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Descriptions on Surfaces in Four-Space, I

Normal Forms

Dedicated to Professor Hidetaka TERASAKA
in Commemoration of His Seventy-Seventh Anniversary

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The pivotal purpose of the notes is to understand the general configuration of a closed oriented surface F piecewise-linearly and locally-flatly embedded in the oriented euclidean 4-space R^4 up to ambient isotopies in R^4. The method which we shall adopt to describe the configuration of F in R^4 is usually called the *motion picture method*. (See Fox [1].) Roughly speaking, this method is to cut F by the parallel hyperplanes $R^3[t] = R^3 \times t \subset R^3 \times R^1 = R^4$, $-\infty < t < +\infty$, for example, after having deformed F into a suitable form that we will call a *normal form* so as to become easy to conceive the configuration of F in R^4.

The first attempt of describing the configuration of F in R^4 by the motion picture method was made by R.H.Fox and J.W.Milnor in their unpublished paper [2] in a somewhat unsatisfactory form in 1957.

The most important concept of the motion picture method is the hyperbolic transformation for links that corresponds to the concept of a hyperbolic critical point of a surface in the Morse Theory, which will be discussed in Section 1. In particular, the normal form of a surface in R^4

will be stated as the *closed realizing surface* of a sequence of *fusions* and *fissions* that are the special types of the hyperbolic transformations for links, originally introduced by F.Hosokawa [5] in 1967 in a restricted form.

In Section 2, we will show that any closed surface F can be deformed into a normal form by an ambient isotopy of R^4. In the case of 2-spheres, this responds to a question of R.H.Fox [1, p.134]. Section 3 will be devoted to a normalization of cobordism surfaces between links.

Throughout the notes, spaces and maps will be considered in the piece-wise linear category. (Refer to Hudson [7], Rourke-Sanderson [10], etc.)

These notes are based on lectures by the first author in the topology seminar at the Kobe University in 1975. The authors would like to gratefully acknowledge a number of valuable advices from Professor Fujitsugu Hosokawa, and are grateful to Dr.Takeshi Kaneto for having read the first manuscript.

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0. Notation and Definitions

0.1. We denote by ∂X, $\mathrm{Int}(X)$ and $\mathrm{Cl}(X)$, respectively, the boundary, the interior and the closure of a manifold X. We say that a submanifold X of a manifold Y is *properly embedded* (or *proper*) if $X \cap \partial Y = \partial X$.

The following notation is fixed throughout the note :

R : the set of all real numbers,

$I = \{x \in \mathrm{R} \mid 0 \leq x \leq 1\}$: the closed unit interval,

$R^n = \{(x_1, x_2, \cdots, x_n) \mid x_i \in \mathrm{R}\}$: the euclidean n-space,

$R^n_+ = \{(x_1, x_2, \cdots, x_n) \in R^n \mid x_n \geq 0\}$,

$R^n_- = \{(x_1, x_2, \cdots, x_n) \in R^n \mid x_n \leq 0\}$,

$D^n = \{(x_1, x_2, \cdots, x_n) \in R^n \mid (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} \leq 1\}$: the n-disk,

$S^{n-1} = \{(x_1, x_2, \cdots, x_n) \in R^n \mid (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2} = 1\}$: the $(n-1)$-sphere,

$S^{n-1}_+ = \{(x_1, x_2, \cdots, x_n) \in S^{n-1} \mid x_n \geq 0\}$,

$S^{n-1}_- = \{(x_1, x_2, \cdots, x_n) \in S^{n-1} \mid x_n \leq 0\}$.

We always assume that R^n, D^n and S^{n-1} have the standard piecewise linear structures which are compatible with the affine structures, and we identify R^n with the subspace of R^{n+m} having all components after the n^{th} equal to 0.

0.2. For a subset A of R^3 and an interval J of R^1, we denote by AJ the subset $\{(x,t) \in R^4 \mid x \in A,\ t \in J\}$ of $R^4 = R^3 \times R^1$. If J consists of one point t_0, then the notation $A[t_0] = A \times t_0$ will be used.

0.3. Definitions. Let X and Y be topological spaces.

(1) Two homeomorphisms (or embeddings) $f, g : X \to Y$ are *isotopic* iff there exists a homeomorphism (or embedding) $H : X \times I \to Y \times I$ such that

(i) level preserving (that is, $H(x,t) = (h_t(x), t)$, where $h_t : X \to Y$ is a homeomorphism (or embedding) for all $t \in I$), and

(ii) $h_0 = f$ and $h_1 = g$.

(2) An *isotopic deformation* of Y is a homeomorphism $H : Y \times I \to Y \times I$ such that

(i) level preserving (that is, $H(y,t) = (h_t(y), t)$, where h_t :

$Y \to Y$ is a homeomorphism), and

(ii) starts with the identity ; $h_0 = 1_Y$.

(3) Two embeddings $f, g : X \to Y$ are *ambient isotopic* iff there exists an isotopic deformation $H : Y \times I \to Y \times I$ of Y with $h_1 f = g$.

(4) Two subspaces X_1 and X_2 of Y are *ambient isotopic* iff there exists an isotopic deformation $H : Y \times I \to Y \times I$ of Y with $h_1(X_1) = X_2$.

In the cases (3) and (4), we will call the isotopic deformation H or $\{h_t\}_{t \in I}$ the *ambient isotopy* of Y between f and g (resp. X_1 and X_2).

Throughout the note, an ambient isotopy of a space will mean an isotopy *with compact support*, unless otherwise stated.

0.4. Definitions. (1) A subspace $\ell = k_1 \cup \cdots \cup k_\mu$ in R^3 (or S^3) is an (oriented) ***link*** *with* μ *components* iff ℓ is homeomorphic with a disjoint union $S^1 \cup \cdots \cup S^1$ of μ (oriented) 1-spheres. An (oriented) link with 1 component is called an (oriented) *knot*.

(2) Two (oriented) knots or links ℓ, ℓ' are *equivalent* (or *of the same type*) iff there exists an (orientation preserving) homeomorphism $\psi : R^3 \to R^3$ (or $S^3 \to S^3$) such that $\psi(\ell) = \ell'$ (and $\psi|_\ell$ is also orientation preserving). The equivalence class of a knot or link is called its *knot type* or *link type*.

(3) An (oriented) link $\ell = k_1 \cup \cdots \cup k_\mu$ in R^3 (or S^3) is called *trivial* (or *unknotted*) iff there exists a disjoint union $D_1^2 \cup \cdots \cup D_\mu^2$ of 2-disks in R^3 (or S^3) with $\partial D_i^2 = k_i$, $i=1,\cdots,\mu$. Trivial links constitute the *trivial type*.

(4) A link ℓ is said to be *splittable* iff there is a 2-sphere $S^2 \subset R^3 - \ell$ such that both components of $R^3 - S^2$ contain points of ℓ. More precisely, we say that ℓ is *splittable* into λ sublinks $\ell_1, \cdots, \ell_\lambda$,

iff there are λ disjoint 3-disks $D_1^3 \cup \cdots \cup D_\lambda^3$ in R^3 such that $\mathrm{Int}(D_i^3) \supset \ell_i$ for $i=1,\cdots,\lambda$; and ℓ is denoted by $\ell = \ell_1 \circ \cdots \circ \ell_\lambda$. If ℓ_i is a knot for all i, $\ell = k_1 \circ \cdots \circ k_\mu$ is said to be *completely splittable*.

Other terminology in Knot Theory is refered to, for example, Rolfsen [9] and Suzuki [11].

0.5. Definition. For a subcomplex P of a manifold M, by $N(P\,;M)$ we denote a regular neighborhood of P in M, that is, we construct its second derived and take the closed star of P.

Let F be a proper 2-manifold in a 4-manifold M. For a point $x \in F$, we have a knot $\partial N(x\,;F)$ in the 3-sphere $\partial N(x\,;M)$. The knot type $\kappa(x)$ of this knot $\partial N(x\,;F)$ in $\partial N(x\,;M)$ is called the *singularity* or *local knot type* of F at x. When $\kappa(x)$ is of trivial type, we may say that F is *locally flat* at x. We say F is *locally flat* in M iff it is locally flat at each point. When $\kappa(x)$ is of non-trivial type, we may say that F is *locally knotted* at x. It should be noted that the locally knotted points occur only at the vertices of F.

For the local knots, we refer the reader to Fox-Milnor [3].

0.6. Throughout the note, by a *surface* we mean a compact, oriented (and connected or not) 2-dimensional manifold F, with boundary which may be empty, *locally flatly* embedded in the oriented R^4.

1. Hyperbolic Transformations and the Realizing Surfaces in R^4

Let ℓ be an oriented link in the oriented R^3. An oriented band (= 2-disk) B in R^3 is said to *span the link* ℓ *by the attaching arcs* $\{\alpha\,,\alpha'\}$, if α and α' are disjoint connected arcs on ℓ such that

$B \cap \ell = \partial B \cap \ell = \alpha \cup \alpha'$ and if the link $\mathrm{Cl}(\ell \cup \partial B - (\alpha \cup \alpha'))$ has the orientation compatible with that of $\ell - (\alpha \cup \alpha')$ and $\partial B - (\alpha \cup \alpha')$, see Fig. 1.

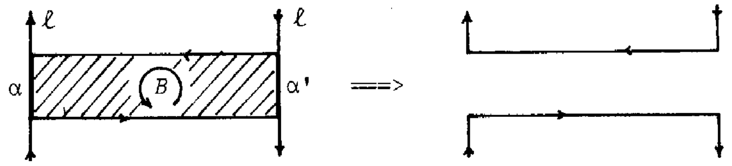


Fig. 1

In general, let $B_1, \cdots, B_m$ be mutually disjoint oriented bands in R^3 such that each B_i spans ℓ by the attaching arcs $\{\alpha_i, \alpha_i'\}$.

1.1. Definition. The oriented link

$$h(\ell\,;\,B_1,\cdots,B_m) = \mathrm{Cl}(\ell \cup \partial B_1 \cup \cdots \cup \partial B_m - (\alpha_1 \cup \alpha_1' \cup \cdots \cup \alpha_m \cup \alpha_m'))$$

is called the *link obtained from* ℓ *by the hyperbolic transformations along the bands* $B_1, \cdots, B_m$.

Consider the sequence $\ell_0 \to \ell_1 \to \cdots \to \ell_n$ of oriented links such that ℓ_{i+1} is obtained from ℓ_i by the hyperbolic transformations along the bands $B^i = \{B_1^i, \cdots, B_{m_i}^i\}$; $\ell_{i+1} = h(\ell_i\,;\,B^i)$ for $i=0,1,\cdots,n-1$.

For a closed interval $[a\,,b]$, let $a = t_0 < t_1 < \cdots < t_n = b$ be the finite subdivision with $t_{i+1} - t_i = |a-b|/n$, $i=0,1,\cdots,n-1$. Now, we construct a proper surface $F_{t_i}^{t_{i+1}} = F_{t_i}^{t_{i+1}}(\ell_i\,,\ell_{i+1}\,;\,B^i)$ in $R^3[t_i\,,t_{i+1}]$ as follows :

$$F_{t_i}^{t_{i+1}} \cap R^3[t] = \begin{cases} \ell_i[t] & \text{for } t_i \le t < (t_i + t_{i+1})/2, \\ (\ell_i \cup B_1^i \cup \cdots \cup B_{m_i}^i)[t] & \text{for } t = (t_i + t_{i+1})/2, \\ \ell_{i+1}[t] & \text{for } (t_i + t_{i+1})/2 < t \le t_{i+1}. \end{cases}$$

Clearly, $F_{t_i}^{t_{i+1}}$ is a locally flat orientable surface in $R^3[t_i\,,t_{i+1}]$.

We will take such orientations on $F_{t_i}^{t_{i+1}}$ and $R^3[t_i, t_{i+1}]$ as induce the orientations of $\ell_{i+1}[t_{i+1}]$ and $R^3[t_{i+1}]$, respectively, under the identification $(\ell_{i+1}[t_{i+1}] \subset R^3[t_{i+1}]) \equiv (\ell_{i+1} \subset R^3)$.

Then the union $F_a^b = F_{t_0}^{t_1} \cup F_{t_1}^{t_2} \cup \cdots \cup F_{t_{n-1}}^{t_n}$ is a locally flat, proper oriented surface in the oriented $R^3[a, b]$.

1.2. **Definition.** The oriented surface $F_a^b = F_a^b(\ell_0, \ell_1, \cdots, \ell_n; B^0, \cdots, B^{n-1})$ is called the *realizing surface in the oriented* $R^3[a, b]$ of the sequence $\ell_0 \to \ell_1 \to \cdots \to \ell_n$ of links.

It is obvious that, given a sequence $\ell_0 \to \ell_1 \to \cdots \to \ell_n$ of hyperbolic transformations, the realizing surface F_a^b in the oriented $R^3[a, b]$ is uniquely constructed.

Now suppose further ℓ_0 and ℓ_n are trivial links with μ_0 and μ_n components, respectively. Then there exist mutually disjoint μ_0 2-disks $D_1, \cdots, D_{\mu_0}$ in R^3 with $\partial(D_1 \cup \cdots \cup D_{\mu_0}) = \ell_0$ and mutually disjoint μ_n 2-disks $D'_1, \cdots, D'_{\mu_n}$ in R^3 with $\partial(D'_1 \cup \cdots \cup D'_{\mu_n}) = \ell_n$.

We define a closed oriented surface $\overline{F}_a^b$ in $R^3[a, b]$ as

$$\overline{F}_a^b = F_a^b \cup (D_1 \cup \cdots \cup D_{\mu_0})[a] \cup (D'_1 \cup \cdots \cup D'_{\mu_n})[b],$$

so that its orientation is coherent with that of oriented F_a^b.

1.3. **Definition.** The oriented closed surface $\overline{F}_a^b$ is called the *closed realizing surface in the oriented* $R^3[a, b]$ of the sequence $\ell_0 \to \ell_1 \to \cdots \to \ell_n$ with trivial links ℓ_0, ℓ_n.

It should be noted that the closed realizing surface $\overline{F}_a^b$ is not in general uniquely determined up to isotopies of $R^3[a, b]$. (That is, $\overline{F}_a^b$ depends upon the choices of 2-disks $D_1, \cdots, D_{\mu_0}, D_1', \cdots, D_{\mu_n}'$.)

1.4. Example. Let D_1, D_1' be 2-disks in R^3 such that $D_1 \cap D_1' = \partial D_1 = \partial D_1' = C_1$ is a 1-sphere. Since $D_1 \cup D_1'$ is a 2-sphere, there exists a 3-disk B^3 in R^3 whose boundary is $D_1 \cup D_1'$. Let D_2 be a 2-disk in the interior of B^3, and let $\partial D_2 = C_2$. Now we consider the following closed surfaces in $R^3[0, 1]$, as shown in Fig. 2.

$$\overline{F}_0^1 = (D_1[0] \cup C_1[0, 1] \cup D_1[1]) \cup (D_2[0] \cup C_2[0, 1] \cup D_2[1]),$$

$$\overline{F}'^1_0 = (D_1[0] \cup C_1[0, 1] \cup D_1'[1]) \cup (D_2[0] \cup C_2[0, 1] \cup D_2[1]).$$

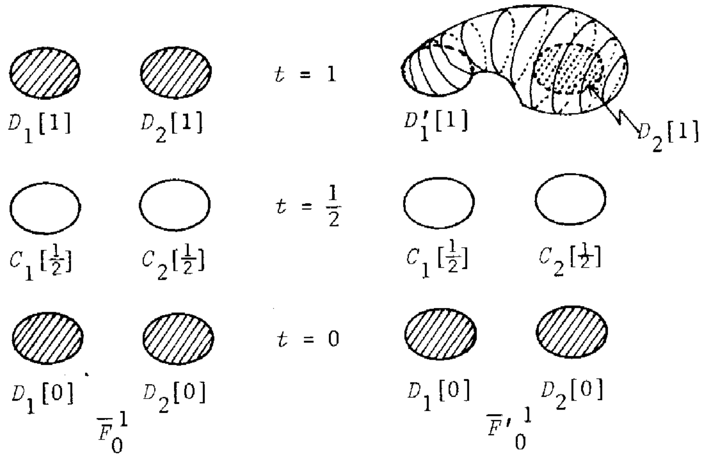


Fig. 2

It is impossible to carry $\overline{F}_0^1$ onto $\overline{F}'^1_0$ by isotopies of $R^3[0, 1]$. In fact, $\overline{F}_0^1$ bounds disjoint two 3-disks $D_1[0, 1] \cup D_2[0, 1]$ in $R^3[0, 1]$, but $\overline{F}'^1_0$ never bounds disjoint two 3-disks in $R^3[0, 1]$. To see this, let $p : R^3[0, 1] - C_2[0, 1] \to R^3[1] - C_2[1]$ be the natural projection. The induced homomorphism $p_* : H_2(R^3[0, 1] - C_2[0, 1]; Z) \to H_2(R^3[1] - C_2[1]; Z)$ sends the homology class represented by the 2-sphere $D_1[0] \cup C_1[0, 1]$

$\cup D_1'[1]$ to the homology class represented by the 2-sphere $(D_1 \cup D_1')[1]$. Since $(D_1 \cup D_1')[1]$ represents a non-zero homology class in $H_2(R^3[1] - C_2[1]\,;Z)$, we obtain that the 2-sphere $D_1[0] \cup C_1[0,1] \cup D_1'[1]$ is not homologous to zero in $R^3[0,1] - C_2[0,1]$. In particular, $D_1[0] \cup C_1[0,1] \cup D_1'[1]$ does not bound any 3-disk in $R^3[0,1] - (D_2[0] \cup C_2[0,1] \cup D_2[1]) \subset R^3[0,1] - C_2[0,1]$. This implies that $\overline{F}'^{\,1}_0$ does not bound disjoint two 3-disks in $R^3[0,1]$, as desired.

