and the identities on $\tilde{\Delta}^T(\ell)$ are obtained by a similar consideration to Theorem 1.1 (3), showing (3). (4) is also obtained by a similar consideration to Theorem 1.1 (4). □

5 A symmetric construction of immersed S^2-links

Every link ℓ^0 in $\mathbf{R}^3$ bounds an immersed disk system L^+ in $\mathbf{R}^3[0, +\infty)$. The double L of L^+ in $\mathbf{R}^4$ is an immersed S^2-link with ℓ^0 as a regular cross-section, which is called an *immersed symmetric S^2-link* with ℓ^0 as the *regular symmetric cross-section.* We have $c(L) = 2c(L^+)$. The 4-*dimensional clasp number* $c^4(\ell^0)$ of the link ℓ^0 is the minimal number of the double point number $c(L^+)$ for all immersed disk systems L^+ with $\partial L^+ = \ell^0$ (see [14]). By definition, we have $c(L) \geqq 2c^4(\ell^0)$ for all immersed symmetric S^2-links L with ℓ^0 the regular symmetric cross-section and the equality is realized by some L. Let $u(\ell^0)$ be the unlinking number of a link ℓ^0, namely the minimal number of crossing changes needed to obtain a trivial link o from ℓ^0. Then it is seen from [14] and Lemma 3.3 that

$$u(\ell^0) \geqq c^4(\ell^0) \geqq \kappa(\ell^0) \geqq \tilde{\kappa}(\ell^0).$$

A unique immersed disk system L^+ can be constructed from every unlinking operation on a link ℓ^0. The resulting immersed symmetric S^2-link L is an immersed *ribbon* S^2-link L with the symmetric middle cross-section ℓ^0. This immersed ribbon S^2-link L is called an immersed ribbon S^2-link *associated with an unlinking operation on* ℓ^0. Then $c(L) \geqq 2u(\ell^0)$ for immersed ribbon S^2-links L associated with all unlinking operations on ℓ^0 and the equality is realized by some L by taking an unlinking operation with the minimal number of operations on ℓ^0. It is noted that once the places of an unlinking operation on a link ℓ^0 are specified, the immersed ribbon S^2-link L associated with the unlinking operation on ℓ^0 is uniquely constructed from ℓ^0 since any choice of disk systems bounded by a trivial link is independent of the equivalence of L by Horibe-Yanagawa's lemma in [16]. A *lassoing* on a link ℓ^0 is to construct a link ℓ^1 with a local addition of a trivial loop from the link ℓ^0 by the operation given in Fig. 1 (see [22]). The link ℓ^1 depends heavily on a choice of a crossing in diagrams of ℓ^0. By a crossing change in the loop in Fig. 1, the link ℓ^1 changes into a split sum of the link ℓ^0 and a trivial knot o^1. This means that if an immersed symmetric S^2-link L with regular symmetric cross-section ℓ^0 is given, then this crossing change produces an immersed symmetric S^2-link L' with $c(L') = 2 + c(L)$ such that the lassoed link ℓ^1 is a regular symmetric cross-section of L'. The immersed symmetric S^2-link L' is called a *lassoed immersed symmetric S^2-link of* the immersed symmetric S^2-link L. If L is an immersed ribbon S^2-link associated with an unlinking operation on ℓ^0, then the lassoed immersed symmetric S^2-link L' is just an immersed ribbon S^2-link with the lassoed link ℓ^1 as the symmetric middle cross-section. We have the following theorem.

