

## DESCRIPTIONS ON SURFACES IN FOUR-SPACE, II

### SINGULARITIES AND CROSS-SECTIONAL LINKS

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This is a continuation of our previous paper [I]. We retain the definitions and notation in [I], and refer to, for example, Theorem 2.1 of [I] as Theorem I.2.1.

The main purpose of this paper is to give a condition for a link to be the middle, upper or lower cross-sectional link of a surface in the 4-dimensional euclidean space  $R^4$  in the normal form (recall Definition I.1.9.). This paper divides into six sections. Section 1 is preliminaries on normal singular surfaces in the 3-dimensional euclidean space  $R^3$ . In Section 2 we will determine the cross-sectional links of locally flat surfaces in  $R^4$  in the normal form. In particular, we shall show the following (A) and (B):

(A) *The followings are equivalent:*

- (a<sub>1</sub>) *A link  $\ell \subset R^3$  with  $\mu$  components is a ribbon link in the weak sense,*
- (a<sub>2</sub>) *A link  $\ell \subset R^3[0]$  is the middle cross-sectional link of a closed (possibly disconnected) orientable surface  $F \subset R^4$  in the normal form such that the (total) genus of  $F$  plus the number of the components of  $F$  is equal to  $\mu$ ,*
- (a<sub>3</sub>) *A link  $\ell \subset R^3[0]$  is the middle cross-sectional link of a connected orientable surface  $F \subset R^4$  of genus  $\mu - 1$  in the normal form.*

(B) *The followings are equivalent:*

- (b<sub>1</sub>) *A link  $\ell \subset R^3$  with  $\mu$  components is a ribbon link in the strong sense,*
- (b<sub>2</sub>) *A link  $\ell \subset R^3[0]$  is the upper (or lower) cross-sectional link of a surface  $F \subset R^4$  with  $\mu$  components in the normal form,*
- (b<sub>3</sub>) *A link  $\ell \subset R^3[0]$  is the middle cross-sectional link of a surface  $F \subset R^4$  consisting of  $\mu$  2-spheres in the normal form.*

The definitions of ribbon links in the weak sense and in the strong sense are given in 2.2. In Section 3, we shall investigate the cross-sectional links of non-locally flat surfaces in  $R^4$  in the normal form. Section 4 is a construction of a 3-manifold in  $R^4$  bounded by a locally flat surface  $F \subset R^4$ , in particular, a ribbon surface. In Section 5, we shall give some topics of 2-knots related to our theme. Final Section is a geometric approach to exploring a difference between a slice

link in the weak (strong) sense and a ribbon link in the weak (strong) sense. In particular, we shall characterize a slice link in the weak (strong) sense in terms of a normal singular 2-disk in  $R^3$  bounded by the link. This characterization was suggested to the authors by F. Hosokawa and T. Yanagawa.

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### Table of Contents

- §1. Preliminaries on Normal Singular Surfaces in  $R^3$
- §2. Cross-Sectional Links of Surfaces in the Normal Form
- §3. Cross-Sectional Links of Non-Locally Flat Surfaces
- §4. 3-Manifolds Bounded by Surfaces in  $R^4$
- §5. Some Topics of 2-Knots
- §6. Slices and Ribbons

#### 1. Preliminaries on Normal Singular Surfaces in $R^3$

Let  $D^*$  be a compact, orientable and oriented surface (possibly disconnected) with boundary  $\partial D^* = \bigcup_{i=1}^m S_i^1$ . Consider a piecewise-linear map  $f: D^* \rightarrow R^3$  such that the restriction  $f|_{\partial D^*}: \partial D^* \rightarrow R^3$  is injective; let  $\ell = f(\partial D^*) = \bigcup_{i=1}^m f(S_i^1)$ . The image  $f(D^*) = D$  is called a *singular surface bounded by the link  $\ell$* . In particular, if the genus of  $D^*$  is 0 and  $D^*$  contains no 2-sphere as a component, then  $D$  is called a *singular disk bounded by the link  $\ell$* . The orientation of  $D$  is chosen so as to be induced from that of  $D^*$  and  $f$ . The set

$$\Sigma(f) = \{x \in D^* \mid f^{-1}f(x) \text{ contains at least two points}\}$$

is called the *singularity of the map  $f: D^* \rightarrow R^3$*  and the image  $\Sigma = f(\Sigma(f))$  is called the *singularity of the singular surface  $D$* . It is easily checked that the singularities  $\Sigma(f)$  and  $\Sigma$  are polyhedra. (See Hudson [24, p. 90].) It should be noticed that the closures of  $\Sigma(f)$  in  $D^*$  and of  $\Sigma$  in  $D$  are subpolyhedra of  $D^*$  and  $D$ , respectively, but neither  $\Sigma(f)$  nor  $\Sigma$  is in general a subpolyhedron of  $D^*$  and  $D$ . See for example, Fig. 2(d) or (e) below. The points of  $Cl(\Sigma(f)) - \Sigma(f)$  and  $Cl(\Sigma) - \Sigma = f(Cl(\Sigma(f)) - \Sigma(f))$  will be called the *branch points* of the map  $f: D^* \rightarrow R^3$  and the singular surface  $D = f(D^*)$ , respectively.

In general the singularities  $\Sigma(f)$  and  $\Sigma$  are too much complicated, but in many case it suffices to treat somewhat simple singularities, called the *normal singularities* defined as follows:

1.1. DEFINITION. The singularity  $\Sigma$  of the singular surface  $D \subset R^3$  is said to be *normal*, iff each point  $x \in \Sigma$  has a small cubic neighborhood  $V(x)$  in  $R^3$  such that the quadruplet

$$(V(x), V(x) \cap D, V(x) \cap \Sigma, x)$$

corresponds to one of the following cases (1), (2) and (3):

(1)  $V(x) \cap \Sigma$  is a proper simple arc in  $V(x)$ , and  $f^{-1}(V(x) \cap D)$  consists of two sheets  $S_1^*$  and  $S_2^*$  such that  $S_1 = f(S_1^*)$  and  $S_2 = f(S_2^*)$  are proper sheets in  $V(x)$  with  $V(x) \cap D = S_1 \cup S_2$  crossing with each other along the arc  $V(x) \cap \Sigma$ , as shown in Fig. 1 (1),

(2)  $V(x) \cap \Sigma$  consists of three proper simple arcs, say  $A_1, A_2, A_3$ , in  $V(x)$  with  $A_1 \cap A_2 \cap A_3 = x$ , and  $f^{-1}(V(x) \cap D)$  consists of three sheets  $S_1^*, S_2^*$  and  $S_3^*$  such that  $S_1 = f(S_1^*), S_2 = f(S_2^*)$  and  $S_3 = f(S_3^*)$  are proper sheets in  $V(x)$  with  $V(x) \cap D = S_1 \cup S_2 \cup S_3$  and each two sheets of  $S_1, S_2$  and  $S_3$  crosses each other along  $A_1, A_2$  or  $A_3$ , as shown in Fig. 1(2),

(3)  $x$  belongs to the link  $\ell = \partial D$ , and  $V(x) \cap \Sigma$  is a simple arc having  $x$  and a point in  $\partial V(x)$  as the end points, and  $f^{-1}(V(x) \cap D)$  consists of two sheets  $S_1^*$  and  $S_2^*$  such that  $S_1 = f(S_1^*)$  is not proper in  $V(x)$  and  $S_2 = f(S_2^*)$  is proper in  $V(x)$  with  $V(x) \cap D = S_1 \cup S_2$  crossing with each other along the arc  $V(x) \cap \Sigma$ , as in Fig. 1 (3), and

(B) each branch point  $b \in Cl(\Sigma) - \Sigma$  has a small neighborhood  $V(b)$  in  $R^3$  such that  $\partial V(b) \cap \Sigma$  consists of two points; see Fig. 2(d), (e) below.

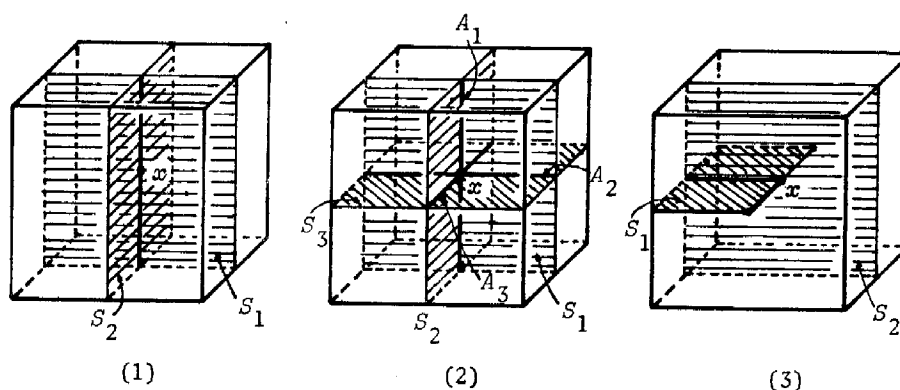


Fig. 1

The singularity  $\Sigma(f) = f^{-1}(\Sigma)$  of the map  $f: D^* \rightarrow R^3$  is said to be *normal* iff  $\Sigma$  is normal.

By the general position argument and obvious modifications, we have the following well-known result:

1.2. PROPOSITION. *In the above notation,  $f$  is homotopic to a piecewise-linear map  $f': D^* \rightarrow R^3$  relative to  $\partial D^*$  whose singularity  $\Sigma(f')$  is normal.  $\square$*

Suppose that the singularity  $\Sigma$  of the singular surface  $D \subset R^3$  is normal.

Clearly, a point  $x$  of  $\Sigma$  as in Case (2) of Definition 1.1 occurs only at finite times. Such a point  $x$  is called a *triple point* of the normal singular surface  $D$ .

Let  $\mathcal{T} = \mathcal{T}(D)$  be the set of triple points of  $D$ ; and set  $\Sigma - \mathcal{T}(D) = \cup_{i=1}^m J_i^\circ$  with  $J_i^\circ$  a connected component. Clearly, for each  $i$ ,  $J_i^\circ$  is a simple arc or loop. If  $J_i^\circ$  and  $J_j^\circ$  ( $i \neq j$ ) have a triple point  $x$  as the common boundary point and if two points  $x_i \in J_i^\circ$  and  $x_j \in J_j^\circ$  sufficiently near to  $x$  are on the same one of  $A_1$ ,  $A_2$  and  $A_3$  in (2) of Definition 1.1, then we join  $J_i^\circ$  and  $J_j^\circ$  with the point  $x$ ; and we perform this modification for all triple points of  $D$ . Then we obtain distinct curves  $J_1, J_2, \dots, J_m$  with  $J_1 \cup J_2 \cup \dots \cup J_m = \Sigma$ . It will be noticed that each triple point of  $D$  corresponds to a self-intersection point of some  $J_i$  or an intersection point of some  $J_i$  and some  $J_j$  ( $i \neq j$ ). It is clear that the normal singular surface  $D$  determines the set  $\mathcal{D}(D) = \{J_1, J_2, \dots, J_m\}$  uniquely; and each  $J_i$  is called a *double line* of  $D$ . From the construction of  $J_i$ , we may canonically specify two curves  $J'_i$  and  $J''_i$  in the preimage  $f^{-1}(J_i)$  in  $D^*$ .

Let  $J \in \mathcal{D}(D) = \{J_1, J_2, \dots, J_m\}$  be a double line of  $D$ .  $J$  must correspond to one of the curves in the following four cases:

- (1) A closed curve  $J$  whose preimage  $f^{-1}(J)$  consists of two closed curves  $J'$  and  $J''$  that lie in the interior  $\text{Int}(D^*)$  of  $D^*$ , see, for example, Fig. 2 (a).
- (2) An arc  $J$  whose preimage  $f^{-1}(J)$  consists of an arc  $J'$  that spans the boundary  $\partial D^*$  of  $D^*$  and an arc  $J''$  that lies in  $\text{Int}(D^*)$ , see, for example, Fig. 2 (b).
- (3) An arc  $J$  whose preimage  $f^{-1}(J)$  consists of two arcs, each of which has one end point in  $\partial D^*$  and another in  $\text{Int}(D^*)$ , see, for example, Fig. 2 (c).
- (4) An arc  $J$  whose preimage  $f^{-1}(J)$  consists of two arcs  $J'$  and  $J''$  such that the closures  $\bar{J}'$  and  $\bar{J}''$  in  $D^*$  have a common end point, which is a branch point of

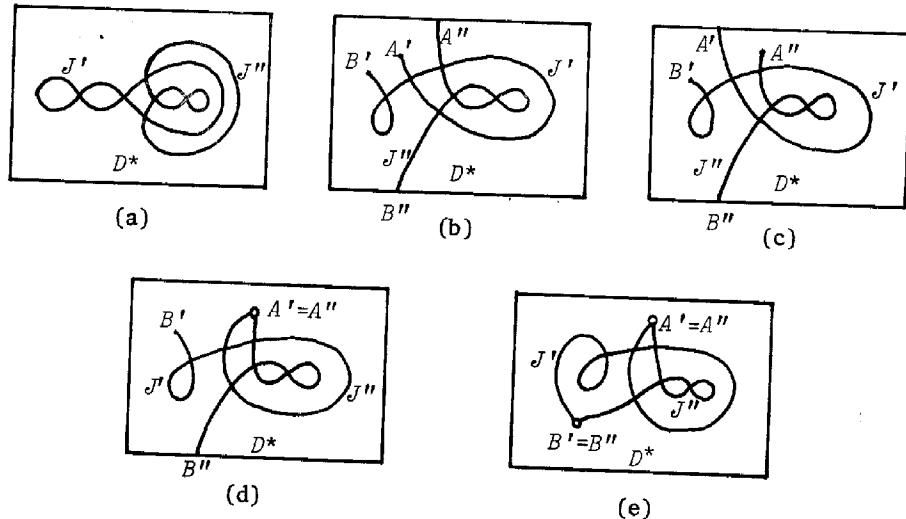


Fig. 2

$f$ , and

- (4<sub>1</sub>)  $J'$  and  $J''$  have just one common end point, see for example, Fig. 2 (d), or  
 (4<sub>2</sub>)  $J'$  and  $J''$  have two common end points, see Fig. 2 (e).

1.3. DEFINITION. A double line  $J \in \mathcal{D}(D)$  of (1), (2), (3) or (4) is called the *loop singularity*, the *ribbon singularity*, the *clasp singularity* or the *branch singularity* of  $D$ , respectively.

1.4. DEFINITION. A double line  $J \in \mathcal{D}(D)$  is said to be *simple*, iff  $J$  has no self-intersection points.

From the construction of the double lines, the following two lemmas are obvious:

1.5 LEMMA. A double line  $J \in \mathcal{D}(D)$  is simple if and only if two curves  $J'$  and  $J''$  of  $f^{-1}(J)$  are simple curves and  $J' \cap J'' = \emptyset$ .  $\square$

1.6. LEMMA. Let  $\mathcal{D}(D)$  be the set of double lines of  $D$ , and let  $J_i^* \subset D^*$  be one of the curves  $f^{-1}(J_i)$ ,  $J_i \in \mathcal{D}(D)$ . If  $J_i^* \cap J_j^* \neq \emptyset$  with  $i \neq j$ , then  $J_i^*$  and  $J_j^*$  cross each other (that is, intersect transversally) in  $D^*$  at any point of  $J_i^* \cap J_j^*$ .  $\square$

1.7. DEFINITION. Let  $D \subset R^3$  be a singular surface whose singularity  $\Sigma$  is normal, and let  $\mathcal{D}(D) = \{J_1, J_2, \dots, J_m\}$  be the set of double lines of  $D$ . For each  $J \in \mathcal{D}(D)$ , let  $J' \cup J'' = f^{-1}(J)$ .

In the cases (2) and (4<sub>1</sub>),  $J'$  or  $J''$  that has an end point in  $\partial D^*$  is called a *b-line*, and the other that has no end points in  $\partial D^*$  is called an *i-line*.

In the cases (1), (3) and (4<sub>2</sub>), we call any one of  $J'$  and  $J''$  a *b-line* and the other an *i-line*.

Suppose that the *b-line* and the *i-line* are specified for the preimage of any double line of  $D$ . A crossing point of two *b-lines*, a *b-line* and an *i-line*, or two *i-lines* is called a point of type  $(b, b)$ ,  $(b, i)$  or  $(i, i)$ , respectively.

1.8. DEFINITION. Let  $D \subset R^3$  be a normal singular surface, and we suppose that the *b-line* and the *i-line* are specified for the preimage of any double line  $J \in \mathcal{D}(D)$ . Let  $p \in \mathcal{T}(D)$  be a triple point of  $D$  with double lines  $J_1, J_2$  and  $J_3$  cutting at  $p$ . (It may happen that  $J_1 = J_2, J_2 = J_3$  or  $J_1 = J_3$ , if  $J_1, J_2$  or  $J_3$  is not simple.) Let  $p^*, p'^*$  and  $p''^*$  be the three points forming the preimage of  $p$ .

A point  $p \in \mathcal{T}(D)$  is called a *triple point of type I*, iff all of the points  $p^*, p'^*, p''^*$  are of type  $(b, i)$ . See, for example, Fig. 3(I). A point  $p \in \mathcal{T}(D)$  is called a *triple point of type II*, iff the points  $p^*, p'^*$  and  $p''^*$  are of type  $(b, b)$ ,  $(b, i)$  and  $(i, i)$ , respectively. See, for example, Fig. 3(II).