However, we can obtain the following :

1.5. Lemma. *The closed realizing surface* $\overline{F}^{\,b}_a$ *is uniquely determined up to isotopies of* $R^3(-\infty,+\infty) = R^4$ *keeping* $R^3[a+\varepsilon, b-\varepsilon]$ *fixed for a sufficiently small positive number* ε.

This lemma is mainly based on the following Horibe-Yanagawa's lemma , which will be found in the Horibe's master thesis [4].

1.6. Lemma (Horibe and Yanagawa). *Let* $S_1, \cdots, S_n$ *be mutually disjoint 2-spheres in* $R^3[0,1]$ *defined by* $S_i = D_i[0] \cup (\partial D_i)[0,1] \cup D_i'[1]$, $i=1,\cdots,n$, *where* D_i *and* D_i' *are 2-disks in* R^3 *with* $\partial D_i = \partial D_i'$. *Then we can find mutually disjoint 3-disks* $B_1, \cdots, B_n$ *in* $R^3[0,+\infty)$ *such that* $\partial B_i = S_i$, $i=1,\cdots,n$.

To prove Lemmas 1.5 and 1.6 precisely, it seems necessary to notice the following Cellular Move Lemma obtained from a general theory of the piecewise-linear topology.

Consider two locally flat n-manifolds M_1, M_2 in a q-manifold Q with $q > n$. We say that M_1 *and* M_2 *differ by an* $(n+1)$*-disk* D, iff $D \subset \mathrm{Int}(Q)$ so that $\mathrm{Cl}(M_1 \cup M_2 - (M_1 \cap M_2)) = \partial D$ and $D \cap M_i = \partial D \cap M_i$ is an

n-disk (i=1, 2); see Fig. 3.

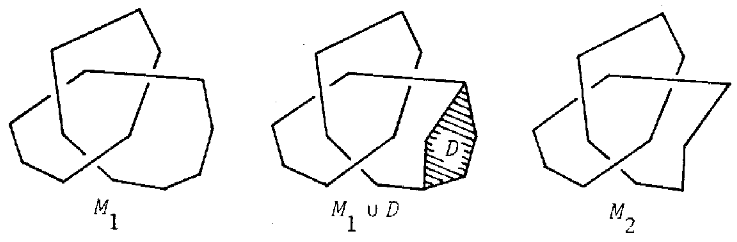


Fig. 3

1.7. **Proposition.** (Cellular Move Lemma). *Let* $M_1^n, M_2^n \subset Q$ *differ by an* $(n+1)$*-disk* D. *There exists an ambient isotopy of* Q *carrying* M_1^n *onto* M_2^n *and keeping* $Q - N(D\,;Q)$ *fixed.* (For the proof, see for example Rourke-Sanderson [10, p.55].)

1.8. **Proof of Lemma 1.5.** Let $\overline{F}_a^b$, $\overline{F}'{}_a^b \subset R^3[a\,,b]$ be two closed realizing surfaces obtained from the same realizing surface $F_a^b \subset R^3[a\,,b]$. Given a small positive number ε, let ε' be a number with $0 < \varepsilon' < \varepsilon$. Consider a closed realizing surface $\overline{F}_{a+\varepsilon'}^{b-\varepsilon'} \subset R^3[a+\varepsilon'\,,b-\varepsilon']$ obtained from $F_a^b \cap R^3[a+\varepsilon'\,,b-\varepsilon']$ by attaching suitable 2-disks in $R^3[a+\varepsilon']$ and $R^3[b-\varepsilon']$. Then $\mathrm{Cl}(\overline{F}_a^b \cup \overline{F}_{a+\varepsilon'}^{b-\varepsilon'} - (\overline{F}_a^b \cap \overline{F}_{a+\varepsilon'}^{b-\varepsilon'}))$ consists of disjoint 2-spheres contained in either $R^3[b-\varepsilon'\,,b]$ or $R^3[a\,,a+\varepsilon']$. The situations of the 2-spheres in $R^3[b-\varepsilon'\,,b]$ or $R^3[a\,,a+\varepsilon']$ are the same as in Horibe and Yanagawa's Lemma (Lemma 1.6), although the interval $[0\,,1]$ has been replaced by $[b-\varepsilon'\,,b]$ or $[a\,,a+\varepsilon']$. Hence there exist disjoint 3-disks in $R^3[b-\varepsilon'\,,+\infty)$ or $R^3(-\infty\,,a+\varepsilon']$ whose boundaries are all of the above 2-spheres. Then the Cellular Move Lemma implies that $\overline{F}_a^b$ is isotopic to $\overline{F}_{a+\varepsilon'}^{b-\varepsilon'}$ by an ambient isotopy of $R^3(-\infty\,,+\infty)$ keeping $R^3[a+\varepsilon\,,b-\varepsilon]$ fixed. We apply a similar argument to $\overline{F}_a^b$ and $\overline{F}_{a+\varepsilon'}^{b-\varepsilon'}$.

Therefore $\overline{F}_a^b$ is ambient isotopic to $\overline{F'}_a^b$ by an ambient isotopy of $R^3(-\infty, +\infty)$ keeping $R^3[a+\varepsilon, b-\varepsilon]$ fixed. This completes the proof. □

1.9. Proof of Lemma 1.6. It suffices to show that there exists a 3-disk $B_n \subset R^3[0, +\infty)$ such that $\partial B_n = S_n$ and $B_n \cap (S_1 \cup \cdots \cup S_{n-1}) = \emptyset$. [In fact, the Cellular Move Lemma, then, assures that the union $S_1 \cup \cdots \cup S_{n-1} \cup S_n$ is ambient isotopic to the union $S_1 \cup \cdots \cup S_{n-1} \cup \partial(D_n[0, \varepsilon])$ for a sufficiently small positive number ε. Next, appeal to the induction on the number n of connected components.]

Consider D_n and $D'_1, \cdots, D'_n$. Note that $D'_1, \cdots, D'_n$ are mutually disjoint and $(\partial D'_1 \cup \cdots \cup \partial D'_{n-1}) \cap D_n = \emptyset$ and $\partial D'_n = \partial D_n$.

By a transversality argument, there is a sufficiently small isotopic deformation $\{h_t\}_{t \in I}$ of R^3 keeping ∂D_n fixed with $h_t(D_n) \cap \partial D'_i = \emptyset$, $i=1,\cdots,n-1$, so that the 2-disk $\tilde{D}_n = h_1(D_n)$ intersects the union $D'_1 \cup \cdots \cup D'_n$ transversally. Now, $D'_i \cap \tilde{D}_n$ consists of only finite number of simple loops for $i=1,\cdots,n-1$, and $D'_n \cap \tilde{D}_n$ consists of simple loops and proper simple arcs, see Fig. 4.

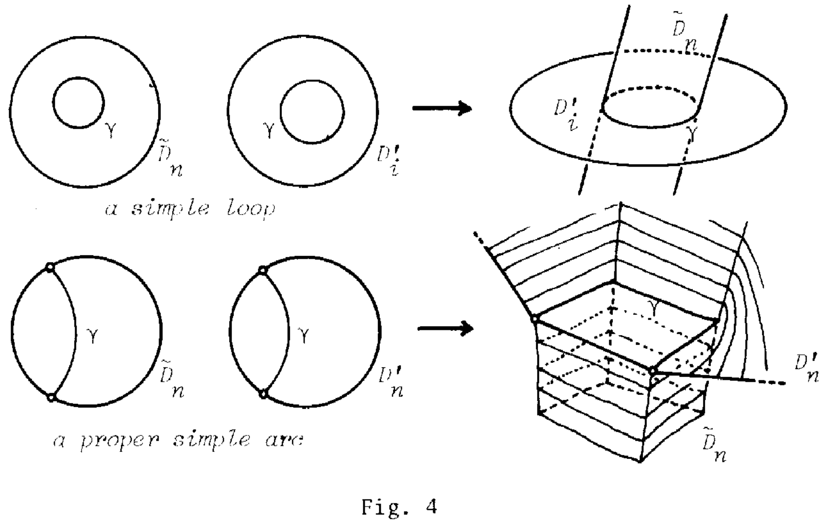


Fig. 4

Let ν be the number of the components of $(D'_1 \cup \cdots \cup D'_n) \cap \tilde{D}_n$.

Let $S'_n = D_n[0] \cup (\partial D_n)[0, 2] \cup D'_n[2]$. Since $S_1 \cup \cdots \cup S_n$ is ambient isotopic to $S_1 \cup \cdots \cup S_{n-1} \cup S'_n$, it suffices to constructs a 3-disk $B'_n \subset R^3[0, +\infty)$ with $\partial B'_n = S'_n$ and $B'_n \cap (S_1 \cup \cdots \cup S_{n-1}) = \emptyset$. We shall construct B'_n by specifying the cross-sections $B'_n \cap R^3[t]$.

Divide the interval $[0, 1]$ into the subintervals $[0, \varepsilon_1]$, $[\varepsilon_1, \varepsilon_2]$, $\cdots$, $[\varepsilon_\nu, 1]$, where $\varepsilon_i = i/(\nu+1)$, $i=1,\cdots,\nu$.

For any t with $0 \le t \le \varepsilon_1$, we define $B'_n \cap R^3[t] = h_{t/\varepsilon_1}(D_n)[t]$. Thus, the part $B'_n \cap R^3[0, \varepsilon_1]$ is constructed. It will be noticed that $B'_n \cap R^3[\varepsilon_1] = \tilde{D}_n[\varepsilon_1]$, and each $D'_i[\varepsilon_1]$, $i=1,\cdots,n$, has only the simple loops or simple arcs of $((D'_1 \cup \cdots \cup D'_n) \cap \tilde{D}_n)[\varepsilon_1]$ as the intersection with $\tilde{D}_n[\varepsilon_1]$. Let $\gamma \subset (D'_1 \cup \cdots \cup D'_n) \cap \tilde{D}_n$ be an innermost loop or arc on some D'_i, and let $\Delta \subset D'_i$ be the 2-disk cut off by γ with $\mathrm{Int}\Delta \cap \tilde{D}_n = \emptyset$. We perform the orientation-preserving cut on $\tilde{D}_n$ along this Δ. By this modification, $\tilde{D}_n$ is divided into one 2-disk $\tilde{D}_n^{(1)}$ and one 2-sphere Σ, see Fig. 5. Now, $B'_n \cap R^3[\varepsilon_1, \varepsilon_2]$ is roughly defined by the realization of this modification into $R^3[\varepsilon_1, \varepsilon_2]$. To be precise, let

$B'_n \cap R^3[t] = \tilde{D}_n[t]$ for $\varepsilon_1 \le t < (\varepsilon_1 + \varepsilon_2)/2$,

$B'_n \cap R^3[t] = (\tilde{D}_n \cup N(\Delta; R^3))[t]$ for $t = (\varepsilon_1 + \varepsilon_2)/2$, where $N(\Delta; R^3)$ is a 3-disk obtained by thickenning the disk Δ so that $N(\Delta; R^3) \cap \tilde{D}_n = \partial N(\Delta; R^3) \cap \tilde{D}_n = N(\partial\Delta; \tilde{D}_n)$, and

$B'_n \cap R^3[t] = \mathrm{Cl}(\tilde{D}_n \cup \partial N(\Delta; R^3) - \partial N(\Delta; R^3) \cap \tilde{D}_n)[t] = (\tilde{D}_n^{(1)} \cup \Sigma)[t]$ for $(\varepsilon_1 + \varepsilon_2)/2 < t \le \varepsilon_2$.

We can continue the orientation-preserving cut so as to obtain one 2-disk $\tilde{D}_n^{(\nu)}$ and ν 2-spheres $\Sigma_1, \cdots, \Sigma_\nu$ so that the union $\tilde{D}_n^{(\nu)} \cup \Sigma_1$

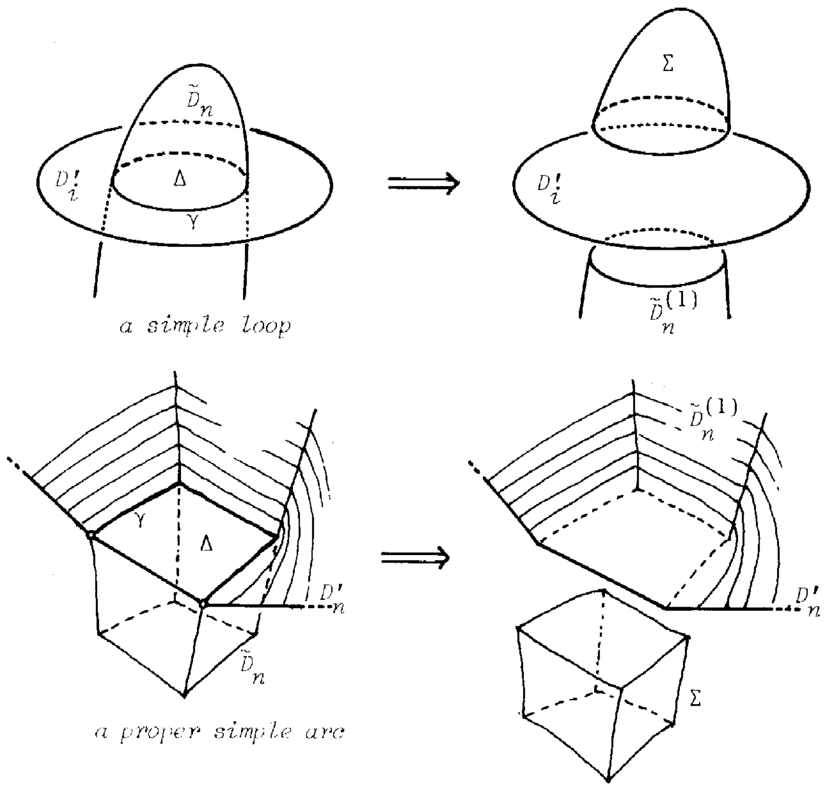


Fig. 5

$\cup \cdots \cup \Sigma_\nu$ no longer intersects the union $D'_1 \cup \cdots \cup D'_{n-1} \cup \mathrm{Int}(D'_n)$. Note that the modifications from $\tilde{D}_n$ to $\tilde{D}^{(\nu)}_n \cup \Sigma_1 \cup \cdots \cup \Sigma_\nu$ have precisely ν times.

Realize these modifications into $R^3[\varepsilon_i, \varepsilon_{i+1}]$ and $R^3[\varepsilon_\nu, 1]$ in good order. Thus, we can construct $B'_n \cap R^3[0, 1]$ which is homeomorphic to a 3-disk with ν open 3-disks removed. Notice that $(B'_n \cap R^3[0, 1]) \cap S_i = \emptyset = R^3(1, +\infty) \cap S_i$, $i=1, \cdots, n-1$.

Since $\tilde{D}^{(\nu)}_n \cup D'_n$, $\Sigma_1, \cdots, \Sigma_\nu$ are mutually disjoint 2-spheres in R^3, from $\Sigma_1, \cdots, \Sigma_\nu$ we can enumerate the 2-spheres, say $\Sigma_1, \cdots, \Sigma_\lambda$ ($\nu \geq \lambda \geq 0$), which contained in the interior of $\tilde{D}^{(\nu)}_n \cup D'_n$ (i.e. the open 3-

disk bounded by $\tilde{D}_n^{(\nu)} \cup D'_n$ in R^3).