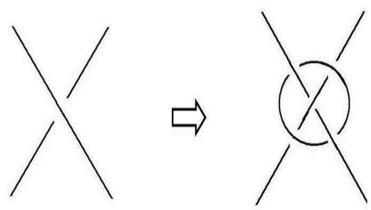

Figure 1: The lassoing operation

Theorem 5.1. For every link ℓ^0, let L be an immersed symmetric S^2-link with regular symmetric cross-section ℓ^0, and L' a lassoed immersed symmetric S^2-link of L with the lassoed link ℓ^1 of ℓ^0 as the regular symmetric cross-section. Let $c(L) = c$ and $c(L') = c' = c + 2$. For every regular symmetric cross-section ℓ of L and every regular symmetric cross-section ℓ' of L', there are rational numbers δ, δ', $\tilde{\delta}$, $\tilde{\delta}' \geqq 1$ such that

$$\begin{aligned} \delta(c - 2\kappa(L)) &= c - 2\kappa(\ell) = c' - 2\kappa(\ell') = \delta'(c' - 2\kappa(L')), \\ \tilde{\delta}(c - 2\tilde{\kappa}(L)) &= c - 2\tilde{\kappa}(\ell) = c' - 2\tilde{\kappa}(\ell') = \tilde{\delta}'(c' - 2\tilde{\kappa}(L')). \end{aligned}$$

Further, if $c = 2c^4(\ell^0) = 2\kappa(\ell^0)$ (or $c = 2c^4(\ell^0) = 2\tilde{\kappa}(\ell^0)$), then we have

$$\begin{gathered} c - 2\kappa(L) = c - 2\kappa(\ell) = c' - 2\kappa(\ell') = c' - 2\kappa(L') = 0 \\ (\text{or } c - 2\tilde{\kappa}(L) = c - 2\tilde{\kappa}(\ell) = c' - 2\tilde{\kappa}(\ell') = c' - 2\tilde{\kappa}(L') = 0, \text{ respectively}). \end{gathered}$$

The following corollary is direct from Theorem 5.1.

Corollary 5.2. For every link ℓ^0, let L be an immersed ribbon S^2-link associated with an unlinking operation on ℓ^0, and L' a lassoed immersed symmetric S^2-link of L with the lassoed link ℓ^1 of ℓ^0 as the symmetric middle cross-section. Let $c(L) = c$ and $c(L') = c' = c + 2$. For every symmetric middle cross-section ℓ of L and every symmetric middle cross-section ℓ' of L', there are rational numbers δ, δ', $\tilde{\delta}$, $\tilde{\delta}' \geqq 1$ such that

$$\begin{aligned} \delta(c - 2\kappa(L)) &= c - 2\kappa(\ell) = c' - 2\kappa(\ell') = \delta'(c' - 2\kappa(L')), \\ \tilde{\delta}(c - 2\tilde{\kappa}(L)) &= c - 2\tilde{\kappa}(\ell) = c' - 2\tilde{\kappa}(\ell') = \tilde{\delta}'(c' - 2\tilde{\kappa}(L')). \end{aligned}$$

Further, if $c = 2u(\ell^0) = 2\kappa(\ell^0)$ (or $c = 2u(\ell^0) = 2\tilde{\kappa}(\ell^0)$), then we have

$$\begin{gathered} c - 2\kappa(L) = c - 2\kappa(\ell) = c' - 2\kappa(\ell') = c' - 2\kappa(L') = 0 \\ (\text{or } c - 2\tilde{\kappa}(L) = c - 2\tilde{\kappa}(\ell) = c' - 2\tilde{\kappa}(\ell') = c' - 2\tilde{\kappa}(L') = 0, \text{ respectively}). \end{gathered}$$

The proof of Theorem 5.1 is given as follows.