Since the preimage of each double line of  $D$  necessarily consists of one *b-line*

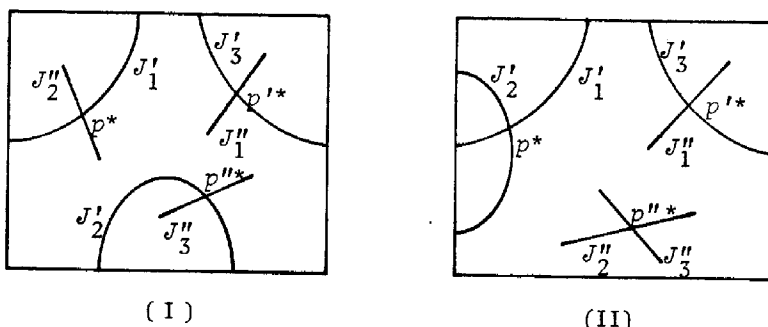
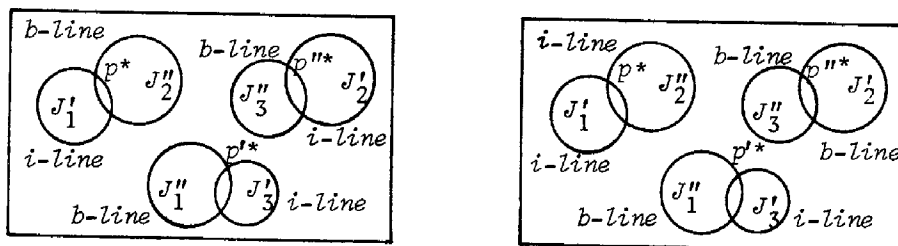


Fig. 3

and one  $i$ -line, we obtain the following:

1.9. PROPOSITION. In 1.8, every triple point  $p \in \mathcal{F}(D)$  is of either type I or type II.  $\square$

1.10. REMARK. In the preimage of a loop singularity (1), a clasp singularity (3) or a branch singularity of (4<sub>2</sub>), the types of related triple points depend on the choices of  $b$ -lines and  $i$ -lines. See Fig. 4.



A triple point  $p$  of type I

A triple point  $p$  of type II

Fig. 4

We record here two existence results of normal singular surfaces in  $R^3$  without proofs.

1.11. PROPOSITION. (SHIBUYA [53, Lemma 1], etc.) Any link  $\ell \subset R^3$  bounds a singular disk  $D \subset R^3$  whose singularity  $\Sigma$  consists of only mutually disjoint, simple clasp singularities.  $\square$

Such a singular disk is called an *elementary disk*, Nakagawa [47]. In general, a knot may bound various elementary disks. The uniqueness of some classes of elementary disks was discussed in Seifert [52] and Nakagawa [47].

1.12. PROPOSITION. (KAPLAN [29]) Any knot  $\ell \subset R^3$  bounds a normal

singular disk  $D \subset R^3$  whose singularity  $\Sigma$  consists of only simple ribbon singularities and which contains only triple points of type I.  $\square$

## 2. Cross-Sectional Links of Surfaces in the Normal Form

Let  $D \subset R^3$  be a normal singular disk given by a piecewise-linear map  $f: D^* \rightarrow R^3, f(D^*) = D$ . By the *component number*  $c(D)$  of  $D$ , we will mean the number of the components of  $D^*$ . Since  $D^*$  is a compact surface of genus 0 containing no 2-sphere as a component,  $c(D)$  does not exceed the number of the components of the link  $\ell = f(\partial D^*) = \partial D$ .

2.1. DEFINITION. A normal singular disk  $D \subset R^3$  is called a *ribbon*, iff the singularity  $\Sigma$  of  $D$  consists of mutually disjoint, simple ribbon singularities alone. Further, a ribbon  $D$  is a *ribbon in the weak sense*, iff  $c(D) = 1$ , and a *ribbon in the strong sense*, iff  $c(D)$  is equal to the number of the components of the link  $\ell = \partial D$ .

2.2. DEFINITION. A link  $\ell \subset R^3$  is called a *ribbon link* iff  $\ell$  bounds a ribbon in  $R^3$ . Further, a link  $\ell$  is a *ribbon link in the weak (resp. strong) sense* iff  $\ell$  bounds a ribbon in the weak (resp. strong) sense. In particular, if  $\ell$  is connected,  $\ell$  is simply called a *ribbon knot*.

The following theorem is essential in this section.

2.3. THEOREM. *The followings are equivalent:*

- (1) A link  $\ell \subset R^3$  bounds a ribbon  $D \subset R^3$  with  $c(D) = \mu$ ,
- (2) There exists a sequence  $\ell \rightarrow \ell' \rightarrow \emptyset$  of hyperbolic transformations of links, such that  $\ell \rightarrow \ell'$  is a fusion from a link  $\ell$  to a link  $\ell'$  with  $\mu$  components and  $\ell' \rightarrow \emptyset$  is a fission from  $\ell'$  to a trivial link  $\emptyset$ .

PROOF. (1) $\Rightarrow$ (2): Consider a ribbon  $D \subset R^3$  with  $c(D) = \mu$  given by a piecewise-linear map  $f: D^* \rightarrow R^3, f(D^*) = D$ , and bounded by the link  $\ell$ . We suppose that  $D$  has  $m$  mutually disjoint, simple ribbon singularities  $\alpha_1, \dots, \alpha_m$ . Let  $\alpha_1^*, \dots, \alpha_m^*$  be the corresponding  $i$ -lines in  $D^*$  and we choose a small regular neighborhood  $N_j = N(\alpha_j^*; D^*)$  of  $\alpha_j^*$  in  $D^*$  for each  $j = 1, \dots, m$ . Then  $D_0 = f(Cl(D^* - \cup_{j=1}^m N_j))$  is a non-singular disk with new resulting boundary components  $O_1 = f(\partial N_1), \dots, O_m = f(\partial N_m)$  as well as the boundary  $\partial D = \ell$ . We note that  $D_0$  has just  $\mu$  components, say  $D_{01}, \dots, D_{0\mu}$ . Let  $\ell_i = \ell \cap \partial D_{0i}$ , and let  $c_i \geq 1$  be the number of the components of  $\ell_i$ . For each  $i$  with  $c_i > 1$ , we choose mutually disjoint  $(c_i - 1)$  bands, say  $\mathcal{B}_i$ , on  $D_{0i}$  that span the link  $\ell_i$  so that the union  $\mathcal{B}_i \cup \ell_i$  is connected. By performing the fusion along these bands  $\mathcal{B}_i$ , the link  $\ell$  is transformed into a link, say  $\ell'$ , with  $\mu$  components. Let  $k_i$  be the component of  $\ell'$  obtained from  $\ell_i$ , and let  $D'_{0i}$  be the corresponding disk obtained from  $D_{0i}$  by splitting  $D_{0i}$  along  $\mathcal{B}_i$ , if  $c_i > 1$ .

Next, let  $\mathcal{O}_i = (O_1 \cup \dots \cup O_m) \cap \partial D_{0i}$ , and let  $c'_i \geq 0$  be the number of the components of  $\mathcal{O}_i$ . For each  $i$  with  $c'_i > 1$ , we choose mutually disjoint  $(c'_i - 1)$  bands, say  $\mathcal{B}'_i$ , on  $D'_{0i}$  that span the knot  $k_i$  so that  $Cl(D'_{0i} - \mathcal{B}'_i)$  consists of mutually disjoint  $c'_i$  annuli and the boundary of each annulus has just one component of  $\mathcal{O}_i$ . By performing the fission along these bands  $\mathcal{B}'_i$ , the link  $\ell'$  is transformed into a trivial link, say  $\mathcal{O}$ , since  $O_1 \cup \dots \cup O_m \subset R^3$  is a trivial link. Thus we obtain a desired sequence  $\ell \rightarrow \ell' \rightarrow \mathcal{O}$ .

(2) $\Rightarrow$ (1): We may assume that the bands  $\mathcal{B}$  used for the hyperbolic transformations of  $\ell \rightarrow \ell' \rightarrow \mathcal{O}$  are mutually disjoint, see Lemma I.1.14. We suppose that  $\mathcal{O}$  has  $\lambda$  components, and let  $\mathcal{E}$  be the union of mutually disjoint  $\lambda$  2-disks in  $R^3$  with  $\partial \mathcal{E} = \mathcal{O}$ . Since  $\mathcal{O} \rightarrow \ell$  is also the hyperbolic transformation along  $\mathcal{B}$ , the union  $\mathcal{E} \cup \mathcal{B}$  may be a ribbon, say  $D$ , if necessary, by deforming the bands  $\mathcal{B}$  into sufficiently narrow bands. (See, for example, Fig. 7 of [I].) Since  $\mathcal{O} \rightarrow \ell'$  is a fusion and  $\ell'$  has  $\mu$  components and  $\ell' \rightarrow \ell$  is a fission,  $c(D) = \mu$ . The boundary  $\partial D$  is clearly of the same type as the link  $\ell$ .

This completes the proof.  $\square$

We have the following three corollaries to Theorem 2.3.

2.4. COROLLARY. *The followings are equivalent:*

- (1) A link  $\ell \subset R^3$  with  $\mu$  components is a ribbon link in the weak sense,
- (2) A link  $\ell \subset R^3[0]$  is the middle cross-sectional link of a locally flat, closed (possibly disconnected) surface  $F \subset R^4$  in the normal form such that the (total) genus of  $F$  plus the number of the components of  $F$  is equal to  $\mu$ ,
- (3) A link  $\ell \subset R^3[0]$  is the middle cross-sectional link of a connected, locally flat, closed surface  $F \subset R^4$  of genus  $\mu - 1$  in the normal form.

PROOF. (1) $\Rightarrow$ (3) follows from Theorem 2.3. In fact, a required surface  $F \subset R^4$  is obtained as the closed realizing surface of  $\mathcal{O} \rightarrow \ell' \rightarrow \ell \rightarrow \ell' \rightarrow \mathcal{O}$ , where  $\ell \rightarrow \ell' \rightarrow \mathcal{O}$  is the sequence in Theorem 2.3(2) and  $\mathcal{O} \rightarrow \ell' \rightarrow \ell$  is its inverse. Since (3) $\Rightarrow$ (2) is obvious, it suffices to show (2) $\Rightarrow$ (1). By Theorem 2.3,  $\ell$  bounds a ribbon  $D'$  with  $c(D') \geq 1$ . If  $c(D') > 1$ , then we connect the components of  $D'$  by  $c(D') - 1$  pipes to obtain a ribbon  $D$  with  $c(D) = 1$ ; this implies (2) $\Rightarrow$ (1).  $\square$

2.5. COROLLARY. *The followings are equivalent:*

- (1) A link  $\ell \subset R^3$  with  $\mu$  components is a ribbon link in the strong sense,
- (2) A link  $\ell \subset R^3[0]$  is the upper (or lower) cross-sectional link of a locally flat, closed surface  $F \subset R^4$  with  $\mu$  components in the normal form,
- (3) A link  $\ell \subset R^3[0]$  is the middle cross-sectional link of a locally flat, closed surface  $F \subset R^4$  consisting of  $\mu$  2-spheres in the normal form.

PROOF. The proof is straightforward from Theorem 2.3.  $\square$

2.6. COROLLARY. *The followings are equivalent:*

- (1) *A knot  $k \subset R^3$  is a ribbon knot,*
- (2) *A knot  $k \subset R^3[0]$  is the upper (or lower) cross-sectional knot of a connected, locally flat, closed surface  $F \subset R^4$  in the normal form,*
- (3) *A knot  $k \subset R^3[0]$  is the middle cross-sectional knot of a locally flat 2-sphere  $F_0 \subset R^4$  in the normal form.  $\square$*

2.7. EXAMPLES. The link in Fig. 5(a) is a ribbon link in the strong

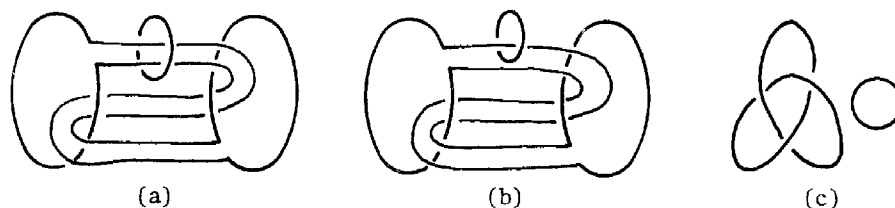


Fig. 5

sense, and the link in Fig. 5(b) is a ribbon link in the weak sense, since these links certainly bound in  $R^3$  ribbons in the strong sense and in the weak sense, respectively. From Corollaries 2.4 and 2.5, it is easily seen that every ribbon link in the strong sense is a ribbon link in the weak sense. However, the converse is not true. In fact, the link in Fig. 5(b) is not a ribbon link in the strong sense. To see this, we need the following sublemma:

2.7.1. SUBLEMMA. *For any oriented link  $\ell = k_1 \cup \dots \cup k_\mu \subset R^3[0]$ , the linking number  $lk(k_i, k_j) = 0$  for  $1 \leq i \neq j \leq \mu$  if and only if there exist mutually disjoint proper (possibly non-locally flat) surfaces  $F_1 \cup \dots \cup F_\mu \subset R^3[0, \infty)$  with  $\partial F_i = k_i$  for  $i = 1, \dots, \mu$ .*

It follows from Sublemma 2.7.1 and Corollary 2.5 that the link in Fig. 5(b) is not a ribbon link in the strong sense.

The link in Fig. 5(c) is not a ribbon link in the weak sense. To see this, suppose the link in Fig. 5(c) is a ribbon link in the weak sense. Then by Corollary 2.4 this link bounds a connected, locally flat surface of genus 0 in  $R^3[0, \infty)$ . This implies that the trefoil knot bounds a locally flat 2-disk in  $R^3[0, \infty)$ . (Such a knot is called a *slice knot*, Fox-Milnor [10], Fox [7]. See Section 6 later.) However, it is known that the trefoil knot never bounds a locally flat 2-disk in  $R^3[0, \infty)$  by using algebraic invariants, e.g., the Alexander polynomial (Fox-Milnor [10]), the signature (Murasugi [46]), etc. Thus, the link in Fig. 5(c) is not a ribbon link in the weak sense.  $\square$

2.7.2. PROOF. of Sublemma 2.7.1. Suppose that  $lk(k_i, k_j) = 0$  for  $1 \leq i \neq j \leq \mu$ . We can easily find a surface  $F_1$  in  $R^3[0]$  with  $\partial F_1 = k_1$  and  $F_1 \cap k_j = \emptyset$  for

$j \geq 2$ . Pushing the interior of  $F_1$  into  $R^3[0, \infty)$ , we have a proper, locally flat surface  $F_1 \subset R^3[0, \infty)$ . Inductively, we can obtain desired surfaces  $F_1 \cup \dots \cup F_\mu \subset R^3[0, \infty)$  with  $\partial F_i = k_i$ . Conversely, we assume that there exist mutually disjoint, proper surfaces  $F_1 \cup \dots \cup F_\mu \subset R^3[0, \infty)$  with  $\partial F_i = k_i$ . Since  $k_i$  bounds the surface  $F_i$  in  $R^3[0, \infty) - F_j$ ,  $i \neq j$ ,  $k_i$  is homologous to 0 in  $R^3[0, \infty) - F_j$ . By using the inclusion isomorphism  $H_1(R^3[0] - k_j; \mathbf{Z}) \cong H_1(R^3[0, \infty) - F_j; \mathbf{Z})$ ,  $k_i$  is homologous 0 in  $R^3[0] - k_j$ . Hence,  $lk(k_i, k_j) = 0$  for  $i \neq j$ . This proves Sublemma 2.7.1.  $\square$

2.8. REMARK. For a ribbon link  $\ell \subset R^3[0]$  in the weak (resp. strong) sense, we have a locally flat, closed surface  $F \subset R^4$  in the normal form by Corollary 2.4(2) or (3) (resp. Corollary 2.5(2) or (3)). It is easily seen that such a surface is not unique up to ambient isotopy of  $R^4$ . In particular, Nakanishi-Nakagawa [49] showed the following:

*For any integer  $n \geq 1$ , there exists a prime ribbon knot  $\mathcal{K} \subset R^3[0]$  such that  $\mathcal{K}$  is the middle cross-sectional knot of  $n$  distinct locally flat 2-spheres  $F_i \subset R^4$  in the normal form,  $i = 1, \dots, n$ .*  $\square$