Divide the interval $[1, 2]$ into the subintervals $[1, \eta_1]$, $[\eta_1, \eta_2]$, $\cdots$, $[\eta_\lambda, 2]$, where $\eta_i = 1 + i/(\lambda+1)$, $i=1,\cdots,\lambda$. In $\Sigma_1, \cdots, \Sigma_\lambda$, we choose an innermost 2-sphere, say Σ_1, which bounds a 3-disk $\Delta_1^3 \subset R^3$ not meeting other Σ_i. Then, we define $B'_n \cap R^3[1, \eta_2]$ as follows :

$$B'_n \cap R^3[t] = \begin{cases} (\tilde{D}_n^{(\nu)} \cup \Sigma_1 \cup \Sigma_2 \cup \cdots \cup \Sigma_\nu)[t] & \text{for } 1 \le t < (1+\eta_1)/2, \\ (\tilde{D}_n^{(\nu)} \cup \Delta_1^3 \cup \Sigma_2 \cup \cdots \cup \Sigma_\nu)[t] & \text{for } t = (1+\eta_1)/2, \\ (\tilde{D}_n^{(\nu)} \cup \Sigma_2 \cup \cdots \cup \Sigma_\nu)[t] & \text{for } (1+\eta_1)/2 < t \le \eta_2. \end{cases}$$

For $R^3[\eta_2, \eta_3]$, $\cdots$, $R^3[\eta_{\lambda-1}, \eta_\lambda]$, we repeat this process. Thus, we obtain $B'_n \cap R^3[0, \eta_\lambda]$ such that $B'_n \cap R^3[\eta_\lambda] = (\tilde{D}_n^{(\nu)} \cup \Sigma_{\lambda+1} \cup \cdots \cup \Sigma_\nu)[\eta_\lambda]$. Since the 2-sphere $\tilde{D}_n^{(\nu)} \cup D'_n$ does not contain $\Sigma_{\lambda+1} \cup \cdots \cup \Sigma_\nu$ in the interior, we can obtain a 3-disk $\Delta^3 \subset R^3$ with $\partial\Delta^3 = \tilde{D}_n^{(\nu)} \cup D'_n$ and $\Delta^3 \cap (\Sigma_{\lambda+1} \cup \cdots \cup \Sigma_\nu) = \emptyset$. Now we define $B'_n \cap R^3[\eta_\lambda, 2]$ as follows :

$$B'_n \cap R^3[t] = \begin{cases} (\tilde{D}_n^{(\nu)} \cup \Sigma_{\lambda+1} \cup \cdots \cup \Sigma_\nu)[t] & \text{for } \eta_\lambda \le t < (\eta_\lambda+2)/2, \\ (\Delta^3 \cup \Sigma_{\lambda+1} \cup \cdots \cup \Sigma_\nu)[t] & \text{for } t = (\eta_\lambda+2)/2, \\ (D'_n \cup \Sigma_{\lambda+1} \cup \cdots \cup \Sigma_\nu)[t] & \text{for } (\eta_\lambda+2)/2 < t \le 2. \end{cases}$$

Next, in $\Sigma_{\lambda+1}, \cdots, \Sigma_\nu$, we choose an innermost one, say $\Sigma_{\lambda+1}$, which bounds a 3-disk $\Delta_{\lambda+1}^3 \subset R^3$ with $\Delta_{\lambda+1}^3 \cap (\Sigma_{\lambda+2} \cup \cdots \cup \Sigma_\nu) = \emptyset$. Let ξ_1 and ξ_2 be numbers with $2 < \xi_1 < \xi_2$, and we define $B'_n \cap R^3[\xi_1, \xi_2]$ as follows:

$$B'_n \cap R^3[t] = \begin{cases} (\Sigma_{\lambda+1} \cup \Sigma_{\lambda+2} \cup \cdots \cup \Sigma_\nu)[t] & \text{for } 2 \le t < \xi_1, \\ (\Delta_{\lambda+1}^3 \cup \Sigma_{\lambda+2} \cup \cdots \cup \Sigma_\nu)[t] & \text{for } t = \xi_1, \\ (\Sigma_{\lambda+2} \cup \cdots \cup \Sigma_\nu)[t] & \text{for } \xi_1 < t \le \xi_2. \end{cases}$$

This process can be continued to eliminate the 2-spheres $\Sigma_{\lambda+2} \cup \cdots \cup \Sigma_\nu$.

Then we obtain a 3-disk B'_n in $R^3[0, +\infty)$ whose boundary coincides with S'_n and which is disjoint from $S_1, \cdots, S_{n-1}$.

This completes the proof of Lemma 1.6. □

Consider an oriented link $\ell \subset R^3$ and mutually disjoint oriented bands $B_1, \cdots, B_m, B'_1, \cdots, B'_{m'}$, $(m, m' \geq 1)$ which span ℓ. Let $\mathcal{B} = \{B_1, \cdots, B_m\}$, $\mathcal{B}' = \{B'_1, \cdots, B'_{m'}\}$ and $\overline{\mathcal{B}} = \mathcal{B} \cup \mathcal{B}'$. If we let $\ell' = h(\ell; \mathcal{B})$ and $\overline{\ell} = h(\ell'; \mathcal{B}')$, then it is easy to see that $\overline{\ell} = h(\ell; \overline{\mathcal{B}})$. For the realizing surfaces F_a^b and F'^b_a in $R^3[a, b]$ of the sequence $\ell \to \overline{\ell}$ and $\ell \to \ell' \to \overline{\ell}$, respectively, we can obtain the following lemma :

1.10. Lemma. *F_a^b and F'^b_a are ambient isotopic in $R^3[a, b]$ keeping the boundary $\partial R^3[a, b]$ fixed.*

Proof. Divide the interval $[a, b]$ into three subintervals $[t_0, t_1]$, $[t_1, t_2]$ and $[t_2, t_3]$, where $a = t_0 < t_1 < t_2 < t_3 = b$. By an isotopic deformation of $R^3[a, b]$ keeping $\partial R^3[a, b]$ fixed, we can assume that F_a^b and F'^b_a are defined as follows :

$$F_a^b \cap R^3[t] = \ell[t] = F'^b_a \cap R^3[t] \quad \text{for} \quad t_0 \leq t < t_1,$$

$$F_a^b \cap R^3[t_1] = (\ell \cup B_1 \cup \cdots \cup B_m \cup B'_1 \cup \cdots \cup B'_{m'})[t_1],$$

$$F'^b_a \cap R^3[t_1] = (\ell \cup B_1 \cup \cdots \cup B_m)[t_1],$$

$$F_a^b \cap R^3[t] = \overline{\ell}[t] \quad \text{and} \quad F'^b_a \cap R^3[t] = \ell'[t] \quad \text{for} \quad t_1 < t < t_2,$$

$$F_a^b \cap R^3[t_2] = \overline{\ell}[t_2], \quad F'^b_a \cap R^3[t_2] = (\ell' \cup B'_1 \cup \cdots \cup B'_{m'})[t_2],$$

$$F_a^b \cap R^3[t] = \overline{\ell}[t] = F'^b_a \cap R^3[t] \quad \text{for} \quad t_2 < t \leq t_3.$$

It is easily checked that F_a^b and F'^b_a differ by 3-disks $B'_1[t_1, t_2]$, $\cdots$, $B'_{m'}[t_1, t_2]$. So by Proposition 1.7 (The Cellular Move Lemma), F_a^b

and F'^{b}_{a} are ambient isotopic in $R^3[a, b]$ keeping $\partial R^3[a, b]$ fixed, which completes the proof. □

1.11. **Definition.** An isotopic deformation (ambient isotopy) $\{h_s\}_{s \in I}$ of $R^3(-\infty, +\infty)$ is said to be *level-preserving* for each $s \in I$ and t with $-\infty < t < +\infty$, $h_s(R^3[t]) = R^3[t]$ holds, and *$[a, b]$-vertical-line-preserving* if for each $s \in I$ and $x \in R^3$ there exists a unique point $x_s \in R^3$ such that $h_s(x[t]) = x_s[t]$ holds for all $t \in [a, b]$.

1.12. Lemma. *Let ℓ_1, ℓ'_1 be links obtained from an oriented link ℓ_0 by the hyperbolic transformations along bands $\{B_i\}$ and $\{B'_j\}$, respectively. If the links with bands* (i.e. *complexes*) $\ell_0 \cup (\cup_i B_i)$ *and* $\ell_0 \cup (\cup_j B'_j)$ *are ambient isotopic in R^3, then the realizing surfaces F^b_a, $F'^b_a \subset R^3[a, b]$ of the sequences $\ell_0 \to \ell_1$, $\ell_0 \to \ell'_1$, respectively, are ambient isotopic by a level-preserving and $[\rho_1, \rho_2]$-vertical-line-preserving isotopic deformation of $R^3(-\infty, +\infty)$, where ρ_1, ρ_2 are arbitrary numbers with $\rho_1 \le a \le b \le \rho_2$. Furthermore, if the ambient isotopy of R^3 carrying $\ell_0 \cup (\cup_i B_i)$ to $\ell_0 \cup (\cup_j B'_j)$ keeps the link ℓ_0 fixed setwise , then the isotopic deformation of $R^3(-\infty, +\infty)$ may be asserted to be level-preserving and $[\xi_1, \xi_2]$-vertical-line-preserving for $a < \xi_1 \le (a+b)/2 < b \le \xi_2$ and to keep $R^3(-\infty, a]$ fixed.*

Proof. Let $\{h_s\}_{s \in I} : R^3 \to R^3$ be the ambient isotopy sending $\ell_0 \cup (\cup_i B_i)$ to $\ell_0 \cup (\cup_j B'_j)$. For a sufficiently small positive number ε, the desired isotopic deformation $\{f_s\}_{s \in I} : R^3(-\infty, +\infty) \to R^3(-\infty, +\infty)$ is defined as follows :

$f_s(x[t]) = h_s(x)[t]$ for $x[t] \in R^3[\rho_1, \rho_2]$,

$f_s(x[t]) = h_{\phi(t,s)}(x)[t]$ for $x[t] \in R^3[\rho_1 - \varepsilon, \rho_1]$, where ϕ is a piecewise linear map from $[\rho_1-\varepsilon,\rho_1]\times[0,1]$ to $[0,1]$ defined by

$$\phi(t,s)=\begin{cases}0 & \text{if } t+\varepsilon s-\rho_1\le 0,\\ (t+\varepsilon s-\rho_1)/\varepsilon & \text{if } t+\varepsilon s-\rho_1>0,\end{cases}$$

$f_s(x[t]) = h_{\psi(t,s)}(x)[t]$ for $x[t]\in R^3[\rho_2,\rho_2+\varepsilon]$, where ψ is a piecewise linear map from $[\rho_2,\rho_2+\varepsilon]\times[0,1]$ to $[0,1]$ defined by

$$\psi(t,s)=\begin{cases}0 & \text{if } \varepsilon s-t+\rho_2\le 0,\\ (\varepsilon s-t+\rho_2)/\varepsilon & \text{if } \varepsilon s-t+\rho_2>0.\end{cases}$$

Finally, let $f_s \mid R^3(-\infty,\rho_1-\varepsilon]\cup R^3[\rho_2+\varepsilon,+\infty)$ be the identity map. This completes the proof. □

1.13. Remark. Let ℓ be a link and B a band spanning ℓ. If we slide the attaching arcs of the band B along the link ℓ or deform the band B itself, then the link with band $\ell\cup B$ is ambient isotopic to the link with band $\ell\cup B'$, B' being the resulting band, in R^3 keeping the link ℓ fixed setwise.

The following is a sort of converse of Lemma 1.10.

1.14. Lemma. *Consider the sequence* $\ell=\ell_0\to\ell_1\to\cdots\to\ell_n=\ell'$ *of oriented links in* R^3 *with* $\ell_{i+1}=h(\ell_i\,;B_{i+1})$ *for a band* $B_{i+1}, i=0,1,\cdots,n-1$. *Then there exist mutually disjoint bands* $B'_1,\cdots,B'_n$ *in* R^3 *spanning* ℓ *such that the realizing surface* $F_a^b(\ell,\ell''\,;\{B'_1,\cdots,B'_n\})\subset R^3[a,b]$ *with* $\ell''=h(\ell\,;\{B'_1,\cdots,B'_n\})$ *is ambient isotopic to the realizing surface* $F_a^b(\ell_0,\ell_1,\cdots,\ell_n\,;B_1,B_2,\cdots,B_n)$ *by an ambient isotopy of* $R^3(-\infty,+\infty)$ *keeping* $R^3(-\infty,a]$ *fixed, level-preserving on* $R^3[b,+\infty)$ *and* $[b,\rho]$*-vertical-line-preserving for a sufficiently large number* ρ.

Proof. We prove our lemma by induction on n. If $n=1$, the assertion is obvious. Let $n \geq 2$ and assume that the assertion holds for the sequence $\ell = \ell_0 \to \ell_1 \to \cdots \to \ell_{n-1}$. From the construction, we can find a level $R^3[t']$, $a < t' < b$, such that the surface $F_a^b(\ell_0, \ell_1, \cdots, \ell_n ; B_1, B_2, \cdots, B_n) \cap R^3[a, t']$ is ambient isotopic to the realizing surface $F_a^{t'}(\ell_0, \ell_1, \cdots, \ell_{n-1} ; B_1, B_2, \cdots, B_{n-1})$ of the sequence $\ell_0 \to \ell_1 \to \cdots \to \ell_{n-1}$ by an ambient isotopy of $R^3(-\infty, +\infty)$ keeping $R^3(-\infty, a] \cup R^3[t', +\infty)$ fixed. From the inductive assumption, $F_a^{t'}(\ell_0, \ell_1, \cdots, \ell_{n-1} ; B_1, B_2, \cdots, B_{n-1})$ is deformed into a realizing surface $F_a^{t'}(\ell_0, \ell''_{n-1} ; \{B'_1, \cdots, B'_{n-1}\})$ with $\ell''_{n-1} = h(\ell_0 ; \{B'_1, \cdots, B'_{n-1}\})$ for mutually disjoint bands $B'_1, \cdots, B'_{n-1}$ by an ambient isotopy of $R^3(-\infty, +\infty)$ keeping $R^3(-\infty, a]$ fixed, level-preserving on $R^3[t', +\infty)$ and $[t', \rho]$-vertical-line-preserving for a large number ρ. This ambient isotopy assures that $F_a^b(\ell_0, \ell_1, \cdots, \ell_n ; B_1, B_2, \cdots, B_n)$ is ambient isotopic to $F_a^b(\ell_0, \ell''_{n-1}, \tilde{\ell}_n ; \{B'_1, \cdots, B'_{n-1}\}, \tilde{B}_n)$ by an ambient isotopy of $R^3(-\infty, +\infty)$ keeping $R^3(-\infty, a]$ fixed, level-preserving on $R^3[b, +\infty)$ and $[b, \rho]$-vertical-line-preserving for a large number ρ, where $\tilde{B}_n$ is the band obtained from B_n by the ambient isotopy of R^3 carrying ℓ_{n-1} to ℓ''_{n-1} and $\tilde{\ell}_n = h(\ell''_{n-1} ; \tilde{B}_n)$.

Let α, α' be the attaching arcs of $\tilde{B}_n$ to ℓ''_{n-1}. We can transform $\tilde{B}_n$ so that $(\alpha \cup \alpha') \cap (B'_1 \cup \cdots \cup B'_{n-1}) = \emptyset$ by sliding α, α' along the link ℓ''_{n-1} and by deforming α, α' into smaller subarcs, if necessary, see Fig. 6.

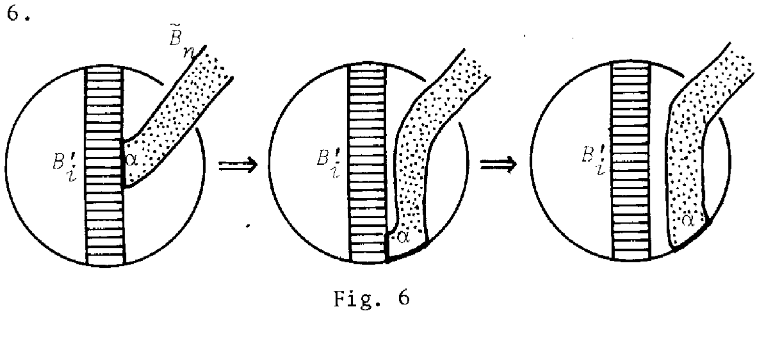


Fig. 6

In general, the transformed $\tilde{B}_n$ intersects $B_1' \cup \cdots \cup B_{n-1}'$. By transversality, we can assume that the intersection $\tilde{B}_n \cap (B_1' \cup \cdots \cup B_{n-1}')$ is a 1-manifold ; that is, it consists of simple loops and simple arcs. Choose a proper simple arc β in $\tilde{B}_n$ starting from α to α' such that β intersects the interior of the 1-manifold $\tilde{B}_n \cap (B_1' \cup \cdots \cup B_{n-1}')$ transversally ; see Fig. 7(a).

