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DISTANCE BETWEEN LINKS BY ZERO-LINKING TWISTS

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1. Introduction

Throughout this paper, by links, we will mean oriented links. For a link L in the 3-sphere S^3, we choose a trivial knot k in S^3 with $k \cap L = \emptyset$ such that the linking number $\mathrm{Link}(k, L) = 0$ in S^3. Doing the $1/d$-surgery of S^3 along k for some non-zero integer d, we obtain a link L' in S^3 from L. The link L' is said to be obtained from the link L by a *zero-linking twist* with twist coefficient d. An important property of a zero-linking twist is that there exists a Seifert surface F for L with $F \cap k = \emptyset$, so that this surface F can also serve as a Seifert surface for L'. Since the crossing change move on a link diagram is a special case of the move of zero-linking twist, we see that L_0 moves to L_1 by a finite sequence of zero-linking twists for any given oriented links L_i $(i = 0, 1)$ with $\#L_0 = \#L_1$. (We denote the number of components of a link L by $\#L$.) Let $d^\tau(L_0, L_1)$ be the least number of zero-linking twists (where the twist coefficients are not fixed) which are needed to move L_0 to L_1. The function d^τ defines a metric function on the set of oriented link types with the same number of components, which we call the τ-*distance* between them. Similarly, the x-*distance* d^{x} between them is defined by using the crossing change moves instead of the zero-linking twists (cf. H. Murakami [8]). Obviously, we have $d^{\mathrm{x}}(L_0, L_1) \geq d^\tau(L_0, L_1)$ for links L_i $(i = 0, 1)$ with $\#L_0 = \#L_1$.

The main purpose of this paper is to estimate the τ-distance $d^\tau(L_0, L_1)$ by two kinds of infinite cyclic covering invariants of the links L_i $(i = 0, 1)$, namely, by the rank invariants on the quotient rings of the integral Laurent polynomial ring $\Lambda = \mathbf{Z}[\mathbf{Z}]$ generalizing Nakanishi's index of [9] and by the local signature invariants of [3]. For the x-distance, a similar estimation by the usual Nakanishi index has been given by Y. Nakanishi [9, 10] and the author [5]. Our estimation by the local signatures is directly obtained from the signature theorem of [4], but it could be also deduced from an estimation by R.A. Litherland [7] using the Atiyah-Singer G-signature theorem of finite branched cyclic coverings of links if we would know the relationship between the local signatures and a Seifert matrix, which will be done in [6]. For the x-distance, a similar estimation by the usual knot signature has been also given by H. Murakami [8].

In a meeting of New KOOK SEMINAR, H. Goda (cf. [2]) announced that the prime knots with up to 10 crossings and the tunnel number one genus one knots move to fibered knots by one crossing change and asked whether every knot can move to a fibered knot by one crossing change. From our estimation by a local-ring-rank, we see that *there are many knots with large τ-distances to the fibered knots*, answering this question in the negative. It would be also interesting to compare this result with a result of J. Stallings in [11] meaning that *every link L can move to a fibered link by a $(+1)$-linking twist with twist coefficient -1* (that is, by the (-1)-surgery of S^3 along a trivial knot k with $k \cap L = \emptyset$ and $\mathrm{Link}(k, L) = +1$). This result will be discussed in §3 together with similar results on the τ-distances to the trivial links and the amphicheiral links.

2. The estimation

We first explain the rank invariants of a Λ-module on the quotient rings of the integral Laurent polynomial ring Λ.

DEFINITION 2.1. A *multiplicative set* of Λ is a subset $S \subset \Lambda - \{0\}$ such that

(1) the units $\pm t^i$ $(i \in \mathbf{Z})$ are in S,
(2) the product gg' of any elements g and g' of S is in S, and
(3) every prime factor of any element $g \in S$ is in S.

For the quotient field $Q(\Lambda)$ of Λ and a multiplicative set S of Λ, let Λ_S be the subring $\{f/g \in Q(\Lambda) \mid f \in \Lambda,\ g \in S\}$ of $Q(\Lambda)$. For a Λ-module H, let H_S be the Λ_S-module $H \bigotimes_\Lambda \Lambda_S$. The following proposition is known (cf. Cartan-Eilenberg [1, p.130]):

PROPOSITION 2.2. *For every multiplicative set S of Λ, we have the following (1) and (2) :*

(1) *Every Λ-exact sequence $0 \to H' \to H \to H'' \to 0$ on Λ-modules H', H and H'' induces a Λ_S-exact sequence $0 \to (H')_S \to H_S \to (H'')_S \to 0$.*

(2) *The kernel of the natural Λ-homomorphism $H \to H_S$ for any Λ-module H is equal to the Λ-submodule $\{x \in H \mid {}^\exists g \in S \text{ with } gx = 0\}$ of H. For every element $y \in H_S$, there is an element $g \in S$ such that gy is in the image of the natural Λ-homomorphism $H \to H_S$.*

For a finitely generated Λ-module H and a multiplicative set S of Λ, let $e_S(H)$ be the least number of Λ_S-generators of H_S (We take $e_S(H) = 0$ when $H = 0$).