Proof of Theorem 5.1. By Theorem 1.1 (1) and Lemma 3.3, it is noted that $c \geqq 2\kappa(L) \geqq 2\kappa(\ell)$, so that $c = 2\kappa(\ell)$ implies $c = 2\kappa(L)$. Assume that $2\kappa(L) = c$. Then we can show that $c = 2\kappa(\ell)$ by a slight generalization of an argument in the proof of Theorem 1.1 since $H_1(X, X^\pm; \mathbf{Z}) = 0$ imply that $H_1(\bar{X}, \bar{X}^\pm; \mathbf{Z})$ are torsion Λ-modules by Lemma 3.1 (4) and hence the retraction $r : X \to X^+$ is used to show that the identities $\beta_1(\bar{X}^\pm) = \beta_1(\bar{X})$ still hold as it is in the case of an immersed ribbon S^2-link. Further, $2\kappa(L) = c$ means $\beta_2(\bar{X}) = 0$ and hence we have

$$\beta(\ell) = \beta_1(\bar{X}^0) = \beta_1(\bar{X}) = \beta_1(L)$$

by the Mayer-Vietoris exact sequence on $(X; X^+, X^-; X^0)$ as it is desired. Thus, we can find a rational number $\delta \geqq 1$ such that $\delta(c - 2\kappa(L)) = c - 2\kappa(\ell)$. Similarly, we can find a rational number $\delta' \geqq 1$ such that $\delta'(c' - 2\kappa(L')) = c' - 2\kappa(\ell')$. By Theorem 1.2 and Lemma 3.3, we can also find rational numbers $\tilde{\delta}, \tilde{\delta}' > 1$ such that

$\tilde{\delta}(c-2\tilde{\kappa}(L)) = c-2\tilde{\kappa}(\ell)$ and $\tilde{\delta}'(c'-2\tilde{\kappa}(L')) = c'-2\tilde{\kappa}(\ell')$. To complete the proof of Theorem 5.1, it suffices to show that

$$c-2\kappa(\ell) = c'-2\kappa(\ell') \quad \text{and} \quad c-2\tilde{\kappa}(\ell) = c'-2\tilde{\kappa}(\ell').$$

Since $c' = c+2$, it suffices to show the following lemma:

Lemma 5.3. $\beta_1(\ell) = \beta_1(\ell')$ and $\tilde{\beta}_1(\ell) = \tilde{\beta}_1(\ell')$.

Proof of Lemma 5.3. Let X^0 and X^1 be the compact exteriors of the links ℓ and ℓ' in S^3. Let T^0 be the one crossing tangle, and T^1 the lassoed tangle, which are illustrated in Fig. 1. Both T^0 and T^1 are in the same 3-ball V in S^3, whose compact exteriors E^0 and E^1 are assumed to be 3-submanifolds of X^0 and X^1, respectively. It is noted that for the 3-ball $V^c = \mathrm{cl}(S^3 \setminus V)$, the compact proper 1-manifold $T = \ell \cap V^c$ coincides with the compact proper 1-manifold $\ell' \cap V^c$. Let E be compact exterior of T in V^c. Then for $i = 0, 1$, the 3-manifold X^i is obtained from E and E^i by pasting along a compact 4-punctured sphere $S(4) (\subset \partial E)$. Let $\bar{E}$, $\bar{E}^i$ and $\bar{S}(4)$ be the lift of $\bar{E}$, $\bar{E}^i$ and $S(4)$ under the covering $\bar{X}^i \to X^i$, respectively, where it is noted that $\bar{E}$ and $\bar{S}(4)$ are independent of i. Since the fundamental group $\pi_1(S(4), x_0)$ for any base point x_0 is a free group of rank 3, it is noted that $\dim_{Q(\Lambda)} H_1(\bar{S}(4); Q(\Lambda)) = 2$. We also have $\dim_{Q(\Lambda)} \mathrm{Im}(i^{\bar{E}}_{\bar{S}(4)})_* = 1$ for the image $\mathrm{Im}(i^{\bar{E}}_{\bar{S}(4)})_*$ of the natural $Q(\Lambda)$-homomorphism $(i^{\bar{E}}_{\bar{S}(4)})_* : H_1(\bar{S}(4); Q(\Lambda)) \to H_1(\bar{E}; Q(\Lambda))$. In fact, consider the exact sequence

$$H_2(\bar{E}, \partial\bar{E}; Q(\Lambda)) \xrightarrow{\partial_*} H_1(\partial\bar{E}; Q(\Lambda)) \xrightarrow{i_*} H_1(\bar{E}; Q(\Lambda)).$$