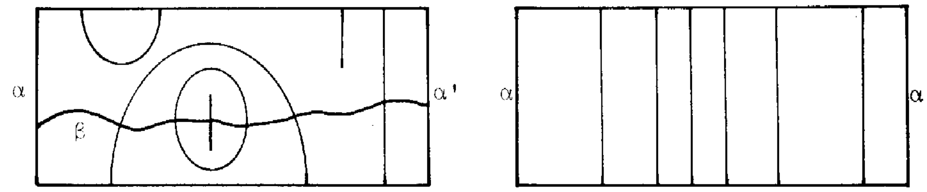


Fig. 7

Replace $\tilde{B}_n$ by a sufficiently small regular neighborhood $N(\beta ; \tilde{B}_n)$. Then for the replaced $\tilde{B}_n$, the intersection $\tilde{B}_n \cap (B_1' \cup \cdots \cup B_{n-1}')$ consists of only proper simple arcs, illustrated in Fig. 7(b). In particular, $\tilde{B}_n \cap (\partial B_1' \cup \cdots \cup \partial B_{n-1}') = \emptyset$.

Let α_i be one of the attaching arcs of B_i' to ℓ_0, and let N_i be the regular neighborhood of $\mathrm{Cl}(\partial B_i' - \alpha_i)$ in B_i' such that $N_i \cap \tilde{B}_n = \emptyset$. Notice that $D_i = \mathrm{Cl}(B_i' - N_i)$ is a 2-disk. It is easy to construct an ambient isotopy of R^3 deforming N_i to B_i' keeping $R^3 - N(D_i; R^3)$ fixed. Using this ambient isotopy, we can transform $\tilde{B}_n$ so that $\tilde{B}_n \cap B_i' = \emptyset$; this transformation is illustrated in Fig. 8.

Thus, the link with band $\ell''_{n-1} \cup \tilde{B}_n$ is ambient isotopic to a link with band $\ell''_{n-1} \cup B_n'$, such that $B_n' \cap (B_1' \cup \cdots \cup B_{n-1}') = \emptyset$ by an ambient isotopy of R^3 keeping ℓ''_{n-1} fixed setwise. Let $\ell'' = h(\ell''_{n-1} ; B_n')$. By Lemma 1.12, $F_a^b(\ell_0, \ell''_{n-1}, \tilde{\ell}_n ; \{B_1', \cdots, B_{n-1}'\}, \tilde{B}_n)$ is ambient isotopic to

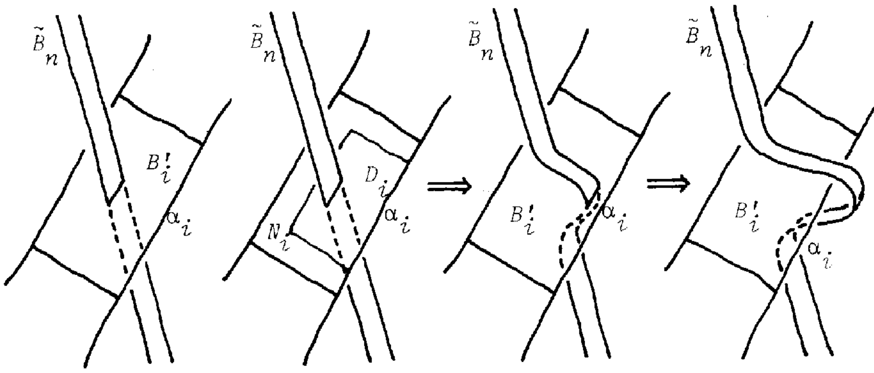


Fig. 8

$F_a^b(\ell_0, \ell''_{n-1}, \ell'' ; \{B'_1, \cdots, B'_{n-1}\}, B'_n)$ by an ambient isotopy of $R^3(-\infty, +\infty)$ keeping $R^3(-\infty, a]$ fixed, level-preserving and $[b, \rho]$-vertical-line-preserving for a large number ρ. Let $R^3[t_1]$ and $R^3[t_2]$, $t_1 < t_2$, be the critical levels such that $R^3[t_1] \cap F_a^b(\ell_0, \ell''_{n-1}, \ell'' ; \{B'_1, \cdots, B'_{n-1}\}, B'_n) = (\ell_0 \cup B'_1 \cup \cdots \cup B'_{n-1})[t_1]$ and $R^3[t_2] \cap F_a^b(\ell_0, \ell''_{n-1}, \ell'' ; \{B'_1, \cdots, B'_{n-1}\}, B'_n) = (\ell''_{n-1} \cup B'_n)[t_2]$. Note that the realizing surfaces $F_a^b(\ell_0, \ell''_{n-1}, \ell'' ; \{B'_1, \cdots, B'_{n-1}\}, B'_n)$ and $F_a^b(\ell_0, \ell'' ; \{B'_1, \cdots, B'_n\})$ differ by the 3-disk $B'_n \times [t_1, t_2]$, where we assume that the critical level of $F_a^b(\ell_0, \ell'' ; \{B'_1, \cdots, B'_n\})$ is $R^3[t_1]$. So, by applying the Cellular Move Lemma (Proposition 1.7), $F_a^b(\ell_0, \ell''_{n-1}, \ell'' ; \{B'_1, \cdots, B'_{n-1}\}, B'_n)$ is ambient isotopic to $F_a^b(\ell_0, \ell'' ; \{B'_1, \cdots, B'_n\})$ keeping $R^3(-\infty, a] \cup R^3[b, +\infty)$ fixed. Combining with the ambient isotopies deforming

$$\begin{aligned} F_a^b(\ell_0, \ell_1, \cdots, \ell_n ; B_1, \cdots, B_n) &\to F_a^b(\ell_0, \ell''_{n-1}, \tilde{\ell}_n ; \{B'_1, \cdots, B'_{n-1}\}, \tilde{B}_n) \\ &\to F_a^b(\ell_0, \ell''_{n-1}, \ell'' ; \{B'_1, \cdots, B'_{n-1}\}, B'_n) \\ &\to F_a^b(\ell_0, \ell'' ; \{B'_1, \cdots, B'_n\}), \end{aligned}$$

the desired ambient isotopy is obtained, and completing the proof. □

Now we would like to introduce two specific types of the hyperbolic transformstions, called *fusion* and *fission*.

Consider an oriented link $\ell \subset R^3$ and mutually disjoint oriented bands $B_1, \cdots, B_m$ $(m \geq 1)$ which span ℓ and let $\ell' = h(\ell; B_1, \cdots, B_m)$. Let ℓ and ℓ' have μ and μ' components, respectively, $(\mu, \mu' \geq 1)$.

1.15. Definition. ℓ' is said to be obtained from ℓ by *m-fusion* (along $B_1, \cdots, B_m$) if $\mu' = \mu - m$. Dually, ℓ' is said to be obtained from ℓ by *m-fission* (along $B_1, \cdots, B_m$) if $\mu' = \mu + m$. The 1-fusion and 1-fission are often called the *simple fusion* and the *simple fission*, respectively. We also say that ℓ' is obtained from ℓ by *complete fusion* (along $B_1, \cdots, B_m$) or by *complete fission* (along $B_1, \cdots, B_m$) according as $\mu' = \mu - m = 1$ or $\mu = \mu' - m = 1$. (See Fig. 9.)

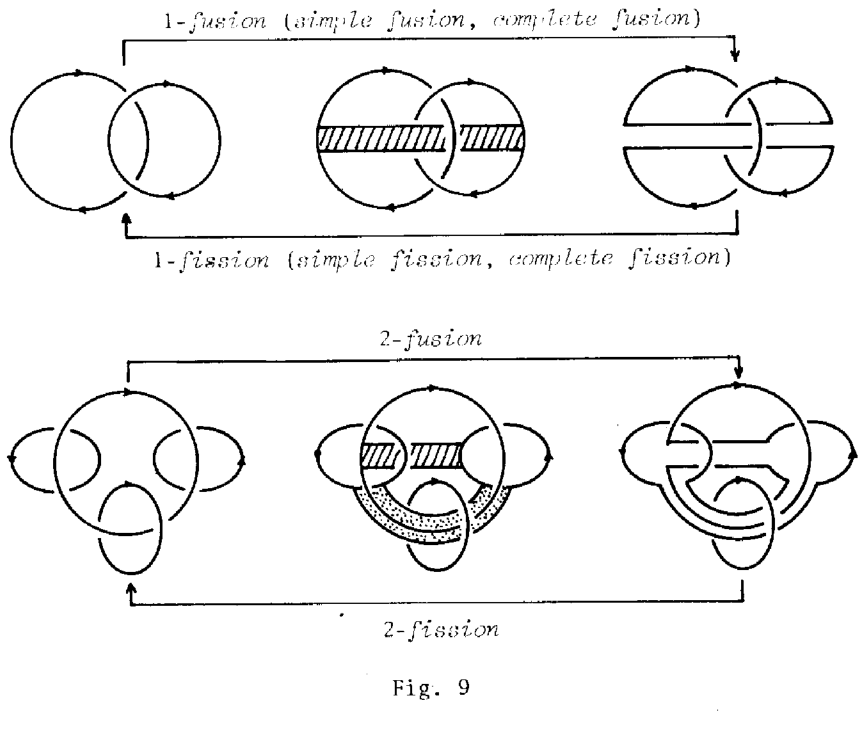


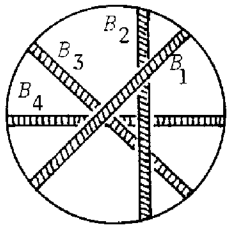
Fig. 9

Clearly, if a transformation $\ell \to \ell'$ is m-fusion, then the inverse transformation $\ell' \to \ell$ is m-fission and conversely. Hence the fusion and fission are dual concepts each other.

1.16. Lemma. *If an oriented knot* k' *is obtained from an oriented knot* k *by a hyperbolic transformations along bands* $B_1, \cdots, B_n$, *then* n *is necessarily even, say* $n = 2m$, *and there exist new mutually disjoint* n *bands* $\overline{B}_1, \cdots, \overline{B}_n$ *which span* k *and such that the realizing surfaces* $F_a^b(k, \ell, k''; \{\overline{B}_1,\cdots,\overline{B}_m\}, \{\overline{B}_{m+1},\cdots,\overline{B}_n\})$ *and* $F_a^b(k, k'; \{B_1,\cdots,B_n\})$ *are ambient isotopic by an ambient isotopy of* $R^3(-\infty, +\infty)$ *keeping* $R^3(-\infty, a]$ *fixed, where* ℓ *is the link obtained from* k *by complete fission along* $\overline{B}_1, \cdots, \overline{B}_m$ *and* k'' *is the knot obtained from* ℓ *by complete fusion along* $\overline{B}_{m+1}, \cdots, \overline{B}_n$. *This ambient isotopy may be level-preserving on* $R^3[b,+\infty)$ *and* $[b,\rho]$*-vertical-line-preserving for an arbitrary* $\rho > b$.

1.17. Remark. In Lemma 1.16, the knot with bands $k \cup B_1 \cup \cdots \cup B_{2m}$ is in general not ambient isotopic to the knot with bands $k \cup \overline{B}_1 \cup \cdots \cup \overline{B}_{2m}$ in R^3. For if so, then the m bands, say $B_1, \cdots, B_m$, corresponding to the m bands $\overline{B}_1, \cdots, \overline{B}_m$ would play the role of complete fission on k. However, the trivial knot with bands $O \cup B_1 \cup B_2 \cup B_3 \cup B_4$, illustrated in Fig. 10, gives a counter-example to this ; that is, no two bands in $\{B_1, B_2, B_3, B_4\}$ play the role of complete fission on O.

Fig. 10



1.18. Proof of Lemma 1.16. Let $\ell^{(1)}$ be $h(k\,;B_1)$. Since $\ell^{(1)}$ is a link with two components and $k' = h(k\,;B_1,\cdots,B_n) = h(\ell^{(1)}\,;B_2,\cdots,B_n)$ is a knot, we can find a band, say B_2, in $\{B_2,\cdots,B_n\}$ such that $k^{(2)} = h(k\,;B_1,B_2) = h(\ell^{(1)}\,;B_2)$ is a knot. Since $\ell^{(3)} = h(k^{(2)}\,;B_3)$ is a link with two components and $k' = h(\ell^{(3)}\,;B_4,\cdots,B_n)$ is a knot, we can find a band, say B_4, in $\{B_4,\cdots,B_n\}$ such that $k^{(4)} = h(\ell^{(3)}\,;B_4)$ is a knot, $\cdots\cdots$.

Since n is finite, the above procedures must be finished by a finite number of times. So, n becomes necessarily even, say $n = 2m$, and we may find a sequence $k = k^{(0)} \to k^{(2)} \to \cdots \to k^{(2m)} = k'$ of knots with $k^{(2i)} = h(k^{(2i-2)}\,;B_{2i-1},B_{2i})$, $i=1,\cdots,m$.

Let $I_1,\cdots,I_m$ be mutually disjoint small connected arcs contained in $k - (B_1 \cup \cdots \cup B_n)$. Slide the attaching arcs of B_1 and B_2 along $k^{(0)} = k$ and deform $B_1 \cup B_2$ themselves into thinner disjoint bands, so that the attaching arcs of the resulting bands $\tilde{B}_1$ and $\tilde{B}_2$ are contained in I_1. Let $\tilde{k}^{(2)}$ be $h(k^{(0)}\,;\tilde{B}_1,\tilde{B}_2)$. Since $k^{(0)} \cup \tilde{B}_1 \cup \tilde{B}_2$ is ambient isotopic to $k^{(0)} \cup B_1 \cup B_2$ by an ambient isotopy of R^3 keeping $k^{(0)}$ fixed setwise, $\tilde{k}^{(2)}$ is a knot. Let $B_3^{(2)}, B_4^{(2)}, \cdots, B_{2m}^{(2)}$ be the bands obtained from $B_3, B_4, \cdots, B_{2m}$ by the isotopic deformation $k^{(0)} \cup B_1 \cup B_2 \to k^{(0)} \cup \tilde{B}_1 \cup \tilde{B}_2$. Slide the attaching arcs of $B_3^{(2)}, B_4^{(2)}$ along $\tilde{k}^{(2)}$ and deform $B_3^{(2)} \cup B_4^{(2)}$ into thinner disjoint bands, so that the attaching arcs of the resulting bands $\tilde{B}_3, \tilde{B}_4$ are contained in I_2. $\tilde{B}_3, \tilde{B}_4$ should be chosen to be disjoint from $\tilde{B}_1, \tilde{B}_2$. Let $\tilde{k}^{(4)}$ be $h(\tilde{k}^{(2)}\,;\tilde{B}_3,\tilde{B}_4)$. Since $\tilde{k}^{(2)} \cup \tilde{B}_3 \cup \tilde{B}_4$ is ambient isotopic to $\tilde{k}^{(2)} \cup B_3^{(2)} \cup B_4^{(2)}$ by an ambient isotopy of R^3 keeping $\tilde{k}^{(2)}$ fixed setwise, $\tilde{k}^{(4)}$ is a knot.

By continuing this modification, we obtain mutually disjoint bands $\tilde{B}_1$, $\tilde{B}_2, \cdots, \tilde{B}_{2m-1}, \tilde{B}_{2m}$, which span $k^{(0)}$ and such that for each i the attaching arcs of $\tilde{B}_{2i-1}$ and $\tilde{B}_{2i}$ are contained in I_i (see for example Fig. 11), and a sequence $k = k^{(0)} \to \tilde{k}^{(0)} \to \tilde{k}^{(2)} \to \cdots \to \tilde{k}^{(2m)}$ of knots with $\tilde{k}^{(2i)} = h(\tilde{k}^{(2i-2)} ; \tilde{B}_{2i-1}, \tilde{B}_{2i})$, $i=1,\cdots,m$.