We call this number the Λ_S-rank of the Λ-module H. Clearly, $e_S(H) \geq e_{S'}(H)$ for any multiplicative set S' of Λ with $S \subset S'$. For every multiplicative set S of Λ, we denote by $\overline{S}$ the smallest multiplicative set of Λ containing S an all non-zero integers. Then $\Lambda_{\overline{S}}$ is a principal ideal domain and $e_{\overline{S}}(H)$ can be easily calculated. Let $\widetilde{E}(L) \to E(L)$ be the infinite cyclic covering over the compact exterior $E(L)$ of an oriented link L in S^3 which is induced from the epimorphism $\gamma_L : \pi_1(E(L)) \to \mathbf{Z}$ sending each oriented meridian of L to $1 \in \mathbf{Z}$. Then $H_1(\widetilde{E}(L))$ is naturally regarded as a finitely generated Λ-module. Let $e_S(L) = e_S(H_1(\widetilde{E}(L)))$. If S is the set of units of Λ, then we have $\Lambda_S = \Lambda$ and $e_S(L)$ is simply denoted by $e(L)$ and is equal to Nakanishi's index of L meaning the minimal size of a square Λ-presentation matrix of $H_1(\widetilde{E}(L))$ (see [5]).

To state the local signature invariants of a link L, let X_L be the 4-manifold obtained from the 4-disk D^4 by attaching along L the 2-handles with the framing uniquely specified by a Seifert surface of L. Let $M_L = \partial X_L$. Then the epimorphism γ_L induces an epimorphism $\gamma : \pi_1(M_L) \to \mathbf{Z}$. From the infinite cyclic covering $\widetilde{M_L} \to M_L$ associated with γ, we can define the integral invariants $\sigma_a^\gamma(M_L)(a \in [-1,1])$ of the pair (M_L, γ) which are zero except a finite number of a (see [3]). In fact, $\sigma_a^\gamma(M_L)$ is the signature of the quadratic form

$$b : TH_1(\widetilde{M_L};\mathbf{R}) \times TH_1(\widetilde{M_L};\mathbf{R}) \to \mathbf{R}$$

restricted to the $p_a(t)$-primary component, where $TH_1(\widetilde{M_L};\mathbf{R})$ denotes the torsion part of $H_1(\widetilde{M_L};\mathbf{R})$ over the real Laurent polynomial ring $\Lambda_{\mathbf{R}} = \Lambda \otimes_{\mathbf{Z}} \mathbf{R}$ and $p_a(t) = t - a$ or $t^2 - 2at + 1$ according to whether $a = \pm 1$ or $-1 < a < 1$. Since the pair (M_L, γ) is uniquely constructed from the link L, we see that these invariants are link invariants, which we call the *local signature invariants* of L. We denote $\sigma_a^\gamma(M_L)$ by $\sigma_a(L)$. Let $\lambda(L)$ denote the signature $\mathrm{sign}(X_L)$ of X_L which is the signature of the *canonical* linking matrix of L (i.e., the linking matrix of L with integral framing given by a Seifert surface). For every $a \in [-1,1]$, let

$$\tau_a(L) = \sum_{x \in [a,1]} \sigma_x(L) \quad \text{and} \quad \overline{\tau}_a(L) = \tau_a(L) + \lambda(L).$$

It turns out that $\overline{\tau}_a(L)$ for every $a \in [-1,1]$ is calculable from a Seifert matrix of L and in particular, $\overline{\tau}_{-1}(L)$ coincides with the usual link signature of L (cf. [6]) (When L is a knot, this fact is directly checked from [3]). Our estimation is stated as follows.