The image $\mathrm{Im}\partial_*$ is self-orthogonal with respect to the non-singular $Q(\Lambda)$-intersection form

$$\mathrm{Int}_{\partial\bar{E}} : H_1(\partial\bar{E}; Q(\Lambda)) \times H_1(\partial\bar{E}; Q(\Lambda)) \to Q(\Lambda).$$

In fact, $\mathrm{Int}_{\partial\bar{E}}(\partial_*(x), y) = 0$ for all $x \in H_2(\bar{E}, \partial\bar{E}; Q(\Lambda))$ if and only if $\mathrm{Int}_{\bar{E}}(x, i_*(y)) = 0$ for all $x \in H_2(\bar{E}, \partial\bar{E}; Q(\Lambda))$ with respect to the non-singular $Q(\Lambda)$-intersection form

$$\mathrm{Int}_{\bar{E}} : H_2(\bar{E}, \partial\bar{E}; Q(\Lambda)) \times H_1(\bar{E}; Q(\Lambda)) \to Q(\Lambda).$$

By Blanchfield duality in [1], the last condition is equivalent to $i_*(y) = 0$, namely $y \in \mathrm{Ker} i_* = \mathrm{Im}\partial_*$. Thus, we have

$$\dim_{Q(\Lambda)} H_1(\partial\bar{E}; Q(\Lambda)) = 2 \dim_{Q(\Lambda)} \mathrm{Im}\partial_*.$$

Because $H_1(\partial\bar{E}, \bar{S}(4); Q(\Lambda)) = 0$ by the excision isomorphism, the natural $Q(\Lambda)$-homomorphism $H_1(\bar{S}(4); Q(\Lambda)) \to H_1(\partial\bar{E}; Q(\Lambda))$ is a $Q(\Lambda)$-isomorphism. This means that $\dim_{Q(\Lambda)} \mathrm{Im}(i^{\bar{E}}_{\bar{S}(4)})_* = 1$. Similarly, we have $\dim_{Q(\Lambda)} \mathrm{Im}(i^{\bar{E}^i}_{\bar{S}(4)})_* = 1$ for the natural $Q(\Lambda)$-homomorphism

$$(i^{\bar{E}^i}_{\bar{S}(4)})_* : H_1(\bar{S}(4); Q(\Lambda)) \to H_1(\bar{E}^i; Q(\Lambda)) \, (i = 0, 1).$$

Since the fundamental group $\pi_1(E^0, x_0)$ for any base point x_0 is a free group of rank 2, it is noted that $\dim_{Q(\Lambda)} H_1(\bar{E}^0; Q(\Lambda)) = 1$. It is also shown that $\dim_{Q(\Lambda)} H_1(\bar{E}^1; Q(\Lambda)) = 1$. In fact, a suitable tangle sum of the one crossing tangle T^0 and the lassoed tangle T^1 gives the Borromean rings 6^3_2 (see Example 5.4 (1) later) whose compact exterior X' has $H_1(\bar{X}'; Q(\Lambda)) = 0$. Hence $H_1(\bar{E}^1, \bar{S}(4); Q(\Lambda)) = 0$, so that $\dim_{Q(\Lambda)} H_1(\bar{E}^1; Q(\Lambda)) = 1$. Let m be a simple loop in $S(4)$ which bounds a disk in V separating the two strings of T^0. Let $\bar{m}$ be a connected lift of m in $\bar{S}(4)$, which represents a non-zero element in $H_1(\bar{S}(4); Q(\Lambda)$ and represents zero in $H_1(\bar{E}^0; Q(\Lambda))$. It is also shown that $\bar{m}$ represents zero in $H_1(\bar{E}^1; Q(\Lambda))$. In fact, a suitable tangle sum of the one crossing tangle T^0 and the lassoed tangle T^1 with two copies of m identified gives a split sum of the Hopf link and a trivial knot whose compact exterior X'' has $\dim_{Q(\Lambda)} H_1(\bar{X}''; Q(\Lambda)) = 1$. If $\bar{m}$ represents a non-zero element in $H_1(\bar{E}^1; Q(\Lambda)) = Q(\Lambda)$, then the Mayer-Vietoris sequence on $(\bar{X}''; \bar{E}^0, \bar{E}^1; \bar{S}(4))$ shows that $\dim_{Q(\Lambda)} H_1(\bar{X}''; Q(\Lambda)) = 0$, which is a contradiction. Thus, the simple loop $\bar{m}$ represents zero in $H_1(\bar{E}^1; Q(\Lambda))$. If $\bar{m}$ represents zero in $H_1(\bar{E}; Q(\Lambda))$, then we have