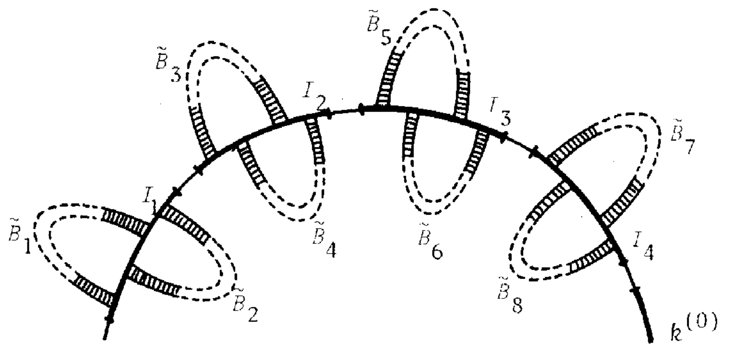


Fig. 11

By applying Lemma 1.12 inductively, the realizing surface $F^{(1)b}_a$ of the sequence $k = k^{(0)} \to k^{(2)} \to \cdots \to k^{(2m)} = k'$ is ambient isotopic to the realizing surface $F^{(2)b}_a$ of the sequence $k = \tilde{k}^{(0)} \to \tilde{k}^{(2)} \to \cdots \to \tilde{k}^{(2m)} = k''$ by an ambient isotopy of $R^3(-\infty, +\infty)$ level-preserving, $[b,\rho]$-vertical-line-preserving for an arbitrary $\rho > b$ and keeping $R^3(-\infty, a]$ fixed.

From Lemma 1.10, it follows that $F^{(1)b}_a$ and $F^{(2)b}_a$ are ambient isotopic, respectively, to the realizing surfaces $F^b_a(k, k' ; \{B_1,\cdots,B_{2m}\})$ and $F^b_a(k, k'' ; \{\tilde{B}_1,\cdots,\tilde{B}_{2m}\})$ by ambient isotopies of $R^3(-\infty,+\infty)$ keeping $R^3(-\infty, a] \cup R^3[b, +\infty)$ fixed.

Now let $\overline{B}_i = \tilde{B}_{2i-1}$ and $\overline{B}_{i+m} = \tilde{B}_{2i}$, $i=1,\cdots,m$, and $\ell = h(k ; \overline{B}_1,\cdots, \overline{B}_m)$. It is easy to check that ℓ is a link with $(m+1)$ components ;

that is, ℓ is obtained from k by complete fission along the bands $\overline{B}_1$, $\cdots$, $\overline{B}_m$. Also, note that k'' is obtained from ℓ by complete fusion along $\overline{B}_{m+1}$, $\cdots$, $\overline{B}_{2m}$. By Lemma 1.10 again, $F_a^b(k, k''; \{\tilde{B}_1, \cdots, \tilde{B}_{2m}\})$ is ambient isotopic to $F_a^b(k, \ell, k''; \{\overline{B}_1, \cdots, \overline{B}_m\}, \{\overline{B}_{m+1}, \cdots, \overline{B}_{2m}\})$ by an ambient isotopy of $R^3(-\infty, +\infty)$ keeping $R^3(-\infty, a] \cup R^3[b, +\infty)$ fixed. Combining these ambient isotopies of $R^3(-\infty, +\infty)$, we have a required ambient isotopy between $F_a^b(k, k'; \{B_1, \cdots, B_n\})$ and $F_a^b(k, \ell, k''; \{\overline{B}_1, \cdots, \overline{B}_m\}, \{\overline{B}_{m+1}, \cdots, \overline{B}_{2m}\})$, and completing the proof of Lemma 1.16. □

Consider trivial links O, O', knots k, k' and a link ℓ in R^3 such that k is obtained from O by complete fusion, ℓ is obtained from k by complete fission, k' is obtained from ℓ by complete fusion and O' is obtained from k' by complete fission.

1.19. Definition. The closed realizing surface $\overline{F}_a^b$ in $R^3[a, b]$ of the sequence $O \to k \to \ell \to k' \to O'$ is called a *(connected) surface in the normal form.* The link ℓ is called the *middle cross-sectional link* of $\overline{F}_a^b$, and the knots k and k' are called the *lower* and *upper cross-sectional knots*, respectively.

In case that O or O' is a knot (i.e. connected), we have $O = k$ or $O' = k'$ in the above sequence $O \to k \to \ell \to k' \to O'$, respectively; and in case that ℓ is a knot, we have $k = \ell = k'$ and sometimes we call the knot $k = \ell = k'$ the *equatorial cross-sectional knot* of $\overline{F}_a^b$.

2. Deforming a Surface into a Surface in the Normal Form

The main purpose of this section is to show the following theorem :

2.1. Theorem. *Any locally flat, connected, closed and oriented surface in* $R^3(-\infty, +\infty)$ *can be deformed into a surface in the normal form by an ambient isotopy of* $R^3(-\infty, +\infty)$ *(with compact support). Further, the middle cross-sectional link of this deformed surface has the genus plus one components.*

The proof will be done in 2.13.

Now we consider the situation that a *locally flat, closed, oriented* and *polyhedral surface* F is given in the oriented 4-space $R^3(-\infty, +\infty)$. To prove Theorem 2.1, we must deform F by an ambient isotopy of $R^3(-\infty, +\infty)$ so that the intersections of F with parallel hyperplanes $R^3[t]$, $-\infty < t < +\infty$, come to be as simple as possible and then must describe the changing of the configuration as t increases from $-\infty$ to $+\infty$.

An intersection $F \cap R^3[t]$ is an *ordinary cross-section* of $F \subset R^4$ if $F \cap R^3[t]$ is an empty set or a link in $R^3[t]$. If $F \cap R^3[t]$ is a link, then the orientation of the link will be chosen so as to be induced from that of the bounded oriented surface $F \cap R^3(-\infty, t]$. The orientation of the hyperplane $R^3[t]$ will be chosen so as to be induced from $R^3(-\infty, t]$.

An intersection $F \cap R^3[t]$ is an *exceptional cross-section* of $F \subset R^4$ if it is not an ordinary cross-section.

It will be noticed that the exceptional cross-sections of F appear for a finite number of hyperplanes ; more concretely, at most for those hyperplanes which pass through a vertex of a triangulated F (as a sub-complex of a triangulated $R^4 = R^3(-\infty, +\infty)$). In fact, if for a level $R^3[t_0]$, $F \cap R^3[t_0]$ is non-empty and does not contain any vertex of the triangulated F, then let $[a_1, a_2, a_3]$ be a 2-simplex of F such that $[a_1, a_2, a_3] \cap R^3[t_0] \neq \emptyset$. Let t_i, $i=1,2,3$, be the fourth coordinate of a_i.

Without loss of generality, we may assume that $t_1 > t_0$, $t_2 > t_0$ and $t_3 < t_0$. It is directly checked that $[a_1,a_2,a_3] \cap R^3[t_0]$ is a 1-disk linearly and properly embedded in $[a_1,a_2,a_3]$. Since F has no boundary and $F \cap R^3[t_0]$ is compact, it follows that $F \cap R^3[t_0]$ is a closed 1-manifold i.e. the disjoint union of a finite number of 1-spheres.

2.2. Lemma. *F is ambient isotopic to a triangulated surface in $R^4 = R^3(-\infty, +\infty)$ by a sufficiently small deformation, so that any two vertices of it have neither the same coordinate (x_1,x_2,x_3) nor the same fourth coordinate t.*

Proof. Choose a large 4-cube $C^4 \subset R^3(-\infty, +\infty)$ containing F in its interior. Let J be a triangulation of C^4 having a triangulation K of F as a subcomplex. Let $v_1, \cdots, v_r$ be the vertices of K and $v_1, \cdots, v_r, v_{r+1}, \cdots, v_s$ be the vertices of J.

Select points $v'_1, \cdots, v'_r$ in the interior $\mathrm{Int}(C^4)$ so that

(1) v'_i is sufficiently close to v_i, $i=1,\cdots,r$,

(2) the half-open segment $(v_i, v'_i] = \{(1-u)v_i + uv'_i \mid 0 < u \leq 1\}$ intersects with no affine hyperplane of $R^3(-\infty, +\infty) = R^4$ generated by subsets of $\{v_1, \cdots, v_s\} \cup \{v'_1, \cdots, v'_{i-1}\}$, $i=1,\cdots,r$, and

(3) any two points of $v'_1, \cdots, v'_r$ have neither the same coordinate (x_1,x_2,x_3) nor the same fourth coordinate t.

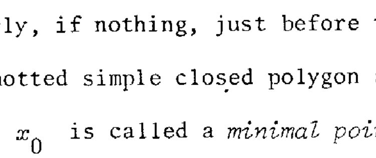
Now we define a piecewise linear homeomorphism $h_u : J \to C^4$, $0 \leq u \leq 1$, by putting $h_u(v_i) = (1-u)v_i + uv'_i$, $i=1,\cdots,r$, and $h_u(v_j) = v_j$, for $j=r+1,\cdots,s$. Notice that the piecewise linear homeomorphism h_1 and the complex J give a simplicial complex structure on C^4 so that $h_1(K)$ is a subcomplex of the complex $h_1(J) = C^4$ having only $v'_1, \cdots, v'_r$ as the vertices of $h_1(K)$.

By letting $h_u \mid R^3(-\infty, +\infty) - C^4$ be the identity map, the family $\{h_u\}_{u \in I}$ gives a required ambient isotopy of $R^3(-\infty, +\infty)$ carrying $F = |K|$ onto $|h_1(K)|$, and completing the proof of Lemma 2.2. □

Let F be a deformed surface as in Lemma 2.2. If $F \cap R^3[t_0]$ is an exceptional cross-section, then it is easy to see that $F \cap R^3[t_0]$ is a polygonal graph with just one exceptional point x_0 which has no neighborhood in $F \cap R^3[t_0]$ homeomorphic to an interval ; this point x_0 is, so called, a *critical point* of F.

To examine exceptional cross-sections, we introduce the following three simple types of critical points, called *elementary critical points*.

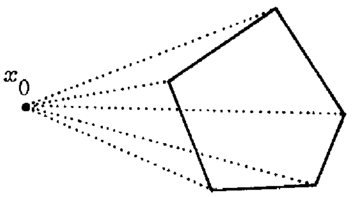
2.3. Definition. In the change of the local configuration of $F \cap R^3[t]$ as t increases past the exceptional level $t = t_0$, if a small unknotted simple closed polygon shrinks to a point x_0 and disappears, then the point x_0 is called a *maximal point* of F ; see Fig. 12.



$t < t_0$ $t = t_0$ $t > t_0$

Fig. 12

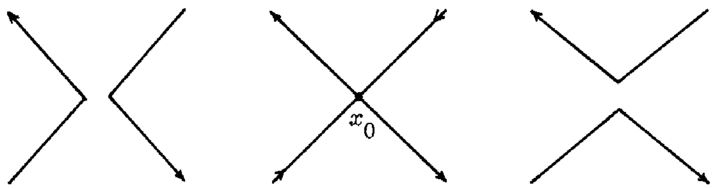
Similarly, if nothing, just before to, a point x_0 at $t = t_0$ and a small unknotted simple closed polygon appears just after $t = t_0$, then the point x_0 is called a *minimal point* of F ; see Fig. 13.



$t < t_0$ $t = t_0$ $t > t_0$

Fig. 13

If two polygonal arcs approach each other and cross at a point x_0 and then two arcs go away just as in Fig. 14, then the point x_0 is called a *saddle point* of F.

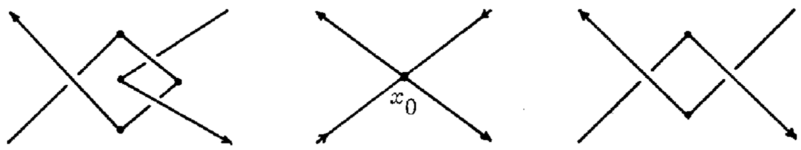


$t < t_0$ $t = t_0$ $t > t_0$

Fig. 14

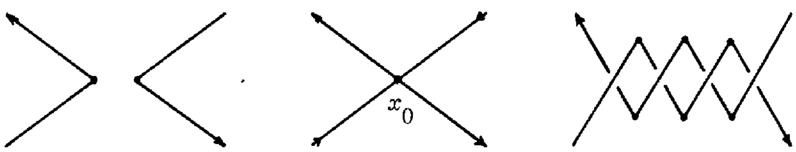
2.4. **Examples.** Here are a few examples of critical points which are not elementary critical points. (Fig. 15 (a), (b), (c)).

(a)



$t < t_0$ $t = t_0$ $t > t_0$

(b)



$t < t_0$ $t = t_0$ $t > t_0$

(c)

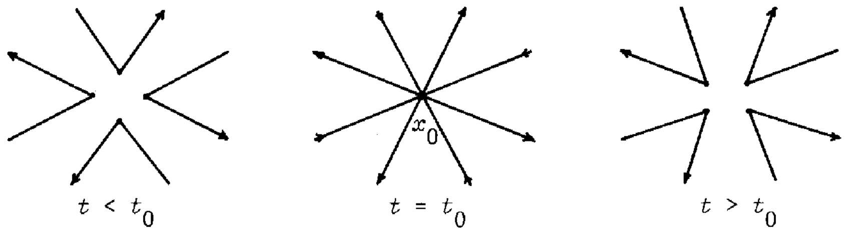


Fig. 15

Note that in each example the critical point x_0 is certainly a locally flat point.

2.5. **Lemma.** *F is ambient isotopic in $R^3(-\infty, +\infty)$ by a sufficiently small deformation to a surface, which has only elementary critical points in distinct levels.*

Proof. Let $v_1, \cdots, v_r$ be vertices of a triangulated F. By Lemma 2.2, we can assume that no two points in $\{v_1, \cdots, v_r\}$ have the same coordinate (x_1, x_2, x_3) or the same fourth coordinate t. Now we take sufficiently small cylindrical neighborhoods $N_i[a_i, b_i]$ of the vertices v_i in $R^3(-\infty, +\infty)$, where each N_i is a convex linear 3-disk in R^3 such that the bottom $N_i[a_i]$ and the top $N_i[b_i]$ are disjoint from F. For each i, we remove the 2-disk $F \cap N_i[a_i, b_i]$ and replace it by a cone $\bar{v}_i * \{F \cap \partial(N_i[a_i, b_i])\}$, where $\bar{v}_i$ is an interior point of $N_i[b_i]$.

The resulting surface $\bar{F}$ can be triangulated by introducing new vertices $\hat{v}_i^1, \cdots, \hat{v}_i^{k_i}$ on the polygonal curve $F \cap \partial(N_i[a_i, b_i])$ for each i. We may choose N_i so that no two vertices of $\hat{v}_i^1, \cdots, \hat{v}_i^{k_i}$ have the same fourth coordinate, by using Lemma 2.2. It is eaily checked that $\bar{F}$ is ambient isotopic to F in $R^3(-\infty, +\infty)$ by a sufficiently small deformation and that each vertex $\bar{v}_i$ is a maximal point of $\bar{F}$.

Since for each j, $1 \leq j \leq k_i$, the closed star neighborhood $\mathrm{St}(\hat{v}_i^j, F)$ is a 2-disk consisting either of four convex linear 2-disks or of one (not necessarily convex) linear 2-disk and two convex linear 2-disks (2-simplexes) according as $\hat{v}_i^j$ is in the 1-skeleton $F^{(1)}$ of F or not, it follows that the number $n(\hat{v}_i^j)$ of the intersection points $\mathrm{Lk}(\hat{v}_i^j, \overline{F}) \cap R^3[t_i^j]$ is 0, 2 or 4, where t_i^j is the fourth coordinate of $\hat{v}_i^j$. (Note that no other vertex of $\overline{F}$ lies in $R^3[t_i^j]$.) If the number $n(\hat{v}_i^j)$ is 0, then this vertex $\hat{v}_i^j$ is clearly a minimal (or maximal) point. If $n(\hat{v}_i^j) = 2$, then it is easy to see that $\overline{F} \cap R^3[t_i^j]$ is an ordinary cross-section. If $n(\hat{v}_i^j) = 4$, then the only two possibilities described in the following Fig.16 occur, since $\hat{v}_i^j$ is a locally flat point.