THEOREM 2.3. *For arbitrary two links L_i $(i = 0,1)$ with $\#L_0 = \#L_1$, every multiplicative set S of Λ, and every $a \in [-1,1]$, we have*

$$d^{\times}(L_0, L_1) \geq d^{\tau}(L_0, L_1) \geq \max(|e_S(L_0) - e_S(L_1)|, |\overline{\tau}_a(L_0) - \overline{\tau}_a(L_1)|/2).$$

PROOF. From the property of the triangle inequality, it suffices to prove the case of $d^{\tau}(L_0, L_1) = 1$. Let a link L_0 move to a link L_1 by the zero-linking twist with twist coefficient d along a trivial knot k. Let k' be a meridian of k in $S^3 - L$. Then the link L_0 changes into the link L_1 in the surgery manifold $\chi(S^3; (k, 0), (k', -d))$ which is homeomorphic to S^3. Let

$$W = E(L_0) \times I \cup h \cup h'$$

be the surgery trace from $E(L_0)$ to $E(L_1)$ where $I = [0, 1]$ and h and h' are the 2-handles whose framings are 0 and $-d$ and whose attaching core circles are k and k', respectively. The assumption of zero-linking twist assures that the infinite cyclic coverings $\widetilde{E}(L_i) \to E(L_i)$ $(i = 0, 1)$ extends to an infinite cyclic covering $p : \widetilde{W} \to W$. Then the natural homomorphisms $H_1(\widetilde{E}(L_i)) \to H_1(\widetilde{W})$ $(i = 0, 1)$ are onto. Let $s = e_S(L_1)$. We choose elements x_i $(i = 1, 2, \ldots, s)$ of $H_1(\widetilde{E}(L_1))$ which generate $H_1(\widetilde{E}(L_1))_S$ over Λ_S (see Proposition 2.2). For each i, we represent x_i by a simple loop in $\widetilde{E}(L_1)$ which is a lift of a simple loop ℓ_i in $E(L_1) - (h \cup h') = E(L_0) \times 1 - (h \cup h')$ so that ℓ_i $(i = 1, 2, \ldots, s)$ are mutually disjoint. Let W^+ be the union of W and mutually disjoint 2-handles h_i'' $(i = 1, 2, \ldots, s)$ attaching to $E(L_0) \times 1$ (with any framings) along ℓ_i $(i = 1, 2, \ldots, s)$. Let $p^+ : \widetilde{W}^+ \to W^+$ be the infinite cyclic covering extending the covering $p : \widetilde{W} \to W$. Note that the natural Λ-epimorphism $H_1(\widetilde{E}(L_1)) \to H_1(\widetilde{W})$ induces a natural Λ_S-epimorphism $H_1(\widetilde{E}(L_1))_S \to H_1(\widetilde{W})_S$. Applying Proposition 2.2(1) to the Mayer-Vietoris sequence on the triplet $(\widetilde{W}^+; \widetilde{W}, \bigcup_{i=1}^{s} (p^+)^{-1}(h_i''))$, we have that $H_1(\widetilde{W}^+)_S = 0$. Then the Λ_S-homomorphism $\partial_S : H_2(\widetilde{W}^+, \widetilde{E}(L_0))_S \to H_1(\widetilde{E}(L_0))_S$ induced from the boundary homomorphism $\partial : H_2(\widetilde{W}^+, \widetilde{E}(L_0)) \to H_1(\widetilde{E}(L_0))$ is onto. Since $H_2(\widetilde{W}^+, \widetilde{E}(L_0))_S \cong H_2(\widetilde{W}^+, \widetilde{E}(L_0) \times I)_S$ is a free Λ_S-module of rank $2 + s$ with a basis represented by lifts of the cores of the 2-handles h, h' and h_i'' $(i = 1, 2, \ldots, s)$. Since k' which is the boundary of the core of h' is null-homotopic in $E(L_0) \times 1$, the core of h' does not contribute to $H_1(\widetilde{E}(L_0))_S$. Hence we have $1 + s \geq e_S(L_0)$. That is, $1 \geq e_S(L_0) - e_S(L_1)$. Similarlly, $1 \geq e_S(L_1) - e_S(L_0)$ and we have $1 \geq |e_S(L_0) - e_S(L_1)|$.

To obtain the signature estimation, we construct a compact oriented 4-manifold X with $\partial X = (-M_{L_0}) \cup M_{L_1}$ by adding $\mathrm{cl}(M_{L_0} - E(L_0)) \times I$ to W. Then the epimorphisms $\gamma_{L_i} : \pi_1(E(L_i)) \to \mathbf{Z}$ $(i = 0, 1)$ induce epimorphisms $\gamma : \pi_1(X) \to \mathbf{Z}$ and $\gamma_i : \pi_1(M_{L_i}) \to \mathbf{Z}$ $(i = 0, 1)$. The signature theorem of [4] implies