$$\dim_{Q(\Lambda)} H_1(\bar{X}^0; Q(\Lambda)) = \dim_{Q(\Lambda)} H_1(\bar{E}; Q(\Lambda)) = \dim_{Q(\Lambda)} H_1(\bar{X}^1; Q(\Lambda))$$

by the Mayer-Vietoris sequences on $(\bar{X}^0; \bar{E}, \bar{E}^0; \bar{S}(4))$ and $(\bar{X}^1; \bar{E}, \bar{E}^1; \bar{S}(4))$. If $\bar{m}$ represents zero in $H_1(\bar{E}; Q(\Lambda))$, then the Mayer-Vietoris sequences on $(\bar{X}^0; \bar{E}, \bar{E}^0; \bar{S}(4))$ and $(\bar{X}^1; \bar{E}, \bar{E}^1; \bar{S}(4))$ show that

$$\dim_{Q(\Lambda)} H_1(\bar{X}^0; Q(\Lambda)) = \dim_{Q(\Lambda)} H_1(\bar{E}; Q(\Lambda)) - 1 = \dim_{Q(\Lambda)} H_1(\bar{X}^1; Q(\Lambda)).$$

Therefore, we have $\beta_1(\ell) = \beta_1(\ell')$. By an analogous infinite cyclic version argument of this argument, we also have $\tilde{\beta}_1(\ell) = \tilde{\beta}_1(\ell')$. □

This completes the proof of Theorem 5.1. □

Here are examples of immersed ribbon S^2-links associated with an unlinking operation on the links ℓ^0 illustrated in Fig. 2 together with the notation of "linkinfo"[1] in the bracket, whose Alexander polynomials are given in [20] and whose unlinking number information is in [14].

[1]http://www.indiana.edu/ linkinfo/.