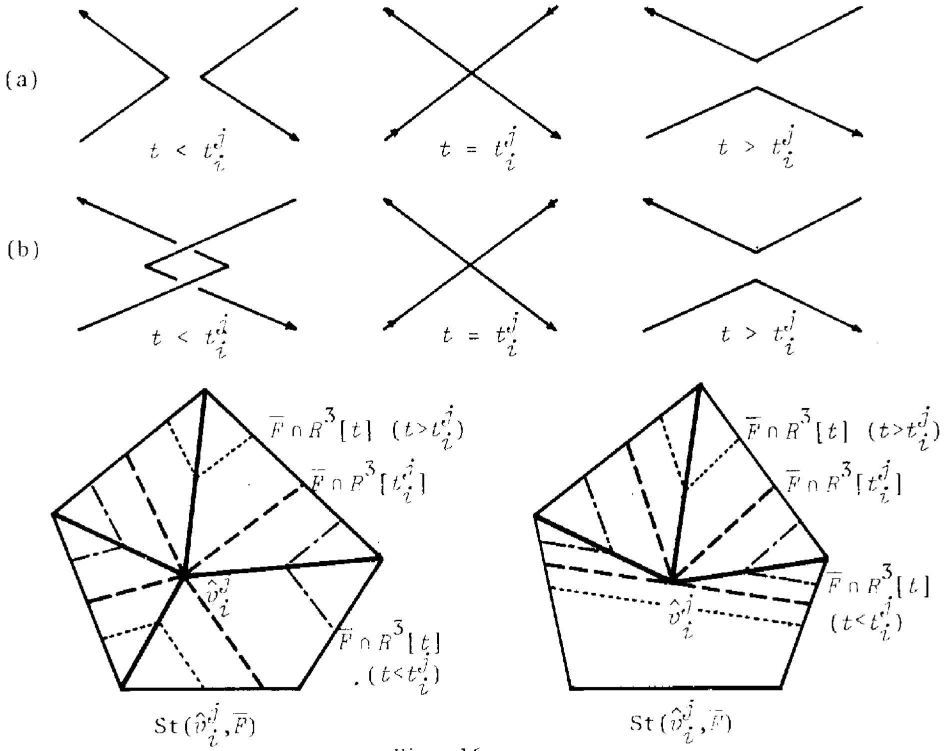


Fig. 16

Now we shall show that the case (b) can be reduced to the case (a). In fact, this follows from the Cellular Move Lemma (Proposition 1.7), since the configuration of the case (b) and the following configuration differ by a 3-disk.

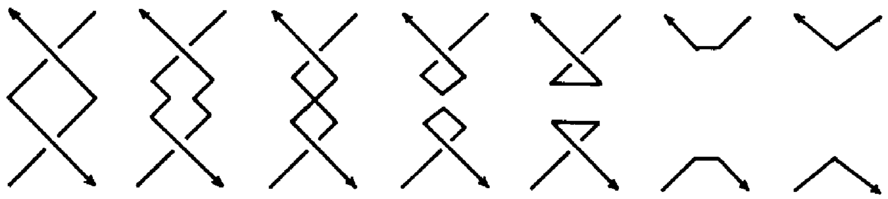


Fig. 17

The 3-disk is illustrated in the following Fig. 18 :

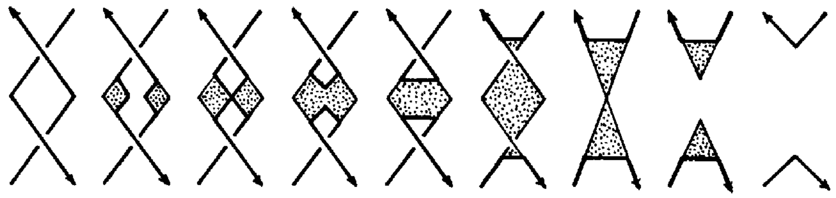


Fig. 18

Since the vertex $\hat{v}_i^j$ of the case (a) is, by definition, a saddle point, and we complete the proof of Lemma 2.5. □

From the combinatorial point of view, it is often convenient to think of *elementary critical bands* instead of elementary critical points.

2.6. Definition. In the change of the local configuration of a surface at t passing the exceptional level $t = t_0$ increasingly, if an unknotted, oriented, simple closed polygon comes to bound an oriented 2-disk B and disappears, then the oriented 2-disk B is called a *maximal band* ; see Fig. 19 below.

If an oriented 2-disk B appears at $t = t_0$ and an unknotted simple closed polygon is left just after $t = t_0$, then the oriented 2-disk B is called a *minimal band* ; see Fig. 20.

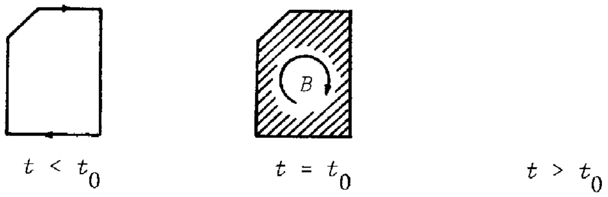


Fig. 19 : A Maximal Band B

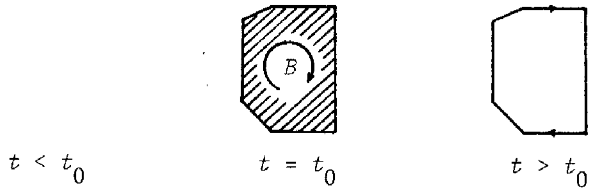


Fig. 20 : A Minimal Band B

If two arcs approach each other and an oriented band B comes to span two arcs and then two arcs are left as in Fig. 21, then this oriented band B is called a *saddle band.*

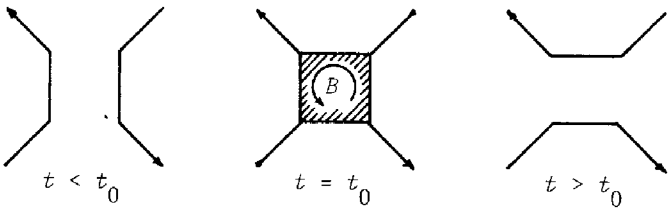


Fig. 21 : A Saddle Band B

2.7. Lemma. *F is ambient isotopic in $R^3(-\infty, +\infty)$ by a sufficiently small deformation to a surface, which has only elementary critical bands in distinct levels.*

Proof. By Lemma 2.5, we can assume that $F \subset R^3(-\infty, +\infty)$ has only elementary critical points at distinct levels. Let v be a maximal point of F at a level $R^3[t_0]$. By definition, for a sufficiently small $\varepsilon > 0$, an unknotted simple closed polygon S_ε occurs in the level $R^3[t_0 - \varepsilon]$ such that the cone $C(v) = v * S_\varepsilon$ is a part of F. Let B_ε be a 2-disk in $R^3[t_0 - \varepsilon]$ bounded by S_ε with $B_\varepsilon \cap (F - S_\varepsilon) = \emptyset$. Since the surfaces F and $\{F - C(v)\} \cup B_\varepsilon$ differ by a 3-disk $v * B_\varepsilon$, F is ambient isotopic to $\{F - C(v)\} \cup B_\varepsilon$ by the Cellular Move Lemma (Proposition 1.7). It is easily checked that B_ε is a maximal band of the surface $\{F - C(v)\} \cup B_\varepsilon$. We may perform this deformation on all of the maximal points of F. For minimal points of F, the same argument may be applied, if we use a sufficiently small $\varepsilon' < 0$ instead of $\varepsilon > 0$.

Let v be a saddle point of F at a level $R^3[t_0]$. For a sufficiently small $\varepsilon > 0$, we choose points w_-, $w'_- \in F \cap R^3[t_0 - \varepsilon]$, w_+, $w'_+ \in F \cap R^3[t_0 + \varepsilon]$ and v_1, v_2, v_3, $v_4 \in F \cap R^3[t_0]$, as illustrated in Fig. 22, in a neighborhood of v in $R^3(-\infty, +\infty)$. Consider four 3-simplexes $\Delta_1^3 =$

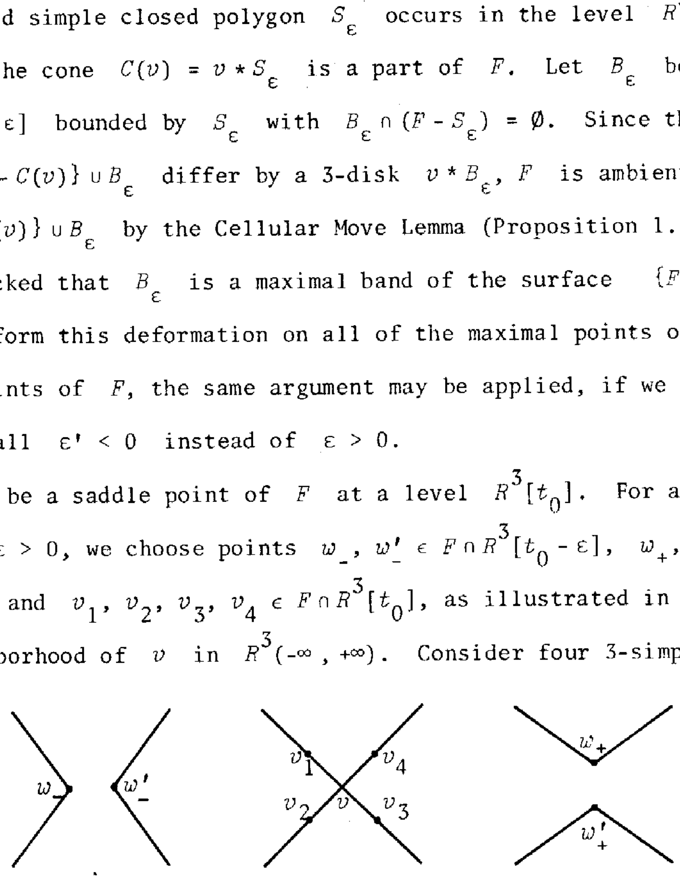


Fig. 22

$[w_-, v_1, v_2, v]$, $\Delta_2^3 = [w'_-, v_3, v_4, v]$, $\Delta_3^3 = [w_+, v_1, v_4, v]$ and $\Delta_4^3 = [w'_+, v_2, v_3, v]$. By applying the Cellular Move Lemma to these four 3-simplexes one by one, F can be deformed to the surface

$$(F - \Delta_1^3 - \Delta_2^3 - \Delta_3^3 - \Delta_4^3) \cup ([w_-, v_1, v_2] \cup [w'_-, v_3, v_4] \cup [w_+, v_1, v_4] \cup [w'_+, v_2, v_3]) \cup B,$$

where $B = [v_1, v_2, v] \cup [v_2, v_3, v] \cup [v_3, v_4, v] \cup [v_4, v_1, v]$. It is easy to see that B is a 2-disk in $R^3[t_0]$ and that B is a saddle band of this resulting surface. We may perform this deformation on all of the saddle points of F.

This completes the proof of Lemma 2.7. □

Lemma 2.7 can be also stated as follows :

2.8. **Lemma.** *F is ambient isotopic in $R^3(-\infty, +\infty)$ to the closed realizing surface $\overline{F}_a^b \subset R^3[a, b]$ of a sequence $O \to \ell_1 \to \ell_2 \to \cdots \to \ell_m \to O'$, such that O and O' are trivial links and each of the transformations $O \to \ell_1$, $\ell_1 \to \ell_2$, $\cdots$, $\ell_m \to O'$ is either a simple fusion or a simple fission.*

Proof. By Lemma 2.7, we may consider that F has only elementary critical bands in distinct levels. Without loss of generality, we can assume that for given a, b $(a < b)$, F is contained in $R^3(a, b)$.

Let $B_1^+, \cdots, B_r^+$ be the maximal bands of F in the levels $R^3[t_1^+]$, $\cdots$, $R^3[t_r^+]$, respectively, with $t_1^+ < \cdots < t_r^+$. Choose mutually disjoint 3-disks $B_1^3, \cdots, B_r^3$ such that

(1) $B_i^3 \subset R^3[t_i^+, b]$ and $B_i^3 \cap (F - B_1^+ - \cdots - B_r^+) = \emptyset$, $i=1,\cdots,r$,

(2) For each i and any $t \in [t_i^+, b]$, $B_i^3 \cap R^3[t]$ is a 2-disk and, in particular, $B_i^3 \cap R^3[t_i^+] = B_i^+$.

One can easily obtain these 3-disks by choosing for each i a polygonal simple arc from B_i^+ to $R^3[b]$ which intersects $R^3[t]$ in a single point for all $t \in [t_i^+, b]$. (Note that $R^3[t_i^+, b] - F$ is connected.)

By applying the Cellular Move Lemma (Proposition 1.7) to these 3-disks $B_1^3, \cdots, B_r^3$, F is ambient isotopic to a surface in $R^3(a, b]$, which has the maximal bands in the level $R^3[b]$ and the other elementary critical bands in $R^3(a, b)$.

We can also apply the similar deformation to the minimal bands of F, and so we may assume that the maximal bands of F are all contained in the level $R^3[b]$ and the minimal bands of F are all contained in the level $R^3[a]$ and the saddle bands are contained in $R^3(a, b)$ at distinct levels.

To complete the proof, we need the following sublemmas :

2.8.1. Sublemma. *Let* $\alpha < \beta$. *Assume that* $F \subset R^3(-\infty, +\infty)$ *is a surface such that for any* $t \in [\alpha, \beta]$, $F \cap R^3[t]$ *is an ordinary cross-section. Then there exists an ambient isotopy* $\{h_s\}_{s \in I}$ *of* $R^3(-\infty, +\infty)$ *keeping* $R^3(-\infty, \alpha]$ *fixed, level-preserving on* $R^3(-\infty, +\infty)$, $[\beta, \rho]$*-vertical-line-preserving for an arbitrary* $\rho \geq \beta$, *such that* $h_1(F \cap R^3[\alpha, \beta]) = \ell_\alpha[\alpha, \beta]$, *where* $\ell_\alpha \subset R^3$ *is the link obtained from the link* $F \cap R^3[\alpha] \subset R^3[\alpha]$ *under the projection* $R^3[\alpha] \to R^3$, $(x, \alpha) \to x$.

2.8.2. Sublemma. *Assume that* $F \subset R^3(-\infty, +\infty)$ *is a surface such that* $F \cap R^3[\gamma]$ *is an exceptional cross-section with a single saddle band* B_γ. *Let* $\ell_\gamma \subset R^3$ *be the link obtained from the link* $\mathrm{Cl}(F \cap R^3(-\infty, \gamma] - B_\gamma) \cap R^3[\gamma]$ *in* $R^3[\gamma]$ *by projecting* $R^3[\gamma]$ *to* R^3.

Then for a sufficiently small $\varepsilon > 0$, *there exists a level-preserving ambient isotopy* $\{h_s\}_{s \in I}$ *of* $R^3(-\infty, +\infty)$ *keeping* $R^3(-\infty, \gamma - 2\varepsilon] \cup R^3[\gamma + 2\varepsilon, +\infty)$ *fixed, so that* $h_1(F \cap R^3[\gamma - \varepsilon, \gamma + \varepsilon])$ *is the realizing surface in* $R^3[\gamma - \varepsilon, \gamma + \varepsilon]$ *of the sequence* $\ell_\gamma \to \ell'_\gamma$ *with* $\ell'_\gamma = h(\ell_\gamma, B^*_\gamma)$, *where* $B^*_\gamma[\gamma] = B_\gamma$.

By assuming these sublemmas, we proceed the proof of Lemma 2.8. Let $a < \gamma_1 < \cdots < \gamma_m < b$ be such that the saddle bands of F occur only at the level $R^3[\gamma_i]$, $i = 1, \cdots, m$. Let F' be the compact surface obtained by removing all minimal and maximal bands. Let $\varepsilon_1, \cdots, \varepsilon_m$ be sufficiently small positive numbers. By Sublemma 2.8.2, we may assume that for each i, $F \cap R^3[\gamma_i - \varepsilon_i, \gamma_i + \varepsilon_i]$ is the realizing surface in $R^3[\gamma_i - \varepsilon_i, \gamma_i + \varepsilon_i]$ of some hyperbolic transfromation of a link along a single band.

By Sublemma 2.8.1, there exist level-preserving ambient isotopies $\{h_s^{(i)}\}_{s \in I}$, $i=0,1,\cdots,m$, of $R^3(-\infty,+\infty)$ such that

(0) $h_s^{(0)}$ is $[\gamma_1 - \varepsilon_1, b]$-vertical-line-preserving and

$$h_1^{(0)}(F' \cap R^3[a, \gamma_1 - \varepsilon_1]) = \ell_a[a, \gamma_1 - \varepsilon_1],$$

(i) $h_s^{(i)}$ is $[\gamma_{i+1} - \varepsilon_{i+1}, b]$-vertical-line-preserving and keeps $R^3 \times (-\infty, \gamma_i + \varepsilon_i]$ fixed, and $h_1^{(i)}(F' \cap R^3[\gamma_i + \varepsilon_i, \gamma_{i+1} - \varepsilon_{i+1}]) = \ell_{(\gamma_i + \varepsilon_i)}[\gamma_i + \varepsilon_i, \gamma_{i+1} - \varepsilon_{i+1}]$, $i=1,\cdots,m-1$,

(m) $h_s^{(m)}$ keeps $R^3(-\infty, \gamma_m + \varepsilon_m]$ fixed and

$$h_1^{(m)}(F' \cap R^3[\gamma_m + \varepsilon_m, b]) = \ell_{(\gamma_m + \varepsilon_m)}[\gamma_m + \varepsilon_m, b].$$

Then the composite ambient isotopy $\{\overline{h}_s\}_{s \in I}$ with $\overline{h}_s = h_s^{(0)} \cdot h_s^{(1)} \cdot \cdots \cdot h_s^{(m)}$ sends F to the desired surface. This proves Lemma 2.8. □

Now we must prove Sublemmas 2.8.1 and 2.8.2. The proofs will be mainly based on the following Isotopy Extension Theorem and Sublemma 2.8.3.