### 3. Cross-Sectional Links of Non-Locally Flat Surfaces

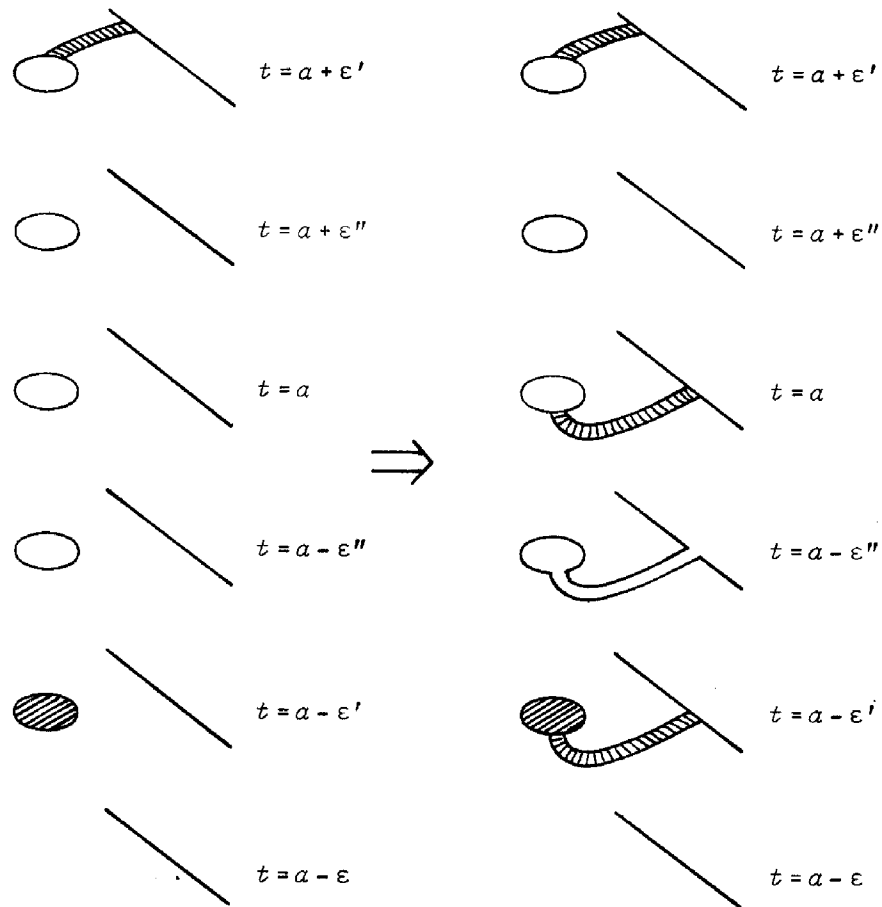
As described in [I, pp 121–122 and Cor. 2.18], a non-locally flat, closed surface (with  $m$  components)  $F \subset R^4$  in the normal form is understood as the disjoint union  $F_1 \cup \dots \cup F_m$  such that, for some fixed  $a < b$ , each  $F_i$  is the closed realizing surface in  $R^3[a, b + 1]$  of some sequence  $\mathcal{O}_i \rightarrow \mathcal{K}_i \rightarrow \ell_i \rightarrow \mathcal{K}'_i \rightarrow \mathcal{O}'_i \circ k_{i1} \circ \dots \circ k_{i\mu_i}$  of knots  $\mathcal{K}_i, \mathcal{K}'_i$  and links  $\mathcal{O}_i, \ell_i, \mathcal{O}'_i \circ k_{i1} \circ \dots \circ k_{i\mu_i}$ ,  $i = 1, \dots, m$ . The links  $\bigcup_{i=1}^m \mathcal{K}_i, \bigcup_{i=1}^m \ell_i$  and  $\bigcup_{i=1}^m \mathcal{K}'_i$  are called the lower, middle and upper cross-sectional links of the surface  $F \subset R^4$  in the normal form, respectively. The knots  $k_{ij}$ ,  $i = 1, \dots, m$ ;  $j = 1, \dots, \mu_i$ , represent the types of the locally knotted points of  $F$  in  $R^4$ .

In this section, we shall give a survey of cross-sectional links of non-locally flat, closed surfaces  $F \subset R^4$  in the normal form.

3.1. PROPOSITION. *A link  $\ell \subset R^3$  is the lower cross-sectional link of a non-locally flat, closed surface  $F \subset R^4$  in the normal form if and only if  $\ell$  is a ribbon link in the strong sense.*

PROOF. If  $\ell$  is the lower cross-sectional link, then there exist a fusion  $\mathcal{O} \rightarrow \ell$  from a trivial link  $\mathcal{O}$  to  $\ell$ , which implies that the inverse  $\ell \rightarrow \mathcal{O}$  is a fission. By Theorem 2.3,  $\ell$  is a ribbon link in the strong sense.

Conversely, suppose that  $\ell \subset R^3[0]$  is a ribbon link with  $m$  components in the strong sense. Let  $k_1 \circ \dots \circ k_m$  be an arbitrary, convexly and completely splitable link. One can find a locally flat, proper surface  $F_1 \cup \dots \cup F_m$  in  $R^3[0, 1]$  such that, for each  $i$ ,  $F_i \cap R^3[0]$  is one component, say  $k_i^?$ , of  $\ell[0] \subset R^3[0]$  and



$$(0 < \epsilon'' < \epsilon' < \epsilon)$$

Fig. 6

$F_i \cap R^3[1] = k_i[1]$ . As in Lemma I.2.7, we may assume that  $F_1 \cup \dots \cup F_m$  has only elementary critical bands. Further, by using a modification illustrated in Fig. 6, we can assume that  $F_1 \cup \dots \cup F_m$  does not contain minimal bands. (It should be noticed that, by this modification, the genus of the resulting surface becomes greater than the original one.) Therefore, by Lemma I.1.14 and Lemma I.2.8, we can find a locally fiat, proper surface  $F'_1 \cup \dots \cup F'_m$  in  $R^3[0, 1]$  such that each  $F'_i$  is the realizing surface in  $R^3[0, 1]$  of a sequence  $k_i^q \rightarrow \ell_i \rightarrow k'_i \rightarrow \mathcal{O}_i \circ k_i$  of complete fissions and a complete fusion, where  $k_i^q[0] = F_i \cap R^3[0]$ ,  $\ell_i$  is a link,  $k'_i$  is a knot and  $\mathcal{O}_i$  is a trivial link. In particular,  $\bigcup_{i=1}^m \mathcal{O}_i \circ k_i$  is a convexly and completely splittable link. Thus, the ribbon link  $\ell$  in the strong sense is the lower

cross-sectional link of a non-locally flat surface in the normal form.  $\square$

Scrutiny of the proof of Proposition 3.1 shows the following more detailed result:

3.2. PROPOSITION. *Given a ribbon link  $\ell \subset R^3$  with  $m$  components in the strong sense and an arbitrary, convexly and completely splittable link  $\cup_{i=1}^m (k_{i1} \circ \dots \circ k_{iu_i})$  with each  $k_{ij}$  knotted, there exists a locally knotted surface  $F = F_1 \cup \dots \cup F_m \subset R^4$  with  $m$  components in the normal form, such that  $\ell$  is the lower cross-sectional link of  $F$ , and for each  $i$ , the local knot types of  $F_i$  are represented by the knots  $k_{i1}, \dots, k_{iu_i}$ .  $\square$*

Any oriented link  $\ell \subset R^3[0]$  always bounds a proper, locally flat, connected and oriented surface in  $R^3[0, \infty)$ . The minimum genus of such a surface is called the 4-dimensional genus of  $\ell$  (Fox [3]), and is denoted by  $g^*(\ell)$ . According to 3.2, for a ribbon link  $\ell = k_1 \cup \dots \cup k_m \subset R^3$  in the strong sense and a convexly and completely splittable link  $\cup_{i=1}^m (k_{i1} \circ \dots \circ k_{iu_i})$  with each  $k_{ij}$  knotted, let  $g_-(k_1 \cup \dots \cup k_m; K_1, \dots, K_m)$  denote the minimum of the genera of the locally knotted surfaces  $F$ 's in  $R^4$  that appear in Proposition 3.2, where  $K_i = \{k_{i1}, \dots, k_{iu_i}\}$  for  $i = 1, \dots, m$ .

3.3. PROPOSITION. *In the above notation, it holds that*

$$g_-(k_1 \cup \dots \cup k_m; K_1, \dots, K_m) \geq \sum_{i=1}^m g^*(k_{i1} \# \dots \# k_{iu_i}),$$

where  $\#$  denotes the usual composition of knots (Fox [7]).  $\square$

3.4. REMARK. Obviously, in 3.3, there exists a ribbon link  $k_1 \cup \dots \cup k_m$  establishing the equality for an given, convexly and completely splittable link  $\cup_{i=1}^m (k_{i1} \circ \dots \circ k_{iu_i})$ . However, the inequality can not always be replaced by the equality. Let  $k \subset R^3$  be a knot with  $g^*(k) = 0$  (i.e. a slice knot) and with non-trivial Alexander polynomial  $\Delta(t)$ . [Take, for example, the stevedore's knot  $6_1$  or the square knot  $3_1 \# -3_1^*$  etc.] For a trivial knot  $O \subset R^3$ , it follows that  $g_-(O; k) \geq 1$ . [For example,  $g_-(O; 6_1) = g_-(O; 3_1 \# -3_1^*) = 1$ .] In fact, this is established by using the following proposition:

3.5. PROPOSITION. *Let  $\ell = k_1 \circ \dots \circ k_m \subset R^3$  be a completely splittable link. If the Alexander polynomial  $\Delta'(t)$  of some complete fusion  $\ell'$  of  $\ell$  is trivial, then the Alexander polynomial  $\Delta_i(t)$  of each component  $k_i$  must be trivial.*

PROOF. It follows from the equality  $\Delta'(t) = \pm t^a F(t) F(t^{-1}) \Delta_1(t) \dots \Delta_m(t)$  for some integer  $a$  and some integral polynomial  $F(t)$  with  $F(1) = \pm 1$ . (cf. Terasaka [57], Fox-Milnor [10].)  $\square$

3.6. LEMMA. *For any link  $\ell \subset R^3$  with  $m$  components, there exists a col-*

lection of mutually disjoint bands  $\mathcal{B} = \{B_1, \dots, B_{m-1}\}$  in  $R^3$ , so that the complete fusion  $\ell = h(\ell; \mathcal{B})$  is a non-trivial knot.

PROOF. It suffices to prove this lemma in the case  $m=2$ . Let  $\ell$  be a regular projection of  $\ell \subset R^3$  in a suitably chosen hyperplane  $R_0^2 \subset R^3$ , and we take an over path on each component of  $\ell(\ell)$ , as in Fig. 7(0). Let  $k_1, k_2$  and  $k_3$  be three

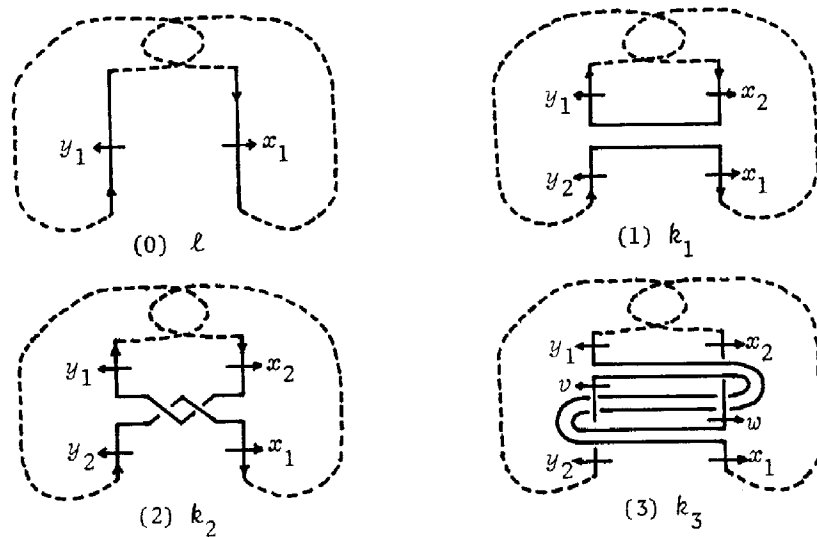


Fig. 7

fusions of  $\ell$ , illustrated in Fig. 7(1), (2) and (3), respectively. From these diagrams, we have the following finite presentations:

$$\pi_1(R^3 - k_1) = \left\langle x_1, x_2, y_1, y_2 \mid r_1, \dots, r_{n+1}, \right. \\ \left. \left. R_1^{(1)} = y_1 x_2^{-1}, R_2^{(1)} = x_2 y_2^{-1} x_1 x_2^{-1} \right\rangle,$$

$$\pi_1(R^3 - k_2) = \left\langle x_1, x_2, y_1, y_2 \mid r_1, \dots, r_{n+1}, \right. \\ \left. \left. R_1^{(2)} = x_1^{-1} y_1 x_1 x_2^{-1}, R_2^{(2)} = y_1 y_2^{-1} y_1^{-1} x_1 \right\rangle,$$

$$\pi_1(R^3 - k_3) = \left\langle x_1, x_2, y_1, y_2, v, w \mid r_1, \dots, r_{n+1}, \right. \\ \left. \left. R_1^{(3)}, R_2^{(3)}, R_3^{(3)}, R_4^{(3)} \right\rangle,$$

$$\text{with } R_1^{(3)} = y_1 v^{-1} w v y_1^{-1} x_2^{-1}, \quad R_2^{(3)} = y_1 x_1 v^{-1} w v^{-1} w^{-1} v y_1^{-1}, \\ R_3^{(3)} = x_2 y_1 w^{-1} v w^{-1} v^{-1} w x_2^{-1}, \quad R_4^{(3)} = x_2 w^{-1} v w x_1^{-1} y_2^{-1} x_1 x_2^{-1}.$$

The Alexander matrices of  $k_1, k_2$  and  $k_3$  are as follows:

	$z_1, \dots, z_n$	$x_1$	$x_2$	$y_1$	$y_2$
$r_1$	$\boxed{\mathbf{M}}$	$a_1$	$a_2$	$b_1$	$b_2$
$\vdots$					
$r_{n+1}$					
$R_1^{(1)}$	$\mathbf{O}$	0	-1	1	0
$R_1^{(1)}$		1	0	0	-1

	$z_1, \dots, z_n$	$x_1$	$x_2$	$y_1$	$y_2$
$r_1$	$\boxed{\mathbf{M}}$	$a_1$	$a_2$	$b_1$	$b_2$
$\vdots$					
$r_{n+1}$					
$R_1^{(2)}$	$\mathbf{O}$	$1-t^{-1}$	-1	$t^{-1}$	0
$R_2^{(2)}$		$t^{-1}$	0	$1-t^{-1}$	-1

	$z_1, \dots, z_n$	$x_1$	$x_2$	$y_1$	$y_2$	$v$	$w$
$r_1$	$\boxed{\mathbf{M}}$	$a_1$	$a_2$	$b_1$	$b_2$	0	0
$\vdots$							
$r_{n+1}$							
$R_1^{(3)}$	$\mathbf{O}$	0	-1	$1-t$	0	$t-1$	1
$R_2^{(3)}$		$t$	0	0	0	$1-2t$	$t-1$
$R_3^{(3)}$		0	0	$t$	0	$t-1$	$1-2t$
$R_4^{(3)}$		$1-t$	0	0	-1	1	$t-1$

Hence the Alexander polynomials  $\Delta_1(t)$ ,  $\Delta_2(t)$  and  $\Delta_3(t)$  of  $k_1$ ,  $k_2$  and  $k_3$ , respectively, are as follows:

$$\begin{aligned}\Delta_1(t) &= \alpha(t) + \beta(t), \\ \Delta_2(t) &= \alpha(t) + t\beta(t) + (t-1)\gamma(t), \\ \Delta_3(t) &= (2t^2 - 5t + 2)\alpha(t) + (2-3t)\beta(t) + 2t(1-t)\gamma(t),\end{aligned}$$

where  $\alpha(t) = \det \|M\mathbf{a}_2\|$ ,  $\beta(t) = \det \|M\mathbf{b}_1\|$  and  $\gamma(t) = \det \|M\mathbf{b}_2\|$ . We shall show that at least one of  $\Delta_1(t)$ ,  $\Delta_2(t)$  and  $\Delta_3(t)$  is not of the form  $\pm t^a$ . Suppose that all of  $\Delta_1(t)$ ,  $\Delta_2(t)$  and  $\Delta_3(t)$  are of the form  $\pm t^a$ . Let  $\mathcal{E}_i = \Delta_i(-1)$ ,  $i=1, 2, 3$ . Clearly,  $|\mathcal{E}_i| = 1$ ,  $i=1, 2, 3$ , and we have:

$$\begin{aligned}\mathcal{E}_1 &= \alpha(-1) + \beta(-1), \\ \mathcal{E}_2 &= \alpha(-1) - \beta(-1) - 2\gamma(-1), \\ \mathcal{E}_3 &= 9\alpha(-1) + 5\beta(-1) - 4\gamma(-1).\end{aligned}$$

Hence  $\mathcal{E}_3 - 2\mathcal{E}_2 = 7\alpha(-1) + 7\beta(-1) = 7\mathcal{E}_1$ . This is a contradiction, since  $|\mathcal{E}_i| = 1$ ,  $i=1, 2, 3$ . This completes the proof.  $\square$

From Corollary 2.4 and Lemma 3.6, we have the following:

3.7. PROPOSITION. Let  $\ell \subset R^3$  be a link that is not a trivial knot.  $\ell$  is

a ribbon link in the weak sense if and only if  $\ell$  is the middle cross-sectional link of a connected, non-locally flat, closed surface  $F \subset R^4$  in the normal form.  $\square$

3.8. PROPOSITION. A link  $\ell = k_1 \cup \dots \cup k_m \subset R^3$  is the upper cross-sectional link of a locally knotted, closed surface  $F = F_1 \cup \dots \cup F_m \subset R^4$  in the normal form such that the types of the locally knotted points of each component  $F_i$  are given by knots  $k_{i1}, \dots, k_{i\mu_i}$ , if and only if  $\ell$  is a fusion of a convexly and completely splittable link

$$(\mathcal{O}^{\lambda_1} \circ k_{11} \circ \dots \circ k_{1\mu_1}) \circ \dots \circ (\mathcal{O}^{\lambda_m} \circ k_{m1} \circ \dots \circ k_{m\mu_m})$$

transforming each sublink  $\mathcal{O}^{\lambda_i} \circ k_{i1} \circ \dots \circ k_{i\mu_i}$  into  $k_i$ ,  $i = 1, \dots, m$ .