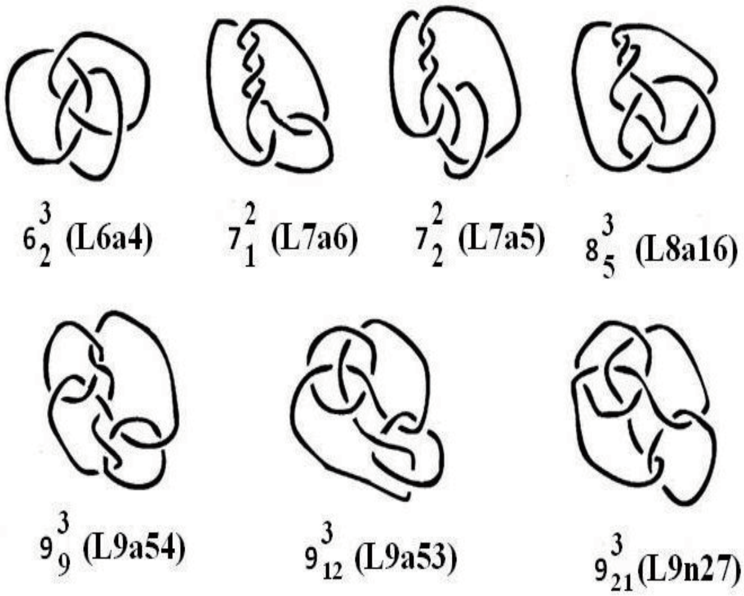

Figure 2: Links used in Example 5.4

Example 5.4. (1) Let $\ell^0 = 6^3_2$ (L6a4) (the Borromean rings) which is a lassoed link of the Hopf link H with $\Delta(H) \doteq 1$. Note that $\kappa(H) = \tilde{\kappa}(H) = 1$. Then $\beta_1(\ell^0) = \tilde{\beta}_1(\ell^0) = 0$ and $u(\ell^0) = \kappa(\ell^0) = \tilde{\kappa}(\ell^0) = 2$ for any orientation of ℓ^0. By Theorem 5.1, the immersed ribbon S^2-link $L = L_{\ell^0}$ has the identities $2\tilde{\kappa}(L) = 2\kappa(L) = c = 4$. Further, for every symmetric middle cross-section ℓ of L, $2\tilde{\kappa}(\ell) = 2\kappa(\ell) = c = 4$. Since $\Delta(\ell^0) \doteq (t_1 - 1)(t_2 - 1)(t_3 - 1) \doteqdot 1$ and $\tilde{\Delta}(\ell^0) \doteq (t-1)^4 \doteqdot 1$, we have

$$\Delta(\ell) \doteqdot 1 \quad \text{and} \quad \tilde{\Delta}(\ell) \doteqdot 1$$

by Theorems 1.1 (4) and 1.2 (4). Further, by Theorem 1.2 (2), Lemma 3.5 and the signs of the unlinking operation, $s_a(\ell) = 0$ for every $a \in [-1, 1)$ and $s_1(\ell) = -\xi^+ = 0, \pm 2$ whereas $s_a(\ell^0) = 0$ for every $a \in [-1, 1]$.

(2) Let $\ell^0 = 7^2_1$ (L7a6). Then $\beta_1(\ell^0) = \tilde{\beta}_1(\ell^0) = 0$ and $\kappa(\ell^0) = \tilde{\kappa}(\ell^0) = 1$ for any orientation of ℓ^0 and $u(\ell^0) \leqq 2$. The Alexander polynomial

$$\begin{aligned} \Delta(\ell^0) &\doteq 1 - t_1 - t_2 + (1 - t_1 - t_2)t_1 t_2 + (t_1 t_2)^2, \\ \tilde{\Delta}(\ell^0) &\doteq (1-t)(1 - 2t - 2t^{-1} + t^2 + t^{-2}) \quad \text{or} \quad (1-t)(3 - 2t - 2t^{-1}) \end{aligned}$$

cannot be written as ff^* up to multiplications of units of Λ and $t_i - 1\,(i = 1, 2)$ or up to multiplications of units of $\tilde{\Lambda}$ and $t - 1$, respectively. Hence by Theorem 1.1 we obtain

$$2\tilde{\kappa}(\ell^0) = 2\kappa(\ell^0) = 2 < c = 2u(\ell^0) = 4.$$

By Theorem 5.1, $\kappa(L) = \tilde{\kappa}(L) = 1$. By Theorem 1.2 (2), for every symmetric middle cross-section ℓ of the immersed ribbon S^2-link $L = L_{\ell^0}$, we have

$$\tilde{\kappa}(\ell) = \kappa(\ell) = \tilde{\kappa}(L) = \kappa(L) = 1,$$

so that $|s_a(\ell) + \xi^+| \leqq 1$ for every $a \in (-1, 1)$.