2.9. Proposition (Isotopy Extension Theorem). *Let M be a closed manifold and Q a manifold without boundary. Given an isotopy $G : M \times [\alpha, \beta] \to Q \times [\alpha, \beta]$ such that for all s, t with $\alpha \le s \le t \le \beta$ the (proper) manifolds pair $(Q \times [s, t], G(M \times [s, t]))$ is locally flat, then there exists an ambient isotopy $H : Q \times [\alpha, \beta] \to Q \times [\alpha, \beta]$ with $H_\alpha = id_Q$ and $G = H \cdot (G_\alpha \times id_{[\alpha,\beta]})$.* (For the proof, see Hudson [7, p.147].)

2.8.3. Sublemma. *Let* $\alpha < \beta$. *Assume that* $F \subset R^3(-\infty, +\infty)$ *is a surface such that for any* $t \in [\alpha, \beta]$, $F \cap R^3[t]$ *is an ordinary cross-section. Then there exists an isotopy* $G : \ell_\alpha \times [\alpha, \beta] \to R^3[\alpha, \beta]$ *with* $G(\ell_\alpha \times t) = F \cap R^3[t]$ *for any* $t \in [\alpha, \beta]$, *such that* $G|_{\ell_\alpha \times \alpha} : \ell_\alpha \times \alpha \to R^3[\alpha]$ *is the inclusion map.*

Proof. Let K be a triangulation of $F \cap R^3[\alpha, \beta]$, and let $\alpha = t_0 < t_1 < \cdots < t_m = \beta$ be numbers such that any vertex of K is contained in some $R^3[t_i]$. By adding new vertices to $K \cap R^3[t_i]$, we can assume that $K \cap R^3[t_i]$, $i=0,1,\cdots,m$, are subcomplexes of K. Hence it suffices to prove Sublemma 2.8.3 for the case that the vertices of the simplicial complex K are contained in either $R^3[\alpha]$ or $R^3[\beta]$. So, we shall consider such a case. Further, without loss of generality, we can assume that any vertex of K is contained in just n 1-simplexes of K, where $n = 3$ or $n \geq 5$. Let v be a vertex of $K \cap R^3[\alpha]$ contained in n 1-simplexes with $n \geq 5$. In these n 1-simplexes, there are just $(n-2)$ 1-simplexes intersecting $R^3[\gamma]$ in single points, where $\gamma = (\alpha+\beta)/2$. Let $u_0, u_1, \cdots, u_{n-3}$ be these $(n-2)$ points, as in Fig. 23. Also,

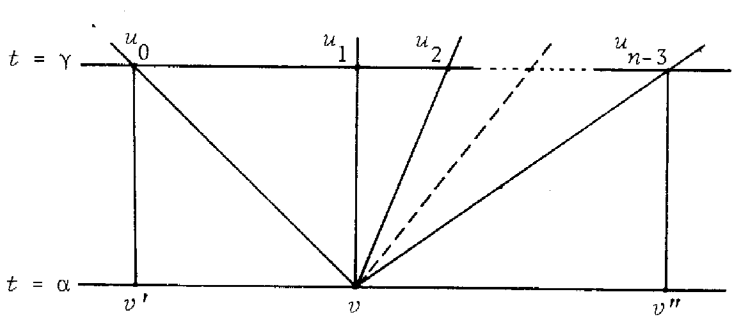


Fig. 23

we choose two points v', v'' in $K \cap R^3[\alpha]$ close to v as in Fig. 23. Given a point $x \in [v', v]$, then we write $x = sv' + (1-s)v$ with $s \in I$.

Let $x' = su_0 + (1-s)v$ and $x'' = su_0 + (1-s)u_1$. Let $\overline{\ell}_x$ be a simple polygon defined by the union $[x , x'] \cup [x' , x'']$. Next, we choose a point u_2^1 in (v , u_3), points u_2^2, u_3^1 in (v , u_4), $\cdots$, and points u_2^{n-5}, u_3^{n-6}, $\cdots$, u_{n-4}^1 in (v , u_{n-3}) so that $t_2^1 > t_3^1 > t_2^2 > \cdots > t_{n-4}^1 > t_{n-5}^2 > \cdots > t_3^{n-6} > t_2^{n-5}$, where t_i^j is the fourth coordinate of u_i^j. Further, let $v_2, v_3, \cdots , v_{n-4}$ be points in (v , v'') in good order. For the case $n = 7$, we illustrated these choices in Fig. 24.

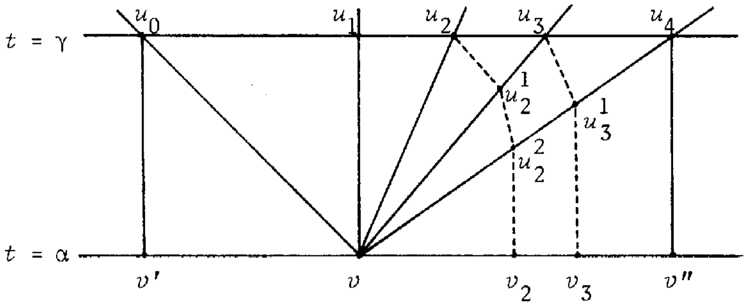


Fig. 24

Given a point $x \in [v , v_2]$, we write $x = sv + (1-s)v_2$ with $s \in I$. By the same way, let $x_{n-5} = sv + (1-s)u_2^{n-5}$, $x_{n-6} = sv + (1-s)u_2^{n-4}$, $\cdots$, $x_1 = sv + (1-s)u_2^1$ and $x' = su_1 + (1-s)u_2$. Let $\overline{\ell}_x$ be a simple polygon defined by the union $[x , x_{n-5}] \cup [x_{n-5}, x_{n-6}] \cup \cdots \cup [x_2 , x_1] \cup [x_1 , x']$.

For any x in $[v_2 , v_3]$, $[v_3 , v_4]$, $\cdots$, or $[v_{n-4} , v'']$, the simple polygon $\overline{\ell}_x$ is defined analogously.

Thus for any $x \in [v' , v] \cup [v , v'']$ the simple proper polygon $\overline{\ell}_x$ in $R^3[\alpha , \gamma]$ is defined. Let $B(v) = \bigcup_{x \in [v',v] \cup [v,v'']} \overline{\ell}_x$. If we remove $B(v)$ for all $v \in K \cap R^3[\alpha]$ with $n \geq 5$ from $F \cap R^3[\alpha , \gamma]$, then the remaining surface consists of only convex-linear 2-disks with four vertices. So, for any $x \in F \cap R^3[\alpha]$ contained in one of these convex-linear 2-disks, we can define $\overline{\ell}_x$ to be the linear 1-disk in an obvious manner.

Thus we defined simple polygons $\bar{\ell}_x$ for all $x \in F \cap R^3[\alpha]$. It should be noted that $\bar{\ell}_x \cap \bar{\ell}_{x'} = \emptyset$ if $x \neq x'$ and $F \cap R^3[\alpha, \gamma] = \bigcup_{x \in F \cap R^3[\alpha]} \bar{\ell}_x$.

Now we define a (piecewise linear) isotopy

$$G' : \ell_\alpha \times [\alpha, \gamma] \to R^3[\alpha, \gamma] \quad \text{by} \quad G'(x' \times t) = \bar{\ell}_{(x'[\alpha])} \cap R^3[t].$$

It will be noted that $G' \mid \ell_\alpha \times \alpha$ is a natural injection and $G'(\ell_\alpha \times t) = F \cap R^3[t]$. Analogously, we can define a (piecewise linear) isotopy

$$G'' : \ell_\gamma \times [\gamma, \beta] \to R^3[\gamma, \beta]$$

with $G'' \mid \ell_\gamma \times \gamma$ being the natural injection and $G''(\ell_\gamma \times t) = F \cap R^3[t]$.

Let $G : \ell_\alpha \times [\alpha, \beta] \to R^3[\alpha, \beta]$ be an isotopy defined by

$$G \mid \ell_\alpha \times [\alpha, \gamma] = G' \quad \text{and} \quad G \mid \ell_\alpha \times [\gamma, \beta] = G'' \cdot (G'_\gamma \times id_{[\gamma, \beta]}),$$

where $G'_\gamma : \ell_\alpha \to \ell_\gamma$ is defined by $G'(x \times \gamma) = G'_\gamma(x)[\gamma]$.

This proves Sublemma 2.8.3. □

2.10. Proof of Sublemma 2.8.1. Since by Sublemma 2.8.3 there is an isotopy $G : \ell_\alpha \times [\alpha, \beta] \to R^3[\alpha, \beta]$ with $G \mid \ell_\alpha \times \alpha$ the natural injection, from the Isotopy Extension Theorem (Proposition 2.9) we obtain an ambient isotopy $H : R^3[\alpha, \beta] \to R^3[\alpha, \beta]$ with $H \mid R^3[\alpha]$ = the identity map and $H \mid \ell_\alpha[\alpha, \beta] = G$.

The desired ambient isotopy $\{h_s\}_{s \in I}$ of $R^3(-\infty, +\infty)$ is, then, defined as follows :

$h_s \mid R^3(-\infty, \alpha] = id.$ for $s \in I$,

$h_s(x[t]) = \tilde{H}^{-1}(x[\alpha + s(t-\alpha)])[t]$ for $x[t] \in R^3[\alpha, \beta]$ and $s \in I$,

where $\tilde{H}^{-1}$ denotes the composite

$$\tilde{H}^{-1} : R^3[\alpha, \beta] \xrightarrow{H^{-1}} R^3[\alpha, \beta] \xrightarrow{projection} R^3,$$

$h_s(x[t]) = \tilde{H}^{-1}(x[\alpha + s(\beta-\alpha)])[t]$ for $x[t] \in R^3[\beta, \rho]$ and $s \in I$,

$h_s(x[t]) = \tilde{H}^{-1}(x[\alpha+\psi(t,s)(\beta-\alpha)])[t]$ for $x[t] \in R^3[\rho,\rho+\varepsilon]$ and $s \in I$,

where $\psi : [\rho , \rho+\varepsilon] \times [0 , 1] \to [0 , 1]$ is defined by

$$\psi(t , s) = \begin{cases} 0 & \text{if } \varepsilon s - t + \rho \le 0, \\ (\varepsilon s - t + \rho)/\varepsilon & \text{if } \varepsilon s - t + \rho > 0, \end{cases}$$

$h_s \mid R^3[\rho+\varepsilon , +\infty) = id.$ for $s \in I$. □

2.11. Proof of Sublemma 2.8.2. For a sufficiently small positive number ε, $\mathrm{Cl}(F \cap R^3(-\infty , \gamma] - B_\gamma) \cap R^3[t]$ is a link for $\gamma - \varepsilon \le t \le \gamma$. So, as in Sublemma 2.8.1, we can deform $\mathrm{Cl}(F \cap R^3[\gamma-\varepsilon , \gamma] - B_\gamma)$ into the product $\ell_\gamma[\gamma-\varepsilon , \gamma]$ by a level-preserving ambient isotopy of $R^3(-\infty , +\infty)$ keeping $R^3[\gamma , +\infty)$ fixed. Similarly, we may assume that $\mathrm{Cl}(F \cap R^3[\gamma,\gamma+\varepsilon] - B_\gamma)$ is the product $\ell'_\gamma[\gamma , \gamma+\varepsilon]$, where $\ell'_\gamma = h(\ell_\gamma ; B^*_\gamma)$. Now the result follows. □

The following follows from the Morse's inequality of a surface in the Morse Theory. (See, for example, Milnor [8, pp.28-31].)

2.12. Proposition. *Suppose a closed (possibly disconnected) surface* $F \subset R^3(-\infty , +\infty)$ *has only elementary critical points, and let* c_+, s, c_- *and* χ *be the numbers of maximal points, saddle points, minimal points and the Euler characteristic of* F. *Then the equality* $c_+ - s + c_- = \chi$ *holds.* □

2.13. Proof of Theorem 2.1. By Lemma 2.8, F is ambient isotopic in $R^3(-\infty , +\infty)$ to the closed realizing surface $\overline{F}{}_a^b \subset R^3[a , b]$ of a sequence $\ell_0 = 0_- \to \ell_1 \to \ell_2 \to \cdots \to \ell_m \to 0_+ = \ell_{m+1}$ with 0_-, 0_+ trivial links and $\ell_{i+1} = h(\ell_i ; B_i)$, $i=0,1,\cdots;m$.

Then from Lemma 1.4 we can further deform $\overline{F}{}_a^b$ into the closed realizing surface $\overline{F}'^{b}_{a}$ of a sequence $O_- \to O'_+$ with $O'_+ = h(O_- ; \{B'_1, \cdots, B'_m\})$ a trivial link. Let O_-, O'_+ have μ_-, μ_+ components, respectively.

Using that $\overline{F}^{b}_{a}$ is connected, we can find $(\mu_- - 1)$ bands $B^-_1, \cdots, B^-_{\mu_- -1}$ in the collection $\{B'_1, \cdots, B'_m\}$ so that $h(O_- ; \{B^-_1, \cdots, B^-_{\mu_- -1}\}) = k_-$ is a knot (i.e. connected). [Proof : Note that the realizing surface F'^{b}_{a} of the sequence $O_- \to O'_+$ is connected, since $\overline{F}'^{b}_{a}$ is connected. Hence the link with bands $O_- \cup B'_1 \cup \cdots \cup B'_m$ is connected. Pick $(\mu_- - 1)$ bands $B^-_1, \cdots, B^-_{\mu_- -1}$ in $\{B'_1, \cdots, B'_m\}$ so that the link with bands $O_- \cup B^-_1 \cup \cdots \cup B^-_{\mu_- -1}$ is also connected. Let $O_- = O_1 \cup \cdots \cup O_{\mu_-}$, where O_i is a connected component of O_-. By changing the subscripts of O_i and B^-_j suitably, we may assume inductively that the link with bands $O_1 \cup \cdots \cup O_i \cup B^-_1 \cup \cdots \cup B^-_{i-1}$ and the component O_{i+1} are connected by a band B^-_i, $i=1, \cdots, \mu_- -1$. This implies that $h(O_- ; \{B^-_1, \cdots, B^-_{\mu_- -1}\}) = k_-$ is a knot.]

Dually, we can also find $(\mu_+ - 1)$ bands $B^+_1, \cdots, B^+_{\mu_+ -1}$ in the remained subcollection $\{B'_1, \cdots, B'_m\} - \{B^-_1, \cdots, B^-_{\mu_- -1}\}$ so that $k_+ = h(O'_+; \{B^+_1, \cdots, B^+_{\mu_+ -1}\})$ is a knot, where we regard the bands $B^+_1, \cdots, B^+_{\mu_+ -1}$ as the spanning bands of the link $O'_+ = h(k_- ; \{B'_1, \cdots, B'_m\} - \{B^-_1, \cdots, B^-_{\mu_- -1}\})$.

Using Lemma 1.10, $\overline{F}'^{b}_{a}$ is ambient isotopic to the closed realizing surface $\overline{F}''^{b}_{a}$ of the sequence $O_- \to k_- \to k_+ \to O'_+$. Then by Lemma 1.16, $\overline{F}''^{b}_{a}$ is ambient isotopic to a surface in the normal form. By Proposition 2.12, the middle cross-sectional link has certainly $(g + 1)$ components, where g is the genus of F. This completes the proof of Theorem 2.1. □

2.14. **Remark.** It should be remarked that, in the sequence $O \to k \to \ell \to k' \to O'$ used in the definition of the normal form, any of the link type ℓ, knot types k, k' and the components of O, the link O' is not uniquely determined by the surface up to ambient isotopy of $R^3(-\infty, +\infty)$, although the number of the components of the middle cross-sectional link is the invariant of the surface.

2.15. Examples. Here are a few examples.

(a) *The standard 2-sphere* : The normal form of the simplest 2-sphere in $R^3(-\infty, +\infty)$ is the following, which has only one minimal band and one maximal band.

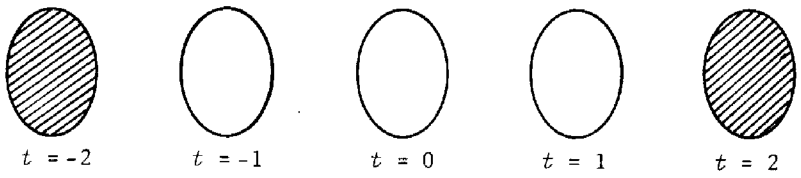


Fig. 25

In general, a (locally flat) 2-sphere K^2 in $R^3(-\infty,+\infty)$ is said to be *unknotted*, if there exists a 3-disk $B^3 \subset R^3(-\infty,+\infty)$ with $\partial B^3 = K^2$. By the well-known homogeneity theorem of manifolds (cf. Rourke-Sanderson [10, p.44] etc.), any two unknotted 2-spheres are ambient isotopic in R^4.