SKETCH PROOF. If  $\ell = k_1 \cup \dots \cup k_m \subset R^3$  is a fusion of the completely splittable link  $\cup_{i=1}^m (\mathcal{O}^{\lambda_i} \circ k_{i1} \circ \dots \circ k_{i\mu_i})$ , it is easily checked that  $lk(k_i, k_j) = 0$  for  $i \neq j$ . Hence  $\ell[0]$  bounds a locally flat surface with  $m$  components in  $R^3(-\infty, 0]$  containing no maximal bands, (cf. Sublemma 2.7.1 and Fig. 6). Hence  $\ell$  is the upper cross-sectional link of a desired locally knotted surface in the normal form.

The converse is easy.  $\square$

The minimal genus of the surface that appears in Proposition 3.8 is denoted by  $g_+(k_1 \cup \dots \cup k_m; K_1, \dots, K_m)$ , where  $K_i = \{k_{i1}, \dots, k_{i\mu_i}\}$ .

3.9. PROPOSITION. In the above notation, it holds that

$$g_+(k_1 \cup \dots \cup k_m; K_1, \dots, K_m) \geq \sum_{i=1}^m g^*(k_{i1} \# \dots \# k_{i\mu_i}).$$

Further, there exists a link  $k_1^q \cup \dots \cup k_m^q$  establishing the equality for any given collection of knots  $K_i = \{k_{i1}, \dots, k_{i\mu_i}\}$ ,  $i = 1, \dots, m$ .  $\square$

Proposition 3.8 asserts that every knot  $k \subset R^3$  is the upper cross-sectional knot of a (possibly locally knotted) connected, closed surface  $F \subset R^4$  in the normal form. The minimal genus of such a surface is called the 4-dimensional ribbon genus of  $k$  and denoted by  $g_r^*(k)$ . More generally, for an arbitrary link  $\ell \subset R^3$ , we define  $g_r^*(\ell)$  to be the minimal genus of locally flat, connected surfaces in  $R^3[0, \infty)$  bounded by the link  $\ell[0]$  and having no minimal bands (or points).

The relations among  $g^*(\ell)$ ,  $g_r^*(\ell)$  and other numerical invariants were studied in Shibuya [53], cf. Murasugi [46], etc.

3.10. PROPOSITION. For any link  $\ell \subset R^3$  with  $\mu$  components, we have:

$$\frac{1}{2}(|\sigma(\ell)| - \mu + 1) \leq g^*(\ell) \text{ and}$$

$$\begin{array}{ccccc}
 g^*(\ell) & \leq & g_r^*(\ell) & \leq & g(\ell) \\
 \wedge & & \wedge & & \wedge \\
 c^*(\ell) & \leq & c_r^*(\ell) & \leq & c(\ell) \\
 \wedge & & \parallel & & \vee \\
 u^*(\ell) & \leq & u_r^*(\ell) & \leq & u(\ell)
 \end{array}$$

Here,  $\sigma(\ell)$  = the Murasugi signature of  $\ell$ ,  
 $g^*(\ell)$  = the 4-dimensional genus of  $\ell$ ,  
 $g_r^*(\ell)$  = the 4-dimensional ribbon genus of  $\ell$ ,  
 $g(\ell)$  = the (3-dimensional) genus of  $\ell$ ,  
 $c^*(\ell)$  = the clasp number in the 4-dimensional sense,  
 $c_r^*(\ell)$  = the clasp number in the 4-dimensional ribbon sense,  
 $c(\ell)$  = the clasp number (in the 3-dimensional sense), cf. (1.11),  
 $u^*(\ell)$  = the unlinking number in the 4-dimensional sense,  
 $u_r^*(\ell)$  = the unlinking number in the 4-dimensional ribbon sense,  
 $u(\ell)$  = the unlinking number (in the 3-dimensional sense).

See Shibuya [53] for the details and the proof.

3.11. REMARK. If a link  $\ell \subset R^3$  is the upper cross-sectional link of a (possibly non-locally flat) closed surface  $F \subset R^4$  in the normal form, then each two components of  $\ell$  has the linking number 0 by Sublemma 2.7.1. However, every link with linking number 0 on each two components is not the upper cross-sectional link of a (possibly non-locally flat) closed surface in  $R^4$  in the normal form. For example, consider the link  $\ell_0 = k_1 \cup k_2 \subset R^3$  in Fig. 8, called the Whitehead link.

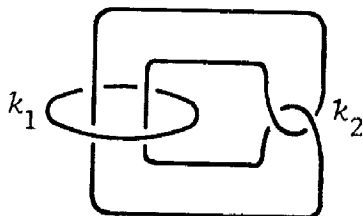


Fig. 8; the Whitehead link

It is obvious that  $lk(k_1, k_2) = 0$ , but we can prove that  $\ell_0$  never bounds in  $R^3[0, \infty)$  disjoint (piecewise-linearly embedded) proper two 2-disks. In fact, if not, by Kawauchi [33, Corollary 4.8] the Alexander polynomial  $A(t)$  of  $\ell_0$  (in the sense of [33]) would satisfy  $A(1) \neq 0$ . However,  $\ell_0$  has  $A(t) = (t-1)^3$  and so  $A(1) = 0$ . The same conclusion can be also derived from calculating the nullity of  $\ell_0$  defined by Murasugi [46], since the nullity is an invariant of topological (and hence piecewise-linear) link concordances, Kauffman-Taylor [28]. (Note that the

nullity  $n(\ell_0)=1$ , and for the trivial link  $\mathcal{O}^2$  with 2 components,  $n(\mathcal{O}^2)=2$ .  $\square$

**4. 3-Manifolds Bounded by Surfaces in  $R^4$**

Throughout this section, a surface in  $R^4=R^3(-\infty, +\infty)$  will mean a *locally flat, closed, orientable and oriented surface*.

First, we consider a surface of a special type, called a *ribbon surface*. This concept was studied by Yajima [58], [59] and Yanagawa [62].

By Corollary I.2.16, a (possibly disconnected) surface  $F \subset R^3(-\infty, +\infty)$  is ambient isotopic to a surface  $F' \subset R^3(-\infty, +\infty)$  in the normal form given by a sequence  $\mathcal{O}_- \rightarrow \ell_- \rightarrow \ell \rightarrow \ell_+ \rightarrow \mathcal{O}_+$  of oriented links.

4.1. DEFINITION. A surface  $F \subset R^3(-\infty, +\infty)$  is called a *ribbon surface* iff  $F$  is ambient isotopic to a surface  $F' \subset R^3(-\infty, +\infty)$  in the normal form given by a sequence  $\mathcal{O}_- \rightarrow \ell_- \rightarrow \ell \rightarrow \ell_+ \rightarrow \mathcal{O}_+$  such that the sequence  $\mathcal{O}_- \rightarrow \ell_- \rightarrow \ell$  is precisely the converse of the sequence  $\ell \rightarrow \ell_+ \rightarrow \mathcal{O}_+$ .

4.2. LEMMA. A ribbon surface  $F = F_1 \cup \dots \cup F_r \subset R^3(-\infty, +\infty)$  of  $r$  components bounds a 3-manifold in  $R^3(-\infty, +\infty)$  homeomorphic to the connected sum  $T_1 \# \dots \# T_r \# q(S^1 \times S^2)$  of solid tori  $T_i$  of genus  $g(F_i)$ ,  $i=1, \dots, r$ , and  $q$  copies of  $S^1 \times S^2$  for some  $q \geq 0$ .

PROOF. We may identify  $F$  with the closed realizing surface  $\bar{F}_a^a$  in  $R^3[-a, a]$  of a sequence  $\mathcal{O} \rightarrow \ell' \rightarrow \ell \rightarrow \ell' \rightarrow \mathcal{O}$ , where  $\mathcal{O} \rightarrow \ell' \rightarrow \ell$  is the converse of  $\ell \rightarrow \ell' \rightarrow \mathcal{O}$ . From the proof of Theorem 2.3, we can assume that the image  $p_+(\bar{F}_a^a \cap R^3[0, a]) = D$  of the projection  $p_+: R^3[0, a] \rightarrow R^3$ ,  $p_+(x, y, z, t) = (x, y, z)$ , is a normal singular disk whose singularity  $\Sigma$  consists of mutually disjoint

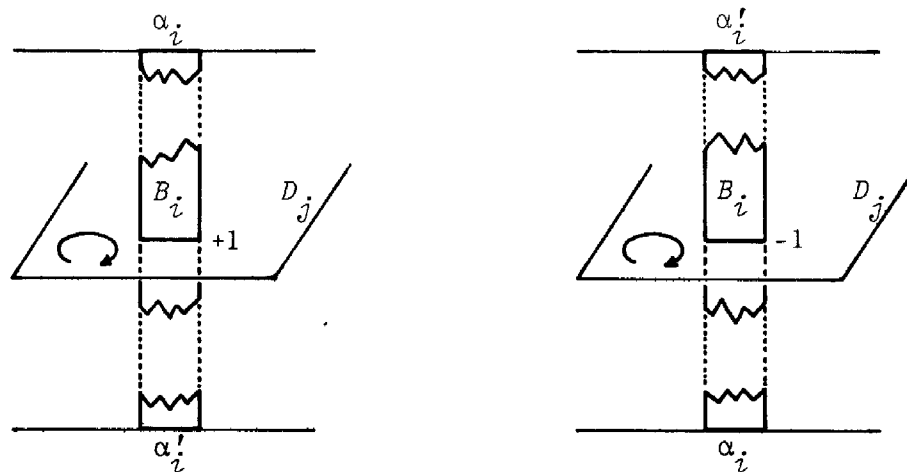


Fig. 9

simple ribbon singularities. Clearly,  $D$  consists of mutually disjoint, oriented 2-disks, say  $D_1, \dots, D_\lambda$ , with  $\partial D_1 \cup \dots \cup \partial D_\lambda = \emptyset$ , and mutually disjoint bands, say  $B_1, \dots, B_m$ , which span the trivial link  $\mathcal{O}$  with  $B_1 \cup \dots \cup B_i$  related to  $\ell \rightarrow \ell'$  and  $B_{i+1} \cup \dots \cup B_m$  related to  $\ell' \rightarrow \mathcal{O}$ . It should be noted that the  $b$ -line of each simple ribbon singularity of  $D$  is in  $B_1 \cup \dots \cup B_m$ . For each  $i, i=1, \dots, m$ , let  $\alpha_i$  and  $\alpha'_i$  be the attaching arcs of  $B_i$  to  $\mathcal{O}$ . Then we can give a sign to each simple ribbon singularity of  $D$  as in Fig. 9. We can assume that the bands  $B_1, \dots, B_m$  are satisfied with one additional condition that, for each  $i, i=1, \dots, m$ , the sum of the signs of simple ribbon singularities of  $D$  along  $B_i$  is zero. (See, for example, a modification shown in Fig. 10 below.)

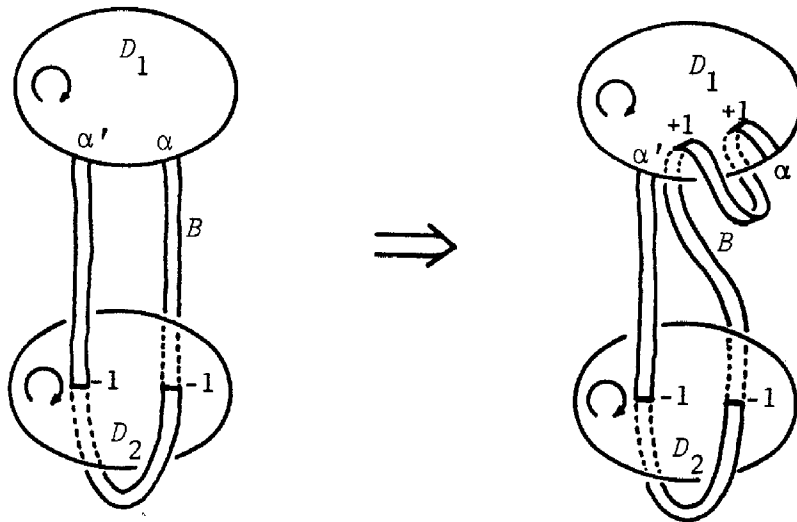


Fig. 10

The additional condition guarantees us that, by removing from  $D_1 \cup \dots \cup D_\lambda$  small neighborhoods  $\Delta_1, \dots, \Delta_{2\mu}$  of simple ribbon singularities, we can attach tubes (=annuli), say  $A_1, \dots, A_\mu$ , surrounding  $B_1, \dots, B_m$ , to the new resulting boundaries of  $Cl(D_1 \cup \dots \cup D_\lambda - (\Delta_1 \cup \dots \cup \Delta_{2\mu}))$  to obtain an oriented surface  $G = (D - (\Delta_1 \cup \dots \cup \Delta_{2\mu})) \cup (A_1 \cup \dots \cup A_\mu)$  bounded by the link  $\ell$ . See Fig. 11 for an illustration of this modification. We now re-index  $\{\Delta_1, \dots, \Delta_{2\mu}\}$  as  $\{\Delta_1, \Delta'_1, \dots, \Delta_\mu, \Delta'_\mu\}$ , so that  $\partial A_i = \partial \Delta_i \cup \partial \Delta'_i, i=1, \dots, \mu$ .

If  $G$  is disconnected, then we choose mutually disjoint 3-disks, say  $\bar{V}_1, \dots, \bar{V}_v$ , in  $R^3$  such that, for each  $j, j=1, \dots, v$ ,  $G \cap \bar{V}_j = G \cap \partial \bar{V}_j$  consists of two 2-disks, say  $\bar{A}_j \cup \bar{A}'_j$ , and  $\bar{A}_j$  and  $\bar{A}'_j$  are contained in the interiors of distinct components of  $G$ , and the resulting surface

$$\bar{G} = Cl(G - (\bar{A}_1 \cup \bar{A}'_1 \cup \dots \cup \bar{A}_v \cup \bar{A}'_v)) \cup (\bar{A}_1 \cap \dots \cap \bar{A}_v)$$

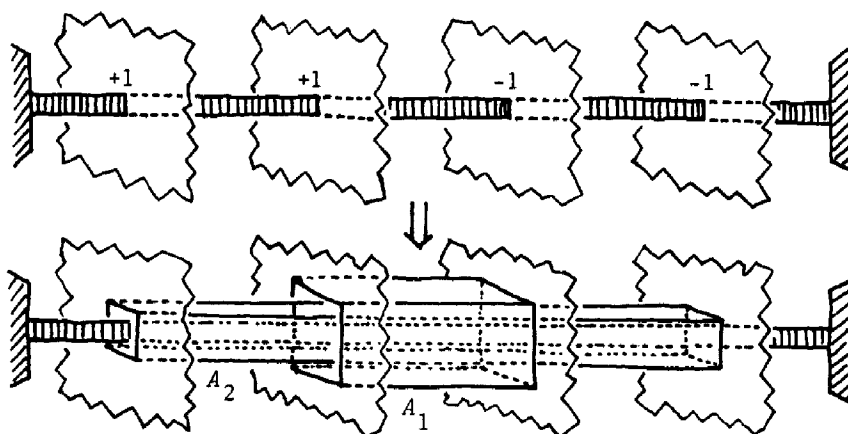


Fig. 11

is an orientable, oriented and connected surface. Here,  $\bar{A}_j = Cl(\partial \bar{V}_j - G \cap \bar{V}_j) = Cl(\partial \bar{V}_j - \bar{A}_j - \bar{A}'_j)$ ,  $j=1, \dots, v$ .