(3) Let $\ell^0 = 7^2_2$ (L7a5). Then we have $\beta_1(\ell^0) = \tilde{\beta}_1(\ell^0) = 0$ and $u(\ell^0) = \kappa(\ell^0) = \tilde{\kappa}(\ell^0) = 1$ for any orientation of ℓ^0. By Theorem 5.1, the immersed ribbon S^2-link $L = L_{\ell^0}$ has the identities $2\tilde{\kappa}(L) = 2\kappa(L) = c = 2$. Further, for every symmetric middle cross-section ℓ of L, $2\tilde{\kappa}(\ell) = 2\kappa(\ell) = c = 2$. Since $\Delta(\ell^0) \doteq (t_1 + t_2^{-1} - 1)(t_1^{-1} + t_2 - 1)$ and $\tilde{\Delta}(\ell^0) \doteq (t-1)(t + t^{-1} - 1)^2$ or $(t-1)(2t-1)(2t^{-1} - 1)$, we have

$$\Delta(\ell) \doteqdot (t_1 + t_2^{-1} - 1)(t_1^{-1} + t_2 - 1), \quad \tilde{\Delta}(\ell) \doteqdot (t + t^{-1} - 1)^2 \text{ or } (2t-1)(2t^{-1} - 1)$$

by Theorems 1.1 (4) and 1.2 (4). Further, by Theorem 1.2 (2) and Lemma 3.5, $s_a(\ell) = 0\, (a \in [-1, 1))$ and $s_1(\ell) = \pm 1$.

(4) Let $\ell^0 = 8^3_5$ (L8a16) which is a lassoed link of the (unoriented) torus link $T(2,4)$ of type $(2,4)$ with $\Delta(T(2,4)) \doteq t_1t_2 + 1$. Note that $\kappa(T(2,4)) = \tilde{\kappa}(T(2,4)) = 1$. Then $\beta_1(\ell^0) = \tilde{\beta}_1(\ell^0) = 0$ and $\kappa(\ell^0) = \tilde{\kappa}(\ell^0) = 2$ for any orientation of ℓ^0 and $u(\ell^0) \leqq 3$. The Alexander polynomial

$$\begin{aligned} \Delta(\ell^0) &\doteq (t_1 - 1)(t_2 - 1)(t_3 - 1)(t_1t_2 + 1), \\ \tilde{\Delta}(\ell^0) &\doteq (1-t)^4(t^2 + 1) \quad \text{or} \quad 2(1-t)^4 \end{aligned}$$

cannot be written as ff^* up to multiplications of units of Λ and $t_i - 1\, (i = 1, 2)$ or up to multiplications of units of $\tilde{\Lambda}$ and $t - 1$, respectively. Hence by Theorem 1.1 we obtain

$$2\kappa(\ell^0) = 2\tilde{\kappa}(\ell^0) = 4 < c = 2u(\ell^0) = 6.$$

By Theorem 5.1, $\kappa(L) = \tilde{\kappa}(L) = 2$. By Theorem 1.2 (2), for every symmetric middle cross-section ℓ of the immersed ribbon S^2-link $L = L_{\ell^0}$, we have

$$\tilde{\kappa}(\ell) = \kappa(\ell) = \tilde{\kappa}(L) = \kappa(L) = 2,$$

so that $|s_a(\ell) + \xi^+| \leqq 1$ for every $a \in (-1, 1)$.

(5) Let $\ell^0 = 9^3_9$ (L9a54). Then $\beta_1(\ell^0) = \tilde{\beta}_1(\ell^0) = 0$ and $\kappa(\ell^0) = \tilde{\kappa}(\ell^0) = 2$ for any orientation of ℓ^0 and $u(\ell^0) \leqq 3$. The Alexander polynomial

$$\begin{aligned} \Delta(\ell^0) &\doteq (t_1 - 1)(t_2 - 1)(t_3 - 1)(t_1^2 - t_1 + 1), \\ \tilde{\Delta}(\ell^0) &\doteq (1-t)^4(t^2 - t + 1) \end{aligned}$$