It is easy to see that the above standard 2-sphere is unknotted.

(b) *J.Stalling's unknotted 2-sphere* (Fig. 26) : This example (b) is, perhaps, the first example, which notices the fact that the middle cross-sectional knot of an unknotted 2-sphere in the normal form may be knotted.

More generally, for any knot k, it can be shown that the *composition* (= the so-called *sum of knots*) $k \# (-k^*)$ *is a middle cross-sectional knot of some unknotted 2-sphere in the normal form*, where $-k^*$ denotes the mirror image of k with reversed orientation of k. In fact, Zeeman [13] showed that for any knot k the 1-twist-spun 2-knot (=2-sphere) is unknotted. With a slight modification of this 2-knot, we obtain an unknotted 2-sphere in the normal form whose middle cross-sectional knot is $k \# (-k^*)$. We are going to give a detail of the matter in the forthcoming paper [14].

(c) *H.Terasaka and F.Hosokawa's unknotted 2-sphere* [12] (Fig. 27) : Note that in the example (c) the middle cross-sectional knot is prime.

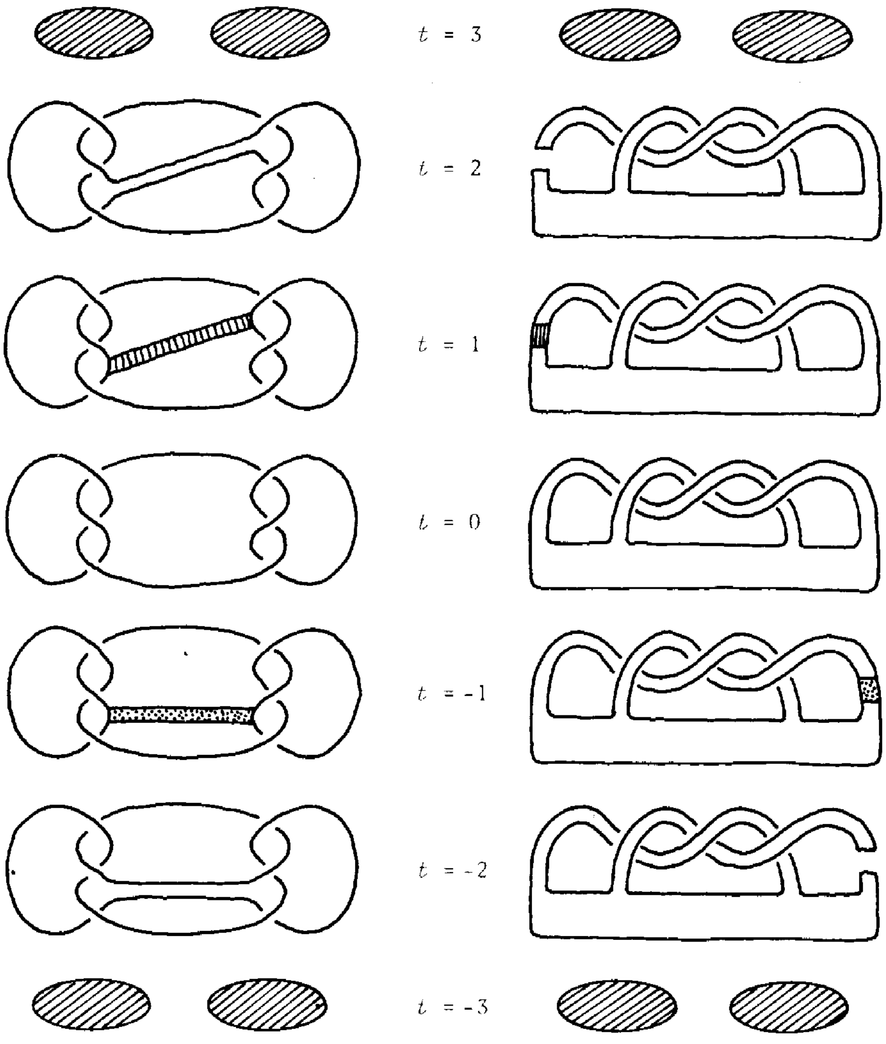


Fig. 26　　　　Fig. 27

(d) An *unknotted* surface of genus 1 (Fig. 28) :

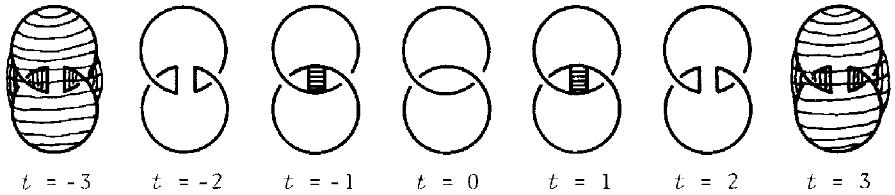


Fig. 28

This surface of genus 1 actually bounds a solid torus of genus 1 in $R^3(-\infty,+\infty)$, as illustrated in the following Fig. 29.

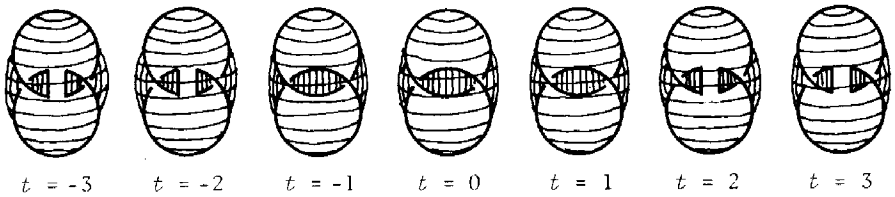


Fig. 29

A connected surface in $R^3(-\infty,+\infty)$ is said to be *unknotted*, if it bounds a solid torus of the same genus in $R^3(-\infty,+\infty)$.

Hosokawa-Kawauchi [6] showed that any two unknotted surfaces of the same genus in $R^3(-\infty,+\infty)$ are ambient isotopic, (cf. [14]). Hence the above surface is ambient isotopic to the following standard surface of genus 1.

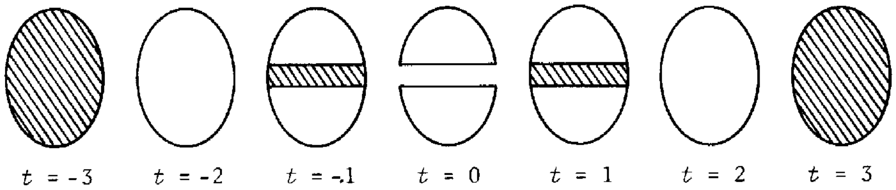


Fig. 30

In Theorem 2.1 we treated a *connected* surface. However, it is easy to see that this assumption was never essential in the proof. We may also derive the following corollary :

2.16. Corollary. *Let* $F \subset R^3(-\infty,+\infty)$ *be a closed oriented (and disconnected) surface of (total) genus* g *with* c *connected components, and let* $a < b$. *Then* F *is ambient isotopic in* $R^3(-\infty,+\infty)$ *to the closed realizing surface* $\overline{F}_a^b \subset R^3[a,b]$ *of some sequence* $0_- \to \ell_- \to \ell \to \ell_+ \to 0_+$, *such that* $0_- \to \ell_-$ *is a fusion from a trivial link* 0_- *to a link* ℓ_- *with* c *components,* $\ell_- \to \ell$ *is a g-fission,* $\ell \to \ell_+$ *is a g-fusion, and* $\ell_+ \to 0_+$ *is a fission to a trivial link* 0_+. *In particular, the middle cross-sectional link* ℓ *has* $(g + c)$ *components.* □

The disconnected surface $\overline{F}_a^b$ in Corollary 2.16 is also called a *surface in the normal form* and the link ℓ is called the *middle cross-sectional link* of $\overline{F}_a^b$ and the links ℓ_- and ℓ_+ are called the *lower* and *upper cross-sectional links*, respectively.

Let $\ell = \ell_1 \circ \cdots \circ \ell_\lambda$ be an oriented link in R^3 which splits into λ sublinks $\ell_1, \cdots, \ell_\lambda$. (Recall Definitions 0.4(4).) If one can find mutually disjoint *convex-linear* 3-disks $D_1^3 \cup \cdots \cup D_\lambda^3$ in R^3 such that $\mathrm{Int}(D_i^3) \supset \ell_i$, $i=1,\cdots,\lambda$, then the link ℓ is said to be *convexly* splittable into $\ell_1, \cdots, \ell_\lambda$. Then the following is obvious :

2.17. Lemma. *Suppose that an oriented link* $\ell = \ell_1 \circ \cdots \circ \ell_\lambda \subset R^3$ *is obtained from an oriented link* $\ell_0 \subset R^3$ *by the hyperbolic transformations along bands* $B_1, \cdots, B_m$; $\ell = h(\ell_0 ; B_1,\cdots,B_m)$. *Then there exists an isotopic deformation* $\{h_s\}_{s \in I}$ *of* R^3 *so that* $\ell' = h(h_1(\ell_0) ; h_1(B_1),$

$\cdots, h_1(B_m)\}$ *is convexly splittable into* $\ell'_1, \cdots, \ell'_\lambda$ *with* $\ell' = h_1(\ell)$ *and* $\ell'_i = h_1(\ell_i)$, $i=1,\cdots,\lambda$. □

Let $O \to k \to \ell \to k' \to \ell^* = O' \circ \ell' = O_1 \circ \cdots \circ O_\lambda \circ k_1 \circ \cdots \circ k_\mu$ be a sequence such that $O \to k$ is a complete fusion from a trivial link O to a knot k, $k \to \ell$ is a complete fission to a link ℓ, $\ell \to k'$ is also a complete fusion to a knot k' and $k' \to \ell^* = O' \circ \ell'$ is a complete fission to a convexly and completely splittable link $\ell^* = O' \circ \ell'$, where $O' = O_1 \circ \cdots \circ O_\lambda$ and $\ell' = k_1 \circ \cdots \circ k_\mu$ are the parts of unknotted and knotted components of ℓ^*, respectively.

For this sequence, the *closed realizing surface* $\hat{F}_a^b$ in $R^3[a, b+1]$ is constructed as follows : First, construct the realizing surface F_a^b in $R^3[a, b]$, and let $\underline{\overline{F}}_a^b$ be a lower closed realizing surface of F_a^b by choosing arbitrary mutually disjoint 2-disks bounded by the trivial link O as usual in $R^3[0]$. Secondly, we take mutually disjoint convex-linear 3-disks $D_1^3, \cdots, D_\lambda^3, D_1'^3, \cdots, D_\mu'^3$ in R^3 such that $\mathrm{Int}(D_i^3) \supset O_i$, $i=1,\cdots,\lambda$, and $\mathrm{Int}(D_j'^3) \supset k_j$, $j=1,\cdots,\mu$. For each i, we take a 2-disk $D_i \subset D_i^3$ with $\partial D_i = O_i$, and for each j we take a point $v_j \in \mathrm{Int}(D_j'^3)$. Now the closed realizing surface $\hat{F}_a^b$ is defined by

$$\hat{F}_a^b = \underline{\overline{F}}_a^b \cup (D_1 \cup \cdots \cup D_\lambda)[b] \cup (D_1' \cup \cdots \cup D_\mu'),$$

where D_j' is the cone with, as the vertex, $v_j[b+1] \in R^3[b+1]$ and, as the base, $k_j[b] \subset R^3[b]$, for each $j=1,\cdots,\mu$.

It should be noted that $\hat{F}_a^b$ is a closed, connected, oriented and *non-locally flat* surface locally knotted at $v_1, \cdots, v_\mu$, such that the local knot type $\kappa(v_j)$ is the knot type of k_j, $j=1,\cdots,\mu$, and that the $\hat{F}_a^b$ is uniquely determined by the sequence up to ambient isotopy of

$R^3(-\infty,+\infty)$. [Note that the non-empty intersection of two convex-linear 3-disks is also a convex-linear 3-disk.] This non-locally flat surface $\hat{F}_a^b$ is also called a *surface in the normal form*, and the link ℓ is called the *middle cross-sectional link* of $\hat{F}_a^b$, and the knots k and k' are called the *lower* and *upper cross-sectional knots* of $\hat{F}_a^b$, respectively.

As a corollary to Theorem 2.1, we also have the following :

2.18. Corollary. *Any closed, connected, oriented and non-locally flat surface $F \subset R^3(-\infty,+\infty)$ is ambient isotopic in $R^3(-\infty,+\infty)$ to the closed realizing surface $\hat{F}_a^b$ of some sequence $0 \to k \to \ell \to k' \to 0' \circ \ell' = 0_1 \circ \cdots \circ 0_\lambda \circ k_1 \circ \cdots \circ k_\mu$, described as above, where $0_1 \circ \cdots \circ 0_\lambda \circ k_1 \circ \cdots \circ k_\mu$ is convexly and completely splittable. Further, the middle cross-sectional link ℓ has the genus of F plus one components, and the link type of $\ell' = k_1 \circ \cdots \circ k_\mu$ is uniquely determined by the given surface F.* □

3. Normalization of Cobordism Surfaces between Links.

We think of arbitrary two oriented links $\ell_i \subset R^3$, $i=0,1$. The union $\ell_0[0] \cup (-\ell_1)[1] \subset R^3[0,1]$, then, necessarily bounds a locally flat, connected, oriented proper surface F in $R^3[0,1]$. We will do a normalization of this surface F.

3.1. Theorem. *If this surface F has genus $g \geq 0$, then there exist trivial links $0_i^{\lambda_i} \subset R^3$ (splitted from the links ℓ_i) $i=0,1$, knots $k, k' \subset R^3$ and a link with $(g+1)$ components $\ell \subset R^3$ accompanied with a sequence $\ell_0 \circ 0_0^{\lambda_0} \to k \to \ell \to k' \to (\ell_1 \circ 0_1^{\lambda_1})^\sim$ of complete fusions and complete fissions, where $(\ell_1 \circ 0_1^{\lambda_1})^\sim$ is ambient isotopic to $\ell_1 \circ 0_1^{\lambda_1}$.*

Proof. By the arguments of Lemmas 2.2, 2.5 and 2.7, one can prove that $F \subset R^3[0\,,1]$ is sufficiently small ambient isotopic to a surface having only elementary critical bands in distinct levels. [Note that the deformations in the proofs of these Lemmas are all local.] Then as stated in Lemma 2.8, by an ambient isotopy of $R^3(-\infty,+\infty)$ keeping $\ell_0[0] \cup (-\ell_1)[1]$ fixed, F is deformed to a surface with only critical bands in $R^3[0\,,1]$ having the maximal bands in the level $R^3[1]$ and the minimal bands in $R^3[0]$ and the saddle bands in $R^3(0\,,1)$. Let F^0 be the proper surface in $R^3[0,1]$ obtained from the new F by removing the interiors of the maximal and minimal bands. Let $\ell_0 \circ O_0^{\lambda_0}$ and $\ell_1 \circ O_1^{\lambda_1}$ be the links representing the links $F^0 \cap R^3[0] \subset R^3[0]$ and $-F^0 \cap R^3[1] \subset R^3[1]$, respectively. By applying Sublemmas 2.8.1 and 2.8.2, $F^0 \subset R^3[0,1]$ is ambient isotopic to the realizing surface $F_0^1 \subset R^3[0,1]$ of a sequence $\ell_0 \circ O_0^{\lambda_0} \to \ell_1' \to \cdots \to \ell_m' \to (\ell_1 \circ O_1^{\lambda_1})'$ of simple fissions or simple fusions, where $(\ell_1 \circ O_1^{\lambda_1})'$ is ambient isotopic to $\ell_1 \circ O_1^{\lambda_1}$ in R^3. Since F_0^1 is connected, the result now follows from an analogous argument of 2.13. This completes the proof. □

3.2. Corollary. *A link $\ell \subset R^3 = R^3[0]$ bounds a connected orientable surface of genus 0 in $R^3[0,+\infty)$ if and only if there exist trivial links O^{λ_0} and O^{λ_1} and a knot k with a sequence $\ell \circ O^{\lambda_0} \to k \to O^{\lambda_1}$ of a complete fusion and a complete fission.* □

3.3. Remark. One can also obtain a suitable version of 3.1 or 3.2 to a result on a disconnected cobordism surface or a non-locally flat cobordism surface. (cf. 2.16 and 2.18.)

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