Let  $0 < t_1 < t_2 < t_3 < t_4 < t_5 < a$ , where the bands of the fusion  $\ell \rightarrow \ell'$  and the bands of the fission  $\ell' \rightarrow \ell$  occur at the levels  $t=t_2$  and  $t=t_3$ , respectively. We define a 3-manifold  $W_+ \subset R^3[0, \infty)$  as follows:

$$W_+ \cap R^3[t] = \begin{cases} \bar{G}[t] & \text{for } 0 \leq t < t_1, \\ (G \cup \bar{V}_1 \cup \dots \cup \bar{V}_v)[t] & \text{for } t = t_1, \\ G[t] & \text{for } t_1 < t \leq t_2, \\ Cl(G - (B_1 \cup \dots \cup B_l))[t] & \text{for } t_2 < t \leq t_3, \\ Cl(G - (B_1 \cup \dots \cup B_m))[t] \\ = (Cl(D_1 \cup \dots \cup D_\lambda - (A_1 \cup A'_1 \cup \dots \cup A_\mu \cup A'_\mu)) \\ \cup (A_1 \cup \dots \cup A_\mu))[t] & \text{for } t_3 < t \leq t_4. \end{cases}$$

Next, we choose an innermost 2-sphere, say  $A_1 \cup A_1' \cup A'_1$ , in the 2-spheres  $A_1 \cup A_1' \cup A'_1, A_2 \cup A_2' \cup A'_2, \dots, A_\mu \cup A_\mu' \cup A'_\mu$ . The 2-sphere  $A_1 \cup A_1' \cup A'_1$ , then, bounds a unique 3-disk  $\bar{V}_1$  in  $R^3$  with  $\bar{V}_1 \cap (A_2 \cup \dots \cup A_\mu) = \emptyset$ . Let  $0 < \varepsilon_1 < \varepsilon_2$  be sufficiently small numbers. We define:

$$W_+ \cap R^3[t] = \begin{cases} (Cl(\bar{D}_1 \cup \dots \cup D_\lambda - (A_1 \cup A'_1 \cup \dots \cup A_\mu \cup A'_\mu)) \\ \cup (A_1 \cup \dots \cup A_\mu))[t] & \text{for } t_4 < t < t_4 + \varepsilon_1, \\ Cl(D_1 \cup \dots \cup D_\lambda - (A_2 \cup A'_2 \cup \dots \cup A_\mu \cup A'_\mu)) \end{cases}$$

$$\begin{cases} \cup (A_2 \cup \dots \cup A_\mu) \cup \mathcal{V}_1 [t] & \text{for } t = t_4 + \varepsilon_1, \\ (\text{Cl}(D_1 \cup \dots \cup D_\lambda - (A_2 \cup A'_2 \cup \dots \cup A_\mu \cup A'_\mu))) \\ \cup (A_2 \cup \dots \cup A_\mu) [t] & \text{for } t_4 + \varepsilon_1 < t \leq t_4 + \varepsilon_2. \end{cases}$$

By the repetition of the procedure, we obtain, at the level  $t=t_5$ ,  $W_+ \cap R^3[t_5] = (D_1 \cup \dots \cup D_\lambda) [t]$ . So, we define  $W_+$  as follows:

$$W_+ \cap R^3[t] = \begin{cases} (D_1 \cup \dots \cup D_\lambda) [t] & \text{for } t_5 \leq t \leq a, \\ \emptyset & \text{for } t > a. \end{cases}$$

From this construction, we have:  $\partial W_+ = (\bar{F}_a^a \cap R^3[0, a]) \cup \bar{G}[0]$ .

Similarly, we can obtain a 3-manifold  $W_-$  in  $R^3[-a, 0]$  which is a copy of  $W_+$  with  $\partial W_- = (\bar{F}_a^a \cap R^3[-a, 0]) \cup \bar{G}[0]$ . Then, the union  $W = W_+ \cup W_-$  is an orientable, connected 3-manifold oriented and bounded by the surface  $\bar{F}_a^a$ . It is easily seen that  $W$  is homeomorphic to the connected sum  $T_1 \# \dots \# T_r \# q(S^1 \times S^2)$  of solid tori  $T_i$  of genus  $g(F_i)$ ,  $i=1, \dots, r$ , and  $q$  ( $\geq 0$ ) copies of  $S^1 \times S^2$ .

This completes the proof of Lemma 4.2.  $\square$

4.3. DEFINITION. Let  $W$  be a 3-manifold in  $R^4$  homeomorphic to the connected sum  $T_1 \# \dots \# T_r \# q(S^1 \times S^2)$  of solid tori  $T_i$  of genus  $g_i$ ,  $i=1, \dots, r$ , and  $q$  copies of  $S^1 \times S^2$ .  $W$  is said to be *semi-unknotted* iff  $W$  contains mutually disjoint 2-spheres  $S_1, S'_1, \dots, S_s, S'_s$ , with  $s=r+q$ , such that

(1) the surface  $S_1 \cup S'_1 \cup \dots \cup S_s \cup S'_s$  is unknotted in  $R^4$ , that is, there exist mutually disjoint 3-disks  $D_1^3, D_1'^3, \dots, D_s^3, D_s'^3$  in  $R^4$  with  $\partial D_i^3 = S_i$ ,  $\partial D_i'^3 = S'_i$ ,  $i=1, \dots, s$ , (See Definition 4.7 below.),

(2)  $S_i \cup S'_i$  bounds a spherical shell  $A_i^3 \cong S^2 \times [0, 1]$  in  $W$  such that  $A_i^3 \cap A_j^3 = \emptyset$  for  $i \neq j$ ,

(3)  $\text{Cl}(W - \cup_{i=1}^s A_i^3)$  is the disjoint union of  $T_1^q, \dots, T_r^q$  and  $S_0^3$ , where  $T_i^q$  is homeomorphic to  $T_i$  with a finite number of open 3-disks removed and  $S_0^3$  is a 3-sphere with a finite number of open 3-disks removed.

4.4. THEOREM. A surface  $F \subset R^4$  is a ribbon surface if and only if  $F$  bounds a semi-unknotted 3-manifold  $W$  in  $R^4$ .

The proof of Theorem 4.4 will be given in (4.12) below.

4.5. COROLLARY. (YANAGAWA [62]) A 2-sphere  $K^2 \subset R^4$  is a ribbon 2-sphere if and only if  $K^2$  bounds a semi-unknotted 3-manifold of a type  $D^3 \# q(S^1 \times S^2)$  ( $q \geq 0$ ) with  $D^3$  a 3-disk.  $\square$

Let  $F \subset R^4$  be a (possibly disconnected) surface. An oriented 3-disk  $B^3$  in  $R^4$  spans  $F$  as a 1-handle, iff  $B^3 \cap F = (\partial B^3) \cap F$  consists of two 2-disks and the

surface  $\text{Cl}(F \cup \partial B^3 - (\partial B^3) \cap F)$  can have an orientation compatible with both the orientations of  $F - (\partial B^3) \cap F$  and  $\partial B^3 - (\partial B^3) \cap F$ .

4.6. DEFINITION. (HOSOKAWA-KAWAUCHI [22]) If  $B_1^3, \dots, B_m^3$  are mutually disjoint, oriented 3-disks in  $R^4$  which span  $F$  as 1-handles, then the resulting closed surface  $h^1(F; B_1^3, \dots, B_m^3) = \text{Cl}(F \cup \partial B_1^3 \cup \dots \cup \partial B_m^3 - F \cap (\partial B_1^3 \cup \dots \cup \partial B_m^3))$  with the orientation induced from that of  $F - F \cap (\partial B_1^3 \cup \dots \cup \partial B_m^3)$  is called the *surface obtained from  $F$  by the hyperboloidal transformations along 1-handles  $B_1^3, \dots, B_m^3$* .

4.7. DEFINITION. Let  $F = F_1 \cup \dots \cup F_\lambda \subset R^4$  be a closed surface with  $F_i$  a component of genus  $g(F_i)$ .  $F$  is said to be *unknotted*, iff there exist mutually disjoint  $\lambda$  solid tori  $T_1, \dots, T_\lambda$  in  $R^4$  of genus  $g(F_i)$  with  $\partial T_i = F_i$ ,  $i = 1, \dots, \lambda$ . If such a collection of solid tori does not exist, then  $F$  is said to be *knotted* in  $R^4$ .

This definition is justified by the following:

4.8. LEMMA. (HOSOKAWA-KAWAUCHI [22, THEOREM 1.2])  $F \subset R^4$  is unknotted if and only if  $F$  is ambient isotopic to the boundary of a regular neighborhood of a 1-complex  $L = L_1 \cup \dots \cup L_\lambda$  in  $R^3[0]$  such that the 1-dimensional Betti number of  $L_i$  is  $g(F_i)$ ,  $i = 1, \dots, \lambda$ .  $\square$

To prove this lemma, we need the following lemma:

4.9. LEMMA. (HOSOKAWA-KAWAUCHI [22, LEMMA 1.4]) Let a 1-sphere  $S^1$  be contained in a 2-sphere  $S^2$  and consider a proper surface  $Y$  in  $S^2 \times [0, 1]$ , homeomorphic to  $S^1 \times [0, 1]$ . If  $Y \cap (S^2 \times 0) = S^1 \times 0$  and  $Y \cap (S^2 \times 1) = S^1 \times 1$ , then  $Y$  is ambient isotopic to  $S^1 \times [0, 1]$  by an isotopic deformation of  $S^2 \times [0, 1]$  keeping  $S^2 \times 0 \cup S^2 \times 1$  fixed.  $\square$

See Hosokawa-Kawauchi [22] for the proof and some topics of unknotted surfaces. We only record the following:

4.10. PROPOSITION. If a surface  $F \subset R^4$  with  $\lambda$  components is unknotted, then the fundamental group  $\pi_1(R^4 - F)$  is a free group of rank  $\lambda$ .  $\square$

4.11. LEMMA. Let  $F \subset R^4$  be an unknotted (possibly disconnected) surface. If  $F' \subset R^4$  is obtained from  $F$  by the hyperboloidal transformations along 1-handles, then  $F'$  is a ribbon surface.

PROOF. We deform  $F$  so that  $F$  is the closed realizing surface  $\bar{F}_{2,2}^2$  of a sequence  $\mathcal{O}' \rightarrow \mathcal{O} \rightarrow \mathcal{O}'$  with  $\mathcal{O}' \rightarrow \mathcal{O}$  a fission of trivial links  $\mathcal{O}'$  and  $\mathcal{O}$  and  $\mathcal{O} \rightarrow \mathcal{O}'$  the converse of  $\mathcal{O}' \rightarrow \mathcal{O}$ . Let  $B_1, \dots, B_m$  be 1-handles used for the hyperboloidal transformations  $F \rightarrow F'$  and let  $\alpha_1, \dots, \alpha_m$  be simple, proper unknotted arcs in  $B_1, \dots, B_m$ , respectively, with  $(B_i, \alpha_i) \cong (D^1 \times D^2, D^1 \times \circ)$  for  $i = 1, \dots, m$ . We can assume that, for each  $i$ , the two attaching points  $\partial \alpha_i$  of  $\alpha_i$  to  $\bar{F}_{2,2}^2$  are in the middle cross-

section  $\bar{F}^2_2 \cap R^3[0]$ . Since the inclusion homomorphism  $\pi_1(R^3[0] - R^3[0] \cap \bar{F}^2_2) \rightarrow \pi_1(R^4 - \bar{F}^2_2)$  is onto,  $\alpha_i$  is isotopic to an arc  $\alpha'_i$  in  $R^3[0]$  by an isotopic deformation of  $R^4$  keeping  $\bar{F}^2_2$  fixed. We take a regular neighborhood of  $\alpha'_i$  of the type  $B'_i[-\varepsilon, \varepsilon]$  in  $R^4$  such that  $B'_i[0] \cap \bar{F}^2_2$  consists of two simple arcs in the boundary of  $B'_i[0]$ , where  $B'_i$  is a 3-disk in  $R^3$  and  $\varepsilon$  is a sufficiently small positive number. We can assume that the 1-handles  $B_1, \dots, B_m$  are properly embedded in  $B'_1[-\varepsilon, \varepsilon], \dots, B'_m[-\varepsilon, \varepsilon]$ , respectively, such that  $B_i \cap \bar{F}^2_2 = B'_i[-\varepsilon, \varepsilon] \cap \bar{F}^2_2$ . (Take triangulations of  $\alpha_i \subset B_i \subset R^4$  so that  $\alpha_i$  is full in both  $B_i$  and  $R^4$ . Next, use the uniqueness theorem of regular neighborhood.)

Using Lemma 4.9, for a certain proper 2-disk  $B^2_i$  in  $B'_i$ , we conclude that  $\text{Cl}(\partial B_i - (\partial B_i) \cap \bar{F}^2_2)$  is ambient isotopic to  $\text{Cl}(\partial(B^2_i[-\varepsilon, \varepsilon]) - \partial(B^2_i \times [-\varepsilon, \varepsilon]) \cap \bar{F}^2_2)$  by an isotopic deformation of  $\partial(B^2_i[-\varepsilon, \varepsilon])$  keeping two 3-disks containing  $\partial(B^2_i[-\varepsilon, \varepsilon]) \cap \bar{F}^2_2$  fixed. Using a collar of  $\partial(B^2_i[-\varepsilon, \varepsilon])$  in  $R^4$ , it follows that  $\text{Cl}(\partial B_i - (\partial B_i) \cap \bar{F}^2_2)$  is ambient isotopic to  $\text{Cl}(\partial(B^2_i[-\varepsilon, \varepsilon]) - \partial(B^2_i[-\varepsilon, \varepsilon]) \cap \bar{F}^2_2)$  by an isotopic deformation of  $R^4$  keeping  $\bar{F}^2_2$  fixed. Hence  $F'$  is regarded as the surface obtained from  $\bar{F}^2_2$  by the hyperboloidal transformations along the 1-handles  $B^2_1[-\varepsilon, \varepsilon], \dots, B^2_m[-\varepsilon, \varepsilon]$ . This shows that  $F'$  is a ribbon surface, and this completes the proof of Lemma 4.11.  $\square$

4.12. PROOF of Theorem 4.4. The 3-manifold constructed in the proof of Lemma 4.2 is certainly a semi-unknotted 2-manifold. Conversely, we assume that a surface  $F \subset R^4$  bounds a semi-unknotted 3-manifold  $W$  as in Definition 4.3. Let  $(\partial T_i) \cap F = F_i$  and  $\partial T_i - F_i = S_1^{(i)} \cup \dots \cup S_{s_i}^{(i)}$ ,  $S_j^{(i)}$  a 2-sphere. Let  $\tilde{T}_i$  be a solid torus in  $T_i$  such that  $\text{Cl}(T_i - \tilde{T}_i) = F_i \times [0, 1]$  and  $\tilde{T}_i \cap S_j^{(i)} = \emptyset$ ,  $j = 1, \dots, s_i$ . Let  $\tilde{F}_i = \partial \tilde{T}_i$ . We regard  $\tilde{F}_i$  as the surface obtained from the boundary  $\partial \Delta_i^3$  of a 3-simplex  $\Delta_i^3$  in  $\text{Int}(T_i)$  by the hyperboloidal transformations along 1-handles  $V_1, \dots, V_{g_i}$  in  $\text{Int}(T_i)$ . We choose mutually disjoint, simple, proper arcs  $\beta_1, \dots, \beta_{s_i}$  in  $\text{Cl}(T_i - \tilde{T}_i)$  such that, for each  $j$ ,  $\beta_j$  spans  $\partial \Delta_i^3 - (V_1 \cup \dots \cup V_{g_i})$  and  $S_j^{(i)}$ , and then we take in  $\text{Cl}(T_i - \tilde{T}_i)$  a small regular neighborhood  $U_j$  of  $\beta_j$  meeting the boundary regularly, with  $U_j \cap U_k = \emptyset$  for  $j \neq k$ . Then  $\text{Cl}(T_i - \tilde{T}_i - U_1 - \dots - U_{s_i})$  is homeomorphic to  $F_i \times [0, 1]$ . Let  $F'_i = \partial \text{Cl}(T_i - \tilde{T}_i - U_1 - \dots - U_{s_i}) - F_i$ . Since  $F = F_1 \cup \dots \cup F_r$  is ambient isotopic to  $F'_1 \cup \dots \cup F'_r$  and  $F'_1 \cup \dots \cup F'_r$  is obtained from a trivial surface  $\cup_{i=1}^r \partial \Delta_i^3 \cup S_1^{(i)} \cup \dots \cup S_{s_i}^{(i)}$  by the hyperboloidal transformations along 1-handles. By Lemma 4.11,  $F'$  is a ribbon surface, and completing the proof.  $\square$

From Theorem 4.4 (4.12) and Lemma 4.11, we also have the following characterization of ribbon surfaces:

4.13. COROLLARY.  $F \subset R^4$  is a ribbon surface if and only if  $F$  is obtained from an unknotted surface consisting of 2-spheres by hyperboloidal transformations along 1-handles.  $\square$

It will be noticed that an unknotted surface in  $R^4$  is a consequence of an unknotted surface consisting of 2-spheres by hyperboloidal transformations along 1-handles.

4.14. COROLLARY. For a ribbon surface  $F \subset R^4$  with non-zero genus, the inclusion homomorphism  $\pi_1(\partial N(F)) \rightarrow \pi_1(\text{Cl}(R^4 - N(F)))$  is not injective, where  $N(F)$  is a regular neighborhood of  $F$  in  $R^4$ .  $\square$

Combining this with the following, we have infinitely many non-ribbon surfaces in  $R^4$ .