cannot be written as ff^* up to multiplications of units of Λ and $t_i - 1\,(i = 1, 2)$ or up to multiplications of units of $\tilde{\Lambda}$ and $t - 1$, respectively. Hence by Theorem 1.1 we obtain

$$2\kappa(\ell^0) = 2\tilde{\kappa}(\ell^0) = 4 < c = 2u(\ell^0) = 6.$$

By Theorem 5.1, $\kappa(L) = \tilde{\kappa}(L) = 2$. By Theorem 1.2 (2), for every symmetric middle cross-section ℓ of the immersed ribbon S^2-link $L = L_{\ell^0}$, we have

$$\tilde{\kappa}(\ell) = \kappa(\ell) = \tilde{\kappa}(L) = \kappa(L) = 2,$$

so that $|s_a(\ell) + \xi^+| \leqq 1$ for every $a \in (-1, 1)$.

(6) $\ell^0 = 9^3_{12}$ (L9a53) which is a lassoed link of the 2-component 4-crossing link 5^2_1 with $\Delta(5^2_1) \doteq (t_1 - 1)(t_2 - 1)$. Note that $\kappa(5^2_1) = \tilde{\kappa}(5^2_1) = 1$. Then $\beta_1(\ell^0) = \tilde{\beta}_1(\ell^0) = 0$ and $u(\ell^0) = \kappa(\ell^0) = \tilde{\kappa}(\ell^0) = 2$ for any orientation of ℓ^0. By Theorem 5.1, the immersed ribbon S^2-link $L = L_{\ell^0}$ has the identities $2\tilde{\kappa}(L) = 2\kappa(L) = c = 4$. Further, for every symmetric middle cross-section ℓ of L, $2\tilde{\kappa}(\ell) = 2\kappa(\ell) = c = 4$. Since $\Delta(\ell^0) \doteq (t_1 - 1)(t_2 - 1)(t_3 - 1)(t_1 - 1)(t_1^{-1} - 1)$ and $\tilde{\Delta}(\ell^0) \doteq (t - 1)^6$, we have

$$\Delta(\ell) \doteqdot 1 \quad \text{and} \quad \tilde{\Delta}(\ell) \doteqdot 1$$

by Theorems 1.1 (4) and 1.2 (4). Further, $s_a(\ell) = 0\,(a \in [-1, 1))$ and $s_1(\ell) = -\xi^+ = 0, \pm 2$ by Theorem 1.2 (2) and Lemma 3.5 although $s_a(\ell^0) = 0\,(a \in [-1, 1])$ by the signs of the unlinking operation.

(7) Let $\ell^0 = 9^3_{21}$ (L9n27) which is a lassoed link of the 2-component trivial link o^2 with $\Delta^T(o^2) \doteq 1$. Note that $\kappa(o^2) = \tilde{\kappa}(o^2) = 0$. Then $\beta_1(\ell^0) = \tilde{\beta}_1(\ell^0) = 1$ and $u(\ell^0) = \kappa(\ell^0) = \tilde{\kappa}(\ell^0) = 1$ for any orientation of ℓ^0. By Theorem 5.1, the immersed ribbon S^2-link $L = L_{\ell^0}$ has the identities $2\tilde{\kappa}(L) = 2\kappa(L) = c = 2$. Further, for every symmetric middle cross-section ℓ of L, $2\tilde{\kappa}(\ell) = 2\kappa(\ell) = c = 2$. Since $\Delta^T(\ell^0) \doteq t_1 - 1$ and $\tilde{\Delta}^T(\ell^0) \doteq (t - 1)^3$, we have

$$\Delta^T(\ell) \doteqdot 1 \quad \text{and} \quad \tilde{\Delta}^T(\ell) \doteqdot 1$$

by Theorems 1.1 (4) and 1.2 (4). Further, by Theorem 1.2 (2) and Lemma 3.5, $s_a(\ell) = 0$ for every $a \in [-1, 1)$ and $s_1(\ell) = \pm 1$.

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