4.15. PROPOSITION (ASANO [1], LIVINGSTON [39]). There exist infinitely many surfaces  $F \subset R^4$  (of genus 1) having the property that the inclusion homomorphism  $\pi_1(\partial N(F)) \rightarrow \pi_1(\text{Cl}(R^4 - N(F)))$  is injective.  $\square$

For general surfaces in  $R^4$ , we have the following weaker result, corresponding to Theorem 4.4. (cf. Suzuki [56, §5])

4.16. THEOREM (GLUCK [11]). An orientable surface  $F \subset R^4$  bounds a compact orientable 3-manifold in  $R^4$ .

PROOF. From Lemmas I.1.14 and I.2.8, we may consider that  $F$  is the closed realizing surface  $\bar{F}_a^b$  of a sequence  $\mathcal{O} \rightarrow \mathcal{O}'$  of trivial links  $\mathcal{O}$  and  $\mathcal{O}'$ , and let  $\underline{F}_a^b$  be the lower closed realizing surface, that is  $\underline{F}_a^b$  is the closure of  $\bar{F}_a^b \cap R^3[a, b]$  in  $R^4$ . We may set  $\mathcal{O}' = h(\mathcal{O}; B_1, \dots, B_m)$ . Let  $D$  be the image of  $\underline{F}_a^b$  under the projection  $p: R^3[a, b] \rightarrow R^3$ . With suitable modifications of bands (see, for example, Fig. 7 of [1]), we can assume that  $D$  is a normal singular surface whose singularity  $\Sigma$  consists of mutually disjoint, simple, ribbon singularities. In fact,  $D$  is the union of mutually disjoint, oriented 2-disks  $D_1, \dots, D_\lambda$  with  $\cup_{i=1}^\lambda \partial D_i = \mathcal{O}$  and the bands  $B_1, \dots, B_m$  spanning  $\mathcal{O}$ . As we have seen in the proof of Lemma 4.2, we can assume that the sum of the signs of simple ribbon singularities of  $D$  along the band  $B_i$  for each  $i$  is zero. In particular, the number of simple ribbon singularities of  $D$  is even, say  $2\mu$ . For small regular neighborhoods  $A_1, \dots, A_{2\mu}$  of these singularities in  $D_1 \cup \dots \cup D_\lambda$ , there exist  $\mu$  tubes  $A_1, \dots, A_\mu$  attached to the new resulting boundaries of  $\text{Cl}(D - A_1 - \dots - A_{2\mu})$  such that the union  $G = (D - A_1 - A'_1 - \dots - A_\mu - A'_\mu) \cup (A_1 \cup \dots \cup A_\mu)$  is an orientable surface bounded by the trivial link  $\mathcal{O}'$ , where  $A_1, A'_1, \dots, A_\mu, A'_\mu$  are the same as  $A_1, \dots, A_{2\mu}$ , but re-indexed so that  $\partial A_i = \partial A_i \cup \partial A'_i$ ,  $i = 1, \dots, \mu$ . We assume that at the level  $t = t_0$  with  $a < t_0 < b$ , the bands of  $\mathcal{O} \rightarrow \mathcal{O}'$  occur. A required 3-manifold, say  $W$ , bounded by  $\bar{F}_a^b$  is constructed as follows: Consider the mutually disjoint 2-spheres  $A_1 \cup A_1' \cup A'_1, \dots, A_\mu \cup A_\mu' \cup A'_\mu$  in  $R^3$ . Find an innermost 2-sphere, say  $A_1 \cup A_1' \cup A'_1$ , in these 2-spheres, and let  $V_1$  be the 3-disk in  $R^3$  bounded by  $A_1 \cup A_1' \cup A'_1$ . Let  $\epsilon_1$  be a sufficiently small positive number, and we take

$$W \cap R^3[a] = (D_1 \cup \cdots \cup D_\lambda \cup F_1)[a],$$

$$W \cap R^3[t] = ((D_1 \cup \cdots \cup D_\lambda - A_1 - A'_1) \cup A_1)[t] \quad \text{for } a < t \leq a + \varepsilon_1.$$

Repeating of this procedure, we obtain, at the level  $t$  with  $a + \varepsilon < t < t_0$ , where  $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_\lambda$ ,

$$W \cap R^3[t] = ((D_1 \cup \cdots \cup D_\lambda - A_1 - A'_1 - \cdots - A_\mu - A'_\mu) \cup (A_1 \cup \cdots \cup A_\mu))[t].$$

At  $t = t_0$ , we have

$$W \cap R^3[t_0] = ((D - A_1 - A'_1 - \cdots - A_\lambda - A'_\lambda) \cup (A_1 \cup \cdots \cup A_\mu))[t_0] = G[t_0].$$

Let  $D'_1, \dots, D'_\nu$  be mutually disjoint  $\nu$  2-disks in  $R^3$  with  $\bigcup_{i=1}^\nu \partial D'_i = \emptyset'$ . Consider the intersection  $G \cap (D'_1 \cup \cdots \cup D'_\nu)$ , which consists of mutually disjoint simple curves. (See Fig. 4 of [I].) Let  $c_1$  be an innermost curve on  $D'_1 \cup \cdots \cup D'_\nu$  in the intersection curves and  $d_1 \subset D'_1 \cup \cdots \cup D'_\nu$  be the 2-disk cut off by  $c_1$  with  $\text{Int}(d_1) \cap (D'_1 \cup \cdots \cup D'_\nu) = \emptyset$ . Let  $\bar{d}_1$  be a 3-disk obtained by thickening  $d_1$  such that  $\bar{d}_1 \cap G = (\partial \bar{d}_1) \cap G$  is an annulus and  $\bar{d}_1 \cup (D'_1 \cup \cdots \cup D'_\nu) = d_1$ . Let  $\varepsilon'_2 > \varepsilon'_1 > 0$  be sufficiently small numbers. Now we define as follows:

$$W \cap R^3[t] = G[t] \quad \text{for } t_0 < t < t_0 + \varepsilon'_1,$$

$$W \cap R^3[t_0 + \varepsilon'_1] = (G \cup \bar{d}_1)[t_0 + \varepsilon'_1],$$

$$W \cap R^3[t] = G_1[t] \quad \text{for } t_0 + \varepsilon'_1 < t \leq t_0 + \varepsilon'_2,$$

where  $G_1 = (G - (\partial \bar{d}_1) \cap G) \cup \text{Cl}(\partial \bar{d}_1 - (\partial \bar{d}_1) \cap G)$ , and it will be noticed that  $G_1 \cap (D'_1 \cup \cdots \cup D'_\nu) = G \cap (D'_1 \cup \cdots \cup D'_\nu) - c_1$ .

Repeating of this procedure, we obtain the manifold  $W \cap R^3[a, t_0 + \varepsilon']$ ,  $t_0 < t_0 + \varepsilon' < b$ , such that  $W \cap R^3[t_0 + \varepsilon'] = G'[t_0 + \varepsilon']$ , where  $G'$  is an oriented surface with  $\partial G' = \emptyset'$  and  $G' \cap \text{Int}(D'_1 \cup \cdots \cup D'_\nu) = \emptyset$ .

Now we need the following proposition due to R. H. Fox [6]; compare Proposition 4.15.

4.17. PROPOSITION. *For any closed orientable (possibly disconnected) surface  $f$  in  $R^3$  with non-zero genus, there exists a 2-disk  $e^2$  in  $R^3$  such that  $e^2 \cap f = \partial e^2$  and this loop  $\partial e^2$  does not bound any 2-disk on  $f$ .  $\square$*

We apply Proposition 4.17 to the closed surface  $G_0 = G' \cup D'_1 \cup \cdots \cup D'_\nu$ . If  $g(G_0) \neq 0$ , then we can find a 2-disk  $e^2$  in  $R^3$  such that  $e^2 \cap G_0 = \partial e^2 \subset \text{Int}(G')$  and  $\partial e^2$  does not bound any 2-disk on  $G'$ . Let  $\bar{e}$  be a 3-disk obtained by thickening  $e^2$  such that  $\bar{e} \cap G' = (\partial \bar{e}) \cap G'$  is an annulus. Let  $\varepsilon''_2 > \varepsilon''_1 > 0$  be sufficiently small numbers with  $t_0 + \varepsilon' < t_0 + \varepsilon''_1 < t_0 + \varepsilon''_2 < b$ , and we define as follows:

$$W \cap R^3[t] = G'[t] \quad \text{for } t_0 + \varepsilon' < t < t_0 + \varepsilon''_1,$$

$$W \cap R^3[t_0 + \varepsilon'_1] = (G' \cup \bar{e})[t_0 + \varepsilon'_1],$$

$$W \cap R^3[t] = G'_1[t] \quad \text{for } t_0 + \varepsilon'_1 < t \leq t_0 + \varepsilon'_2,$$

where  $G'_1 = (G' - \bar{e} \cap G') \cup \text{Cl}(\partial\bar{e} - (\partial\bar{e}) \cap G')$ , and it should be noted that  $\chi(G'_1) = 2 + \chi(G')$ , where  $\chi$  denotes the Euler characteristic.

Repeating of this procedure, finally we obtain the manifold  $W \cap R^3[a, t_0 + \varepsilon'']$ ,  $t_0 + \varepsilon' < t_0 + \varepsilon'' < b$ , such that  $W \cap R^3[t_0 + \varepsilon''] = G''[t_0 + \varepsilon'']$ , where  $g(G'') = 0$  and  $G'' \cap \text{Int}(D'_1 \cup \dots \cup D'_v) = \emptyset$ .

Since  $G''_0 = G'' \cup D'_1 \cup \dots \cup D'_v$  consists of mutually disjoint 2-spheres in  $R^3$ , we can find an innermost 2-sphere, say  $\Sigma_1$ , in  $G''_0$ , which bounds a 3-disk  $\sigma_1$  in  $R^3$  such that  $\text{Int}(\sigma_1) \cap G''_0 = \emptyset$ . Let  $\varepsilon_2^* > \varepsilon_1^* > 0$  be sufficiently small numbers with  $t_0 + \varepsilon'' < t_0 + \varepsilon_1^* < t_0 + \varepsilon_2^* < b$ , and let

$$W \cap R^3[t] = G''[t] \quad \text{for } t_0 + \varepsilon'' < t < t_0 + \varepsilon_1^*,$$

$$W \cap R^3[t_0 + \varepsilon_1^*] = (G'' \cup \sigma_1)[t_0 + \varepsilon_1^*],$$

$$W \cap R^3[t] = (G'' - \Sigma_1)[t] \quad \text{for } t_0 + \varepsilon_1^* < t \leq t_0 + \varepsilon_2^*.$$

Repeating of this procedure, we obtain finally  $W \cap R^3[t_0 + \varepsilon^{**}] = \emptyset$  for a level  $t = t_0 + \varepsilon^{**}$  with  $t_0 + \varepsilon'' < t_0 + \varepsilon^{**} < b$ . By the Cellular Move Lemma (Proposition 1.1.7),  $\partial W$  is ambient isotopic to  $\bar{F}_a^b$ . Hence  $F$  bounds a 3-manifold in  $R^3[a, b]$  ambient isotopic to  $W$ , and this completes the proof of Theorem 4.16.  $\square$

We call a compact, oriented (connected) 3-manifold  $W \subset R^4$  a *Seifert manifold* for the surface  $F \subset R^4$ , provided  $\partial W = F$ .

It is well-known that any compact, connected and orientable 3-manifold  $W^3$  with non-empty boundary can be represented as  $T \cup h^2(D_1) \cup \dots \cup h^2(D_k)$  where  $T$  is a solid-torus,  $D_i$  is a 2-disk for each  $i$  and  $h^2(D_i) \cong D^2 \times D^1$  is a 2-handle along the core  $D_i$ . This representation will be called a *Heegaard splitting* for  $W^3$ , and the *genus* of the splitting is the genus of  $T$  and the smallest possible genus of Heegaard splitting for  $W^3$  will be denoted by  $Hg(W^3)$ .

By Theorem 4.16 and Definition 4.7, we have the following:

4.18. THEOREM (HOSOKAWA-KAWAUCHI [22, THEOREM 2.3]). *For any surface  $F$  in  $R^4$ , there exists a finite number of 1-handles  $B_1, \dots, B_u$  such that the surface  $h^1(F; B_1, \dots, B_u)$  is an unknotted surface of genus  $g(F) + u$ .  $\square$*

4.19. DEFINITION ([23]). The *unknotting number*,  $u(F)$ , of a surface  $F$  in  $R^4$  is the least number of 1-handles  $B_1, \dots, B_u$  for  $F$  so that  $h^1(F; B_1, \dots, B_u)$  is unknotted.

4.20. PROPOSITION ([23]). *Let  $F \subset R^4$  be a connected surface of genus*

$g(F)$ , and let  $W$  be a Seifert manifold for  $F$ . Then it holds that:

$$0 \leq u(F) \leq Hg(W) - g(F) < \infty. \quad \square$$

4.21. THEOREM ([23]). For arbitrary non-negative integers  $n$  and  $u$ , there exists a connected surface  $F$  in  $R^4$  with  $g(F)=n$ ,  $u(F)=u$ .  $\square$

See Hosokawa-Maeda-Suzuki [23] for the proof and some related topics of unknotting number of surfaces in  $R^4$ .

## 5. Some Topics of 2-Knots

By a 2-knot, we mean a locally flat 2-sphere in  $R^4$  (or in  $S^4$ ), considered from the piecewise-linear point of view. General references to the 2-knot theory are Fox [7], Suzuki [56] and Kervaire-Weber [35]. In this section, we shall give some topics of 2-knots related to our theme.

We reformulate the results in §4 on ribbon 2-knots.

5.1. THEOREM. The followings are equivalent:

- (1) A 2-knot  $K \subset R^4$  is a ribbon 2-knot,
- (2) A 2-knot  $K \subset R^4$  has a semi-unknotted Seifert manifold in  $R^4$ ,
- (3) A 2-knot  $K \subset R^4$  is obtained from an unknotted surface consisting of 2-spheres by the hyperboloidal transformations along 1-handles.  $\square$

5.2. EXAMPLES. (a) The 2-knot obtained by spinning an arbitrary knot is a ribbon 2-knot. (See Fox-Milnor [10], etc...)

(b) The 2-knots constructed by Sumners [55] are ribbon 2-knots. (See Omae [50] for the proof, and see also Asano-Marumoto-Yanagawa [2].)

The fundamental group of the complementary domain  $R^4 - K$  of a 2-knot  $K \subset R^4$  is usually called a 2-knot group. Kervaire [34] discussed the characterization problem of knot groups. Cf. Levine [36], Suzuki [56].

5.3. THEOREM (KERVAIRE [34]). If a group  $G$  is a 2-knot group, then

- (0)  $G$  is a finitely presented group,
- (1) the abelianized group  $G/G'$  is infinite cyclic,
- (2) there exists an element  $\mu \in G$  with  $G/\langle\langle \mu \rangle\rangle = 1$ ,
- (3) the second homology group  $H_2(G; \mathbf{Z}) = 1$ .  $\square$

In fact, Kervaire [34, Theorem 1] asserted that these conditions (0), (1), (2) and (3) are sufficient conditions for a group  $G$  to be an  $n$ -knot group with  $n \geq 3$ . However, the problem of characterizing 2-knot groups by algebraic conditions remains unsolved. The following characterization of ribbon 2-knot groups was given by Yajima [59], [60].

5.4. THEOREM (YAJIMA [59], [60]). *A group  $G$  is a ribbon 2-knot group if and only if*

- (1) *the abelianized group  $G/G'$  is infinite cyclic,*
- (4)  *$G$  has a finite Wirtinger presentation of deficiency 1.  $\square$*

A finite presentation  $\langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$  of a group  $G$  is called a *Wirtinger presentation* iff each relator  $r_i$  has the form  $x_{p_i} w_i x_{q_i}^{-1} w_i^{-1}$ , where  $w_i$  is a word in  $x_1, \dots, x_n$ . The condition (4) implies the conditions (0), (1) and (3) of Theorem 5.3. Levine [36] gave a sufficient condition for a group  $G$  to be a 2-knot group, but as Yoshikawa [69] pointed out, this condition is equivalent to Yajima's conditions and the 2-knots constructed in [36] are ribbon 2-knots.

5.5. COROLLARY. *For a ribbon 2-knot group  $G$ , the abelianized commutator subgroup  $G'/G''$  is a torsion-free abelian group and the first elementary ideal  $\mathcal{E}_1$  of  $G$  is principal.  $\square$*

This corollary follows from Theorem 5.4 and the following somewhat general lemma:

5.6. LEMMA. *Let  $G$  be a group with a finite presentation of deficiency 1. If  $G/G'$  is infinite cyclic, then  $G'/G''$  is a torsion-free abelian group and the first elementary ideal  $\mathcal{E}_1$  of  $G$  is principal.*

PROOF. Let  $\langle t, x_1, \dots, x_n \mid r_1, \dots, r_n \rangle$  be a presentation of  $G$  of deficiency 1. We can assume that the epimorphism  $\gamma: G \rightarrow \langle t \rangle$  satisfies  $\gamma(x_i) = 1$  for  $i = 1, \dots, n$ , and  $\gamma(t) = t$ , where  $\langle t \rangle$  is the infinite cyclic group generated by the symbol  $t$ . (Take a pre-abelian presentation of  $G$ . See Magnus-Karras-Solitar [42, pp. 140-149].) Then the  $n \times n$  matrix  $\|\gamma(\partial r_i / \partial x_j)\|$  obtained by the Fox free calculus is a presentation matrix of the  $\mathbf{Z}\langle t \rangle$ -module  $G'/G''$ . (See Kawauchi [32, Lemma 2.6].) Here  $\mathbf{Z}\langle t \rangle$  denotes the integral group ring of the group  $\langle t \rangle$ . The first elementary ideal  $\mathcal{E}_1$  of  $G$  is generated by the so-called Alexander polynomial  $\Delta(t) = \det \|\gamma(\partial r_i / \partial x_j)\|$  of  $G$ . Hence  $\mathcal{E}_1$  is a principal ideal. Further,  $|\Delta(1)| = 1$ , since  $G/G'$  is infinite cyclic. (In fact, notice that the integral matrix  $\|\gamma(\partial r_i / \partial x_j)\|_{t=1}$  is a presentation matrix of  $(G'/G'') \otimes_{t=1} \mathbf{Z} = 0$ . See, for example, Kawauchi [32, Lemma 2.7].) This implies that  $G'/G''$  is a torsion-free abelian group by Crowell [5, Theorem 1.3]. This completes the proof.  $\square$

5.7. COROLLARY. *For a ribbon 2-knot group  $G$ ,  $G/G''$  is a torsion-free group.*

PROOF. This follows from Corollary 5.5 and the short exact sequence  $1 \rightarrow G'/G'' \rightarrow G/G'' \rightarrow G/G' \rightarrow 1$ . (See Marumoto [44] and Hitt [18].)  $\square$

5.8. QUESTION OF T. YANAGAWA. *Is a ribbon 2-knot group necessarily*

*torsion-free?*

Asano-Marumoto-Yanagawa [2] discussed this question and gave an affirmative answer in [2, Corollary 3.8], but they did not give the proof of [2, Proposition 3.2].

5.9. THEOREM (YAJIMA [59]). *There exist infinitely many 2-knots that are not ribbon 2-knots.*

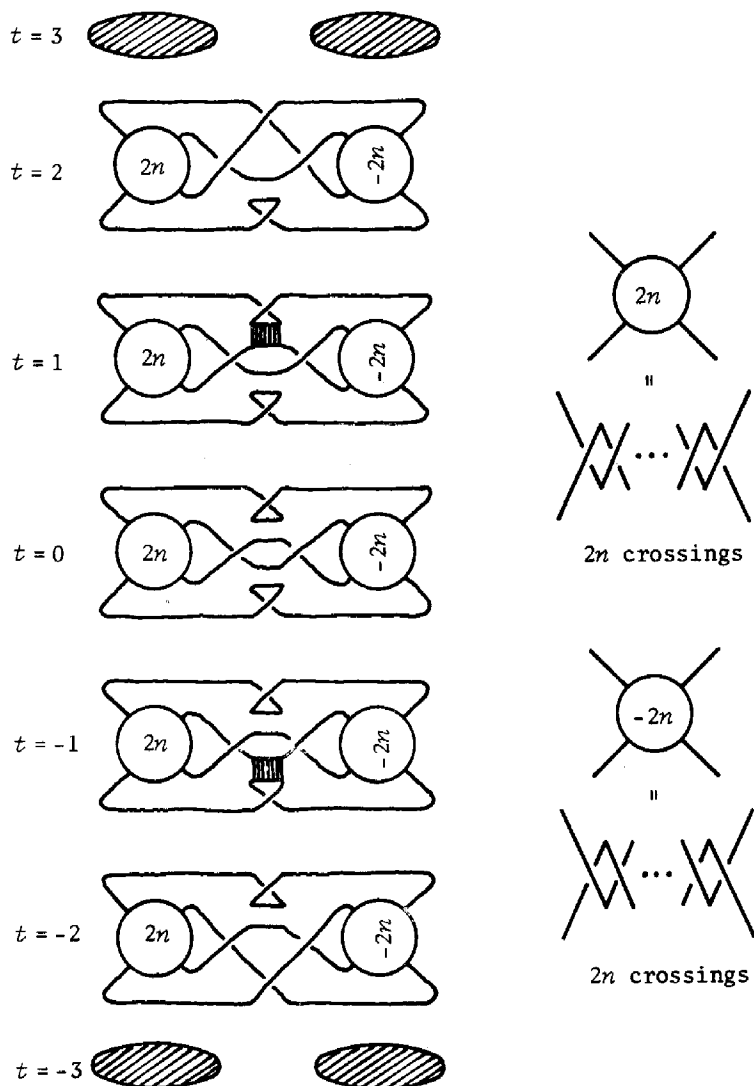


Fig. 12

PROOF. Consider the 2-knots  $K_n \subset R^4$ ,  $n = 1, 2, 3, \dots$ , due to Fox [7, Example 15]), illustrated in Fig. 12. The 2-knot group  $G_n = \pi_1(R^4 - K_n)$  has a presentation  $\langle t, b \mid b^{2n+1} = 1, tbt^{-1} = b^{-1} \rangle$ . Hence the commutator subgroup  $G'_n$  is a cyclic group of order  $2n + 1$ . Therefore,  $G_n$  is not a ribbon 2-knot group by Corollary 5.5.  $\square$

5.10. REMARKS. (a) Kanenobu [27] showed that Fox's 2-knots given in Fig. 12 are 2-twist spun knots [70], see also Litherland [37].

(b) Kanenobu [26] asserted, under an affirmative answer to 5.8, that there exists a non-ribbon 2-knot whose Seifert manifold is homeomorphic to  $D^3 \#_q(S^1 \times S^2)$ . Recall Definition 4.3 and Corollary 4.5.  $\square$

On the second homology group  $H_2(\pi_1(R^4 - F); \mathbf{Z})$  of  $\pi_1(R^4 - F)$  for a closed orientable surface  $F \subset R^4$ , we have the following:

5.11. PROPOSITION.

(1) (MABDA [41]) For the group  $A_5 \times \mathbf{Z}$ ,  $A_5$  the alternating group of degree 5, there exists a locally flat closed surface  $F \subset R^4$  with  $\pi_1(R^4 - F) \cong A_5 \times \mathbf{Z}$  and  $H_2(A_5 \times \mathbf{Z}; \mathbf{Z})$  has order 2.

(2) (GORDON [13]) For any finitely generated abelian group  $A$ , there exists a locally flat closed surface  $F \subset R^4$  with  $H_2(\pi_1(R^4 - F); \mathbf{Z}) \cong A$ .

(3) (LITHERLAND [38]) For any abelian group  $A$  generated by  $2g$  elements, there exists a locally flat closed surface  $F \subset R^4$  of genus  $2g$  such that  $H_2(\pi_1(R^4 - F); \mathbf{Z}) \cong A$ .  $\square$

We also refer the reader to Simon [54], Kanenobu [25], Hillman [17], Yoshikawa [67], [68] for 2-knot groups.

A 2-knot  $K \subset R^4$  (or  $S^4$ ) is said to be *unknotted*, iff there exists a 3-disk  $B^3 \subset R^4$  (or  $S^4$ ) with  $\partial B^3 = K$ ; recall Definition 4.7. (This is equivalent to saying that  $K$  is ambient isotopic to the boundary of a 3-simplex in a triangulated  $R^4$  by the Cellular Move Lemma (Proposition I.1.7).)

The unknotting conjecture is usually stated as follows:

5.12. UNKNOTTING CONJECTURE. For a 2-knot  $K \subset S^4$ , the complement  $S^4 - K$  is homotopy equivalent to the circle  $S^1$  if and only if  $K \subset S^4$  is unknotted.

REMARK. YANAGAWA [63] showed that the unknotting conjecture is true for ribbon 2-knots, however the second-step of the proof of [63, (2.2)'] is incorrect. Thus, the unknotting conjecture is unsettled even for ribbon 2-knots.

5.13. THEOREM (MARUMOTO [43]). Let  $K \subset R^4$  be a ribbon 2-knot obtained from an unknotted surface consisting of two 2-spheres by the hyper-

*boloidal transformation along a 1-handle. If  $\pi_1(R^4 - K)$  is infinite cyclic, then  $K \subset R^4$  is unknotted.*  $\square$

5.14. FOX-HOSOKAWA CONJECTURE FOR UNKNOTTING 2-KNOTS. *A 2-knot  $K \subset R^4$  is unknotted, if  $K$  is ambient isotopic to a 2-knot in the normal form whose middle cross-sectional knot is unknotted.*

REMARK. One of the simplest case of 5.14 was given by Hosokawa [20]. However, it has a gap in the proof of [20, Lemma 2].

We say that a 2-knot satisfying the assumption of 5.14 is a 2-knot of Fox-Hosokawa type. It is easily seen that a 2-knot  $K \subset R^4$  of Fox-Hosokawa type has the infinite cyclic knot group;  $\pi_1(R^4 - K) \cong \pi_1(S^1)$ , Cf. for example, Suzuki [56, §3].

5.15. QUESTION. *If  $\pi_1(R^4 - K) \cong \pi_1(S^1)$ , then is the 2-knot  $K \subset R^4$  of Fox-Hosokawa type?*

The Fox-Hosokawa Conjecture 5.14 is a special case of the Unknotting Conjecture 5.12, since the following theorem shows that, for a 2-knot  $K \subset R^4$  of Fox-Hosokawa type, the complement  $R^4 \cup \{\infty\} - K$  is homotopy equivalent to a circle.

5.16. THEOREM (KAWAUCHI [30]). *For a 2-knot  $K \subset S^4$  if  $\pi_1(S^4 - K) \cong \pi_1(S^1)$ , then the complement  $S^4 - K$  is homotopy equivalent to  $S^1$ .*

PROOF. Let  $N$  be a regular neighborhood of  $K$  in  $S^4$ . Since  $K$  is locally flat in  $S^4$ ,  $N$  is homeomorphic to  $S^2 \times D^2$ , and so the boundary of  $E = \text{Cl}(S^4 - N)$  is homeomorphic to  $S^2 \times S^1$ . We take the universal cover  $\tilde{E}$  of  $E$ , which is obviously an infinite cyclic cover.  $H_2(E; \mathbf{Z}) = 0$  implies that  $H_2(\tilde{E}; \mathbf{Q})$  is finitely generated over the rational number field  $\mathbf{Q}$ . Then by the partial Poincaré duality theorem [31], there is a duality  $\cap \mu: H^i(\tilde{E}; \mathbf{Z}) \cong H_{3-i}(\tilde{E}, \partial\tilde{E}; \mathbf{Z})$ ,  $i=0, 1$ . (Note that  $H_1(\tilde{E}; \mathbf{Z}) = 0$ , since  $\tilde{E}$  is simply connected.) Hence  $H_2(\tilde{E}, \partial\tilde{E}; \mathbf{Z}) = 0$  and  $H_3(\tilde{E}, \partial\tilde{E}; \mathbf{Z}) = \mathbf{Z}$ . As shown in Theorem 4.16, one can find a compact oriented proper 3-manifold  $V \subset E$  such that  $\partial V = S^2 \times x \subset S^2 \times S^1 = \partial E$  for a point  $x \in S^1$ . Let  $\tilde{V} \subset \tilde{E}$  be a lift of  $V$ . (Such a  $\tilde{V}$  certainly exists, since the inclusion  $V \subset E$  induces the trivial homomorphism  $H_1(V; \mathbf{Z}) \rightarrow H_1(E; \mathbf{Z}) = \pi_1(E)$ .) The boundary homomorphism  $\partial: H_3(\tilde{E}, \partial\tilde{E}; \mathbf{Z}) \rightarrow H_2(\partial\tilde{E}; \mathbf{Z})$  sends the homology class  $[\tilde{V}]$  onto a generator  $[\partial\tilde{V}]$  of  $H_2(\partial\tilde{E}; \mathbf{Z}) \cong \mathbf{Z}$ . Since  $H_3(\tilde{E}, \partial\tilde{E}; \mathbf{Z}) \cong \mathbf{Z}$ , the boundary homomorphism  $\partial: H_3(\tilde{E}, \partial\tilde{E}; \mathbf{Z}) \rightarrow H_2(\partial\tilde{E}; \mathbf{Z})$  is an isomorphism. From the exact sequence of the pair  $\partial\tilde{E} \subset \tilde{E}$ , it follows that  $\tilde{H}_*(\tilde{E}; \mathbf{Z}) = 0$ . Since  $\tilde{E}$  is simply connected,  $\tilde{E}$  is contractible, and hence  $E$  is homotopy equivalent to  $S^1$ . This completes the proof.  $\square$

6. Slices and Ribbons

6.1. DEFINITION. A knot  $\mathcal{K} \subset R^3 = R^3[0]$  is called a *slice knot*, iff there exists a 2-knot  $K^2 \subset R^4$  with  $\mathcal{K} = K^2 \cap R^3[0]$ .

Equivalently, a knot  $\mathcal{K} \subset R^3 = R^3[0]$  is a slice knot, iff there is a locally flat, proper 2-disk  $E \subset R^3[0, +\infty)$  with  $\partial E = \mathcal{K}$ . By Corollary 2.6, every ribbon knot is a slice knot. We are interested in the converse question: *Is every slice knot a ribbon knot?*

Here is one criterion to this question, which follows easily from Section 3 of [I], Cf. Fox [9].

6.2. PROPOSITION. A knot  $\mathcal{K} \subset R^3$  is a slice knot if and only if for some integer  $n \geq 0$ , the link  $\mathcal{K} \circ \mathcal{O}^n$  is a ribbon link in the weak sense, where  $\mathcal{K} \circ \mathcal{O}^n$  is the (convexly) completely splittable link consisting of the knot  $\mathcal{K}$  and a trivial link  $\mathcal{O}^n$  with  $n$  components.  $\square$

Let  $\mathcal{K} \subset R^3[0]$  be a slice knot and  $\tilde{E} \subset R^3[0, 2)$  be a locally flat, proper 2-disk with  $\partial \tilde{E} = \mathcal{K}$ . From the argument in [I], we can assume that  $\tilde{E}$  has only elementary critical bands in  $R^3[0, 2)$  such that the intersection  $\tilde{E}' = \tilde{E} \cap R^3[1, 2)$  is a proper 2-disk without minimal bands and the intersection  $\tilde{E} \cap R^3[0, 1]$  is a proper annulus with only minimal bands; see Fig. 13. Since  $\tilde{E}'$  does not have minimal bands, the knot  $\mathcal{K}_0 = \partial \tilde{E}' \subset R^3[1]$  is a ribbon knot. Thus,  $\mathcal{K}_0$  is obtained by a complete fusion from a trivial link  $\mathcal{O}^{m+1}$  with  $m+1$  components for some  $m \geq 0$ .

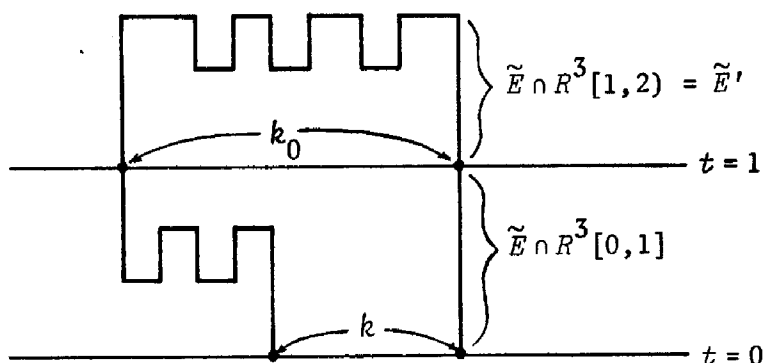


Fig. 13

Let  $\mathcal{D} = \cup_{i=0}^m D_i$  be a union of mutually disjoint, non-singular 2-disks  $D_i$  in  $R^3$  with  $\partial \mathcal{D} = \cup_{i=0}^m \partial D_i = \mathcal{O}^{m+1}$ . Let  $\mathcal{B} = \cup_{j=1}^m B_j$  be a union of mutually disjoint bands  $B_j$  in  $R^3$  used for the complete fusion  $\mathcal{O}^{m+1} \rightarrow \mathcal{K}_0$ . Then we have  $p(E') = \mathcal{D} \cup \mathcal{B}$ , where  $p: R^3(-\infty, +\infty) \rightarrow R^3$  is the natural projection. We may assume

that the minimal bands of  $\tilde{E}$  appear on  $R^3[1/2]$  at the same time. Let  $\tilde{\mathcal{D}} = \cup_{i=1}^n \bar{D}_i$  be a union of mutually disjoint 2-disks in  $R^3$  representing the minimal bands of  $\tilde{E}$ , and let  $\partial\tilde{\mathcal{D}} = \cup_{i=1}^n \partial\bar{D}_i$ . Let  $\tilde{\mathcal{B}} = \cup_{i=1}^n \bar{B}_i$  be a union of mutually disjoint bands  $\bar{B}_i$  in  $R^3$  used for the complete fission  $\mathcal{A}_0 \rightarrow \mathcal{A} \circ \partial\tilde{\mathcal{D}}$ . We now consider the singular 2-disk  $E = \mathcal{D} \cup \mathcal{B} \cup \tilde{\mathcal{B}} \cup \tilde{\mathcal{D}}$  that is the image of  $\tilde{E}$  under the projection  $p$  and whose boundary  $\partial E$  is  $\mathcal{A}$ . First we may assume that the singularity of  $E$  is normal. (One can modify  $E$  by a level preserving isotopy of  $R^3(-\infty, +\infty)$  so that the singularity of the image  $E = p(\tilde{E})$  is normal.) Further, we assume that  $(\partial\tilde{\mathcal{B}}) \cap (\partial\tilde{\mathcal{D}}) = (\partial\tilde{\mathcal{B}}) \cap (\partial D_0)$  and for each  $i, i=1, \dots, n, (\partial\tilde{B}_i) \cap (\partial\tilde{D}_i) = (\partial\tilde{B}_i) \cap (\partial\tilde{D}_i)$ .

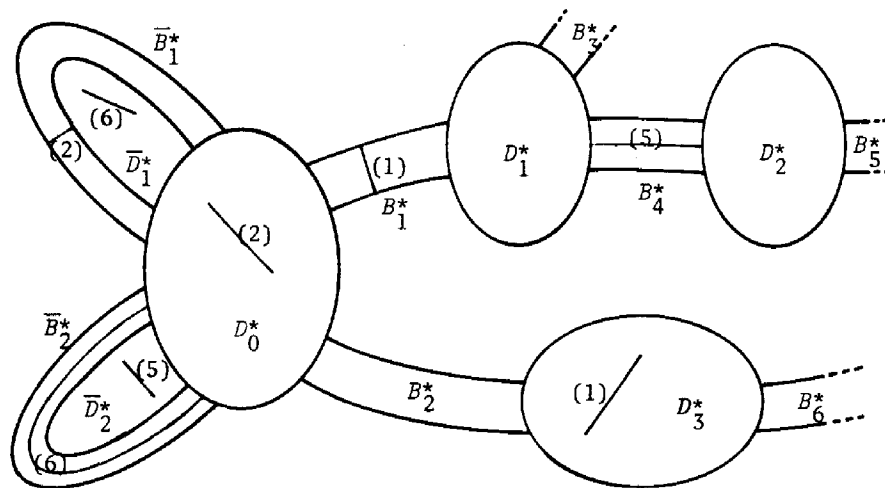


Fig. 14

We suppose that the normal singular 2-disk  $E$  is given by a piecewise-linear map  $f: E^* \rightarrow E$  with  $E^*$  a non-singular 2-disk. Let  $\mathcal{D}^*, \mathcal{B}^*, \tilde{\mathcal{B}}^*$  and  $\tilde{\mathcal{D}}^*$  be the unions of 2-disks in  $E^*$  corresponding to  $\mathcal{D}, \mathcal{B}, \tilde{\mathcal{B}}$  and  $\tilde{\mathcal{D}}$ , respectively, under the map  $f$ ; see Fig. 14 for our situation. Let us observe the singularities between  $\mathcal{D}, \mathcal{B}, \tilde{\mathcal{B}}$  and  $\tilde{\mathcal{D}}$ . There are six different cases. Since  $\tilde{\mathcal{D}}$  is the union of bands relating to the minimal bands of  $\tilde{E}$ , the intersection  $\tilde{\mathcal{D}} \cap \mathcal{A}$  is empty. Thus, we can reduce the types of the singularities to the following types by doing suitable modifications of  $\mathcal{D}, \mathcal{B}, \tilde{\mathcal{B}}$  and  $\tilde{\mathcal{D}}$ , as in Fig. 14:

- (1)  $\mathcal{D} \cap \mathcal{B}$  has only mutually disjoint, simple ribbon singularities,
- (2)  $\mathcal{D} \cap \tilde{\mathcal{B}}$  has only mutually disjoint, simple ribbon singularities,
- (3)  $\mathcal{D} \cap \tilde{\mathcal{D}}$  has mutually disjoint, simple singularities,
- (4)  $\mathcal{B} \cap \tilde{\mathcal{B}}$  is empty by taking narrow bands,
- (5)  $\mathcal{B} \cap \tilde{\mathcal{D}}$  has only mutually disjoint, simple ribbon singularities,
- (6)  $\tilde{\mathcal{B}} \cap \tilde{\mathcal{D}}$  has only mutually disjoint, simple ribbon singularities. We now

claim that:

6.3. LEMMA. *E does not have clasp singularities.*

PROOF. Suppose  $E$  has a clasp singularity  $\alpha$  with  $\partial\alpha = p \cup q$ . As illustrated in Fig. 15,  $\alpha = f(\alpha^*) = f(\alpha'^*)$ ,  $p = f(p^*) = f(p'^*)$  and  $q = f(q^*) = f(q'^*)$  with  $p^*, q^* \in \partial E^*$  and  $p'^*, q'^* \in \text{Int}(E^*)$ . Since  $\bar{\mathcal{D}} \cap \mathcal{L} = \emptyset$  and  $\bar{\mathcal{B}}$  is the union of bands of fissions and  $\mathcal{B}$  is the union of bands of fusions and  $\mathcal{B} \cap \bar{\mathcal{B}} = \emptyset$ , the point  $p'^*$  is contained in none of  $\bar{\mathcal{D}}^*$ ,  $\bar{\mathcal{B}}^*$  and  $\mathcal{B}^*$ . So  $p'^*$  is contained in  $\mathcal{D}^*$  and another point  $p^*$  is not contained in  $\mathcal{D}^*$ . Hence  $p^*$  must be in  $\mathcal{B}^* \cup \bar{\mathcal{B}}^*$ . If  $p^* \in \mathcal{B}^*$ , then  $p \in \mathcal{D} \cap \mathcal{B}$ , which would imply that  $p$  is an endpoint of a ribbon singularity. Thus,  $p^* \notin \mathcal{B}^*$  and  $p^* \in \bar{\mathcal{B}}^*$ . We can prove that  $q^* \in \mathcal{D}^*$  by the similar way to show that  $p'^* \in \mathcal{D}^*$ . Hence  $\alpha^* \cap (\mathcal{D}^* \cap (\bar{\mathcal{B}}^* \cup \bar{\mathcal{D}}^*)) \neq \emptyset$ . However, from the singularity of type (6) above, we see that  $\alpha^* \cap \mathcal{D}^* \cap \bar{\mathcal{B}}^* = \emptyset$ . So  $\alpha^* \cap \mathcal{D}^* \cap \bar{\mathcal{D}}^* \neq \emptyset$ ; we take a point  $r^* \in \alpha^* \cap \mathcal{D}^* \cap \bar{\mathcal{D}}^*$ . Since  $\mathcal{D}$  and  $\bar{\mathcal{D}}$  are non-singular, another point  $r'^*$  corresponding to  $r^*$  is not contained in  $\mathcal{D}^* \cup \bar{\mathcal{D}}^*$ . On the other hand,  $r'^* \in \mathcal{B}^* \cup \bar{\mathcal{B}}^*$ , since  $\mathcal{B}$  and  $\bar{\mathcal{B}}$  are the unions of bands of fusions and fissions, respectively. This is a contradiction. Therefore,  $E$  does not have clasp singularities, completing the proof.  $\square$

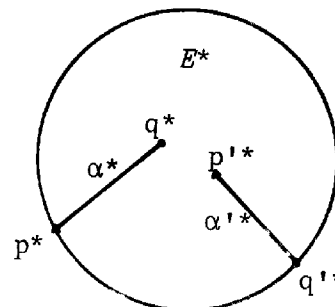


Fig. 15

6.4. LEMMA. *Let  $\alpha$  be a double line on  $E$ . The  $b$ -line in the preimage of  $\alpha$  is contained in either  $\bar{\mathcal{D}}^* \cup \bar{\mathcal{B}}^*$  or  $\mathcal{B}^*$ , and the  $i$ -line is contained in  $\mathcal{D}^* \cup \mathcal{B}^* \cup \bar{\mathcal{B}}^*$ , provided that the  $b$ -line and the  $i$ -line are suitably specified in the case of a loop singularity of Fig. 2(a) or a branch singularity with two branch points as in Fig. 2(e).*

PROOF. We say that a loop  $\alpha^*$  in the preimage of a loop singularity  $\alpha$  is a  $b$ -line, if  $\alpha^*$  is contained in  $\text{Int}(\bar{\mathcal{D}}^*)$ . Since  $\bar{\mathcal{D}} \cap \mathcal{L} = \emptyset$ ,  $\bar{\mathcal{D}}^*$  contains no  $i$ -lines of the ribbon singularity and the branch singularity with a single branch point (Cf. Fig. 2(d)). Suppose that a double line  $\alpha$  intersects with  $\text{Int}(\bar{\mathcal{D}})$ , but is not a loop singularity. Then, the closure of each line  $\beta^* \subset \bar{\mathcal{D}}^*$  in the preimage of  $\alpha \cap \bar{\mathcal{D}}$  intersects with  $\partial\mathcal{D}^*$ . If  $(\partial\beta^*) \cap (\partial\bar{\mathcal{D}}^*) \neq \emptyset$ , it can be shown that each point of  $(\partial\beta^*) \cap (\partial\bar{\mathcal{D}}^*)$  is a branch point by the similar way to the proof of Lemma 6.3. So,  $\beta^*$  cannot be extended into the interior of  $\mathcal{D}^*$ . If  $\partial\beta^* \subset \partial\mathcal{D}^*$ , then  $\alpha$  is a branch line with two branch points, and we say that  $\beta^*(=\alpha^*)$  is a  $b$ -line. Assume

that  $\beta^* \cap (\partial\bar{\mathcal{D}}^* - \partial\mathcal{D}^*) \neq \emptyset$ , and we choose a point  $p^*$  in  $\beta^* \cap (\partial\bar{\mathcal{D}}^* - \partial\mathcal{D}^*)$ . Since  $\bar{\mathcal{B}}^* \cup \bar{\mathcal{D}}^*$  is non-singular and  $\mathcal{B} \cap \bar{\mathcal{B}} = \emptyset$ , another point  $p'^* \in f^{-1}(p)$  is contained in  $\mathcal{D}^*$ , where  $p = f(p^*)$ . Hence the point  $p$  is contained in  $\bar{\mathcal{B}} \cap \mathcal{D}$ . Therefore,  $\beta^*$  will be extended into  $\bar{\mathcal{B}}^*$  toward  $\partial\bar{\mathcal{D}}^* - \partial\mathcal{D}^*$  by the property (2), see Fig. 14. So this extended line  $\alpha^*$  is a  $b$ -line on  $E^*$  and does not intersect with  $\partial\mathcal{D}^*$ . This implies that each  $b$ -line  $\alpha^*$  is contained in either  $\bar{\mathcal{D}}^* \cup \bar{\mathcal{B}}^*$  or  $\mathcal{B}^*$ . (A  $b$ -line contained in  $\mathcal{B}^*$  is in the preimage of  $\mathcal{B} \cap \mathcal{D}$  with property (1), see Fig. 14.). Since a double line that intersects with  $\text{Int}(\bar{\mathcal{D}}^*)$  is a  $b$ -line, we also have that an  $i$ -line is contained in  $\mathcal{B}^* \cup \bar{\mathcal{B}}^* \cup \mathcal{D}^*$ . This completes the proof.  $\square$

6.5. THEOREM.  $E$  is a normal singular 2-disk without a clasp singularity and a triple point of type I.

PROOF. By Lemma 6.3,  $E$  does not have a clasp singularity. Since each of  $\mathcal{D}$ ,  $\bar{\mathcal{D}}$ ,  $\mathcal{B}$  and  $\bar{\mathcal{B}}$  is non-singular and  $\mathcal{B} \cap \bar{\mathcal{B}} = \emptyset$ , we have a point  $p^* \in \bar{\mathcal{D}}^*$  in the preimage of each triple point  $p$ . By Lemma 6.4,  $p^*$  is of type  $(b, b)$ . So, the other two points in the preimage of  $p$  are those of types  $(b, i)$  and  $(i, i)$ , and this completes the proof.  $\square$

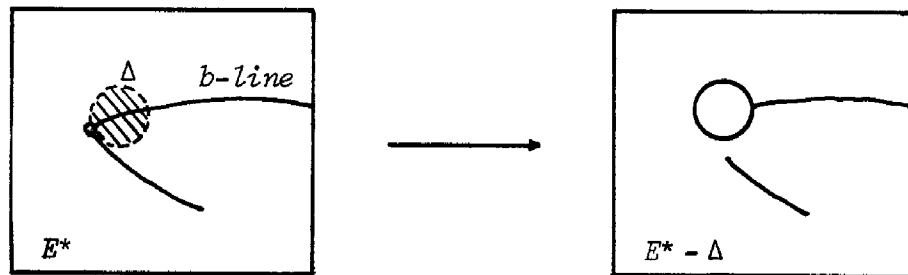
6.6. SUPPLEMENTS. The points of types  $(b, b)$ ,  $(b, i)$  and  $(i, i)$  in the preimage of a triple point in  $E$  are contained in  $\bar{\mathcal{D}}^*$ ,  $\mathcal{B}^* \cup \bar{\mathcal{B}}^*$  and  $\mathcal{D}^*$ , respectively.

The following has been suggested by F. Hosokawa and T. Yanagawa to the authors.

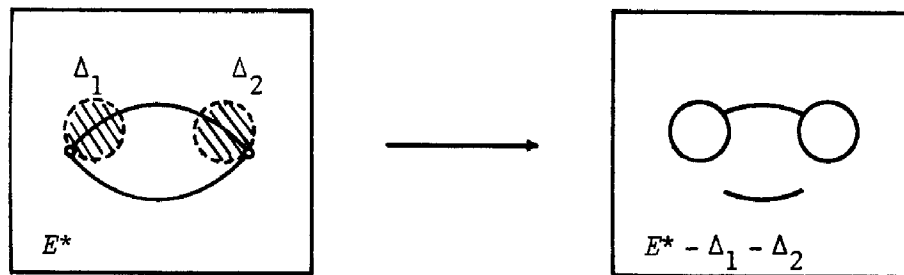
6.7. COROLLARY. A knot  $\mathcal{K} \subset R^3$  is a slice knot if and only if  $\mathcal{K}$  bounds a normal singular 2-disk without a clasp singularity and a triple point of type I in  $R^3$ .

PROOF. By Theorem 6.5, a slice knot certainly bounds a desired normal singular 2-disk. Conversely, we assume that a knot  $\mathcal{K} \subset R^3$  bounds a normal singular 2-disk  $E$  without a clasp singularity and a triple point of type I, where  $E$  is given by a map  $f: E^* \rightarrow E$  such that  $E^*$  is a non-singular 2-disk. Remove from  $E^*$  a small open 2-disk neighborhood of each point of  $(b, b)$ -type. For  $b$ -lines of the branch singularities, further remove small open 2-disks from  $E^*$  as in Fig. 16 below.

Next for a simple loop singularity, we remove from  $E^*$  a small open 2-disk neighborhood of an arbitrary point in the  $b$ -line. Then the resulting holey 2-disk  $E'^*$  gives a ribbon  $E' = f(E'^*)$ . Since  $\partial E' (= f(\partial E'^*))$  is the link  $\mathcal{K} \circ \mathcal{O}^n$  for some  $n$ , from Proposition 6.2 we conclude that  $\mathcal{K}$  is a slice knot. This completes the proof.  $\square$



A branch singularity with a single branch point



A branch singularity with two branch points

Fig. 16

6.8. REMARK. The question "Is every slice knot a ribbon knot?" asks whether one can always eliminate the triple point of type II in  $E$  without changing the knot type of the boundary  $\partial E$ . For example, assume that a slice knot  $\mathcal{K} \subset R^3$  bounds a much simplified singular 2-disk  $E$  with preimage  $E^*$  as shown in Fig. 17.  $E$  contains a single triple point of type II. This triple point can be in fact eliminated. Take a simple arc  $w^*$  in  $E^*$  such that  $w^*$  connects a point of  $\partial E^*$  and the point  $p^*$  of type  $(b, b)$  as shown in Fig. 17. Remove from  $E^*$  a neighborhood of

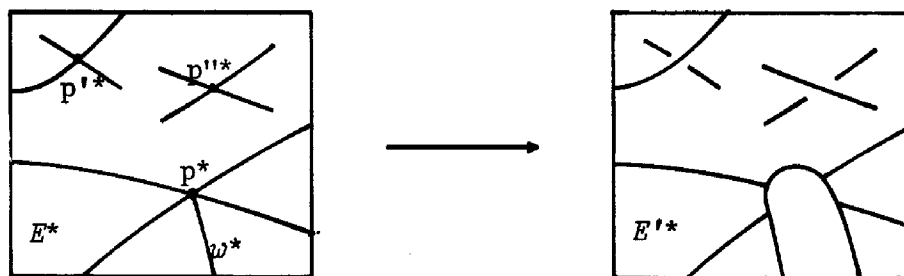


Fig. 17

$w^*$ . The resulting 2-disk  $E'^*$  determines a ribbon  $E'$ . Since  $\partial E'$  is obviously isotopic to the knot  $\mathcal{K}$ , it follows that  $\mathcal{K}$  is a ribbon knot.  $\square$

6.9. DEFINITION. An oriented link  $\ell \subset R^3[0]$  with  $\mu$  components is said to be a *slice link in the weak sense*, iff  $\ell$  bounds a locally flat oriented, proper, connected surface of genus 0 in  $R^3[0, +\infty)$ , and a *slice link in the strong sense*, iff  $\ell$  bounds a union of mutually disjoint,  $\mu$  locally flat proper 2-disks in  $R^3[0, +\infty)$ .

Clearly, a slice link in the strong sense is a slice link in the weak sense, and a ribbon link in the weak (resp. strong) sense is a slice link in the weak (resp. strong) sense. By parallel arguments of Theorem 6.5 and Corollary 6.7, we may also have the following:

6.10. COROLLARY. (1) A link  $\ell \subset R^3$  is a slice link in the weak sense if and only if  $\ell$  bounds in  $R^3$  a normal singular holey 2-disk without a clasp singularity and a triple point of type I.

(2) A link  $\ell \subset R^3$  with  $\mu$  components is a slice link in the strong sense if and only if  $\ell$  bounds in  $R^3$  a normal singular 2-disk with component number  $\mu$  and without a clasp singularity and a triple point of type I.  $\square$

REMARK. We refer the reader to Casson-Gordon [3], [4] and Fox [9] for our problem "Is every slice knot a ribbon knot?".

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Added in Proof. Recently, Cochran [71] has obtained some remarkable results on ribbon 2-knots, and Murakami-Sugishita [72] have discussed on a relation between triple points of normal singular surfaces in  $R^3$  and the cobordism property of knots.

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