$$\tau_{a-0}^{\gamma}(X) - \mathrm{sign}(X) = -\tau_a(L_0) + \tau_a(L_1)$$

for all $a \in (-1, 1]$, where $\tau_{a-0}^{\gamma}(X)$ is a signature invariant of a $\Lambda_{\mathbf{R}}$-intersection matrix $P(t)$ on the $\Lambda_{\mathbf{R}}$-free module $H_2(\widetilde{X}; \mathbf{R})/TH_2(\widetilde{X}; \mathbf{R})$ for the infinite cyclic covering $\widetilde{X} \to X$ associated with γ. We note that $|\tau^{\gamma}_{a-0}(X)|$ does not exceed the $\Lambda_{\mathbf{R}}$-rank of $P(t)$ by definition. Since $H_2(\widetilde{X}, \widetilde{M_{L_0}})$ is a Λ-free module of rank 2, we see that $\Lambda_{\mathbf{R}}$-rank $P(t) \leq 2$, so that $|\tau^{\gamma}_{a-0}(X)| \leq 2$. To calculate the signature sign(X), we consider the oriented 4-manifold $Y = X_{L_0} \cup X$ identifying the two copies of M_{L_0}. By construction, the intersection form on $H_2(Y)$ is represented by a block sum of the matrix $\begin{pmatrix} 0 & 1 \\ 1 & -d \end{pmatrix}$ and the intersection matrix on $H_2(X_{L_1})$. Thus, we have $\mathrm{sign}(Y) = \mathrm{sign}(X_{L_1})$. By the Novikov additivity of signatures, we have $\mathrm{sign}(Y) = \mathrm{sign}(X_{L_0}) + \mathrm{sign}(X)$, so that

$$\mathrm{sign}(X) = \mathrm{sign}(X_{L_1}) - \mathrm{sign}(X_{L_0}) = \lambda(L_1) - \lambda(L_0).$$

Therefore, for $a \in (-1, 1]$, we have

$$\begin{aligned} 1 &\geq |\tau^{\gamma}_{a-0}(X)|/2 \\ &= |\mathrm{sign}(X) - \tau_a(L_0) + \tau_a(L_1)|/2 \\ &= |\overline{\tau}_a(L_0) - \overline{\tau}_a(L_1)|/2. \end{aligned}$$

To obtain the signature estimation at $a = -1$, we can use the signature theorem of [4] to see the identity

$$\begin{aligned} \mathrm{sign}(X^2) - 2\mathrm{sign}(X) &= -\tau_{-1}(L_0) + \tau_{-1}(L_1), \quad \text{so that} \\ \overline{\tau}_{-1}(L_1) - \overline{\tau}_{-1}(L_0) &= \mathrm{sign}(X^2) - \mathrm{sign}(X), \end{aligned}$$

where X^2 denotes the double covering space over X associated with the composite epimorphism $\pi_1(X) \xrightarrow{\gamma} \mathbf{Z} \to \mathbf{Z}_2$. Let $x^!$ be the image of an element $x \in H_2(X; \mathbf{Q})$ under the transfer monomorphism $H_2(X; \mathbf{Q}) \to H_2(X^2; \mathbf{Q})$ which is induced from the chain map $C_\#(X; \mathbf{Q}) \to C_\#(X^2; \mathbf{Q})$ sending each simplex Δ to $(1+t)\Delta'$, where Δ' denotes a lift of Δ and t denotes the non-trivial covering transformation. We note the identities

$$t(x^!) = x^! \quad \text{and} \quad \mathrm{Int}_{X^2}(x^!, y^!) = 2\mathrm{Int}_X(x, y)$$

for all $x, y \in H_2(X; \mathbf{Q})$. Let $A_1 \subset H_2(X^2; \mathbf{Q})$ be the image of the transfer monomorphism which is identified with the $\mathbf{Q}$-vector subspace of all elements x with $tx = x$. Let $M^2_{L_0}$ be the lift of M_{L_0} under the double covering $X^2 \to X$. Using that $H_2(X^2, M^2_{L_0}; \mathbf{Q})$ has a $\mathbf{Q}$-basis represented by the lifts of the cores of the 2-handles h, h' we can write

$$\begin{aligned} H_2(X^2, M^2_{L_0}; \mathbf{Q}) &= B_{-1} \oplus B_1, \quad \text{and} \\ H_2(X^2; \mathbf{Q}) &= A_{-1} \oplus A_0 \oplus A_1, \end{aligned}$$

where $B_{\pm 1}$ denotes the $\mathbf{Q}$-vector subspace $(t\pm 1)H_2(X^2, M^2_{L_0}; \mathbf{Q})$ of dimension 2 and A_{-1} is embedded into B_{-1} by the natural homomorphism $j_*: H_2(X^2, \mathbf{Q}) \to H_2(X^2, M^2_{L_0}; \mathbf{Q})$, and A_0 is a $\mathbf{Q}$-vector subspace whose elements are represented by cycles in $M^2_{L_0}$. For $x \in A_{-1}$, we write $j_*(x) = (t-1)z_x \in B_{-1}$. Then for $y \in A_1$, we have

$$\mathrm{Int}_{X^2}(x,y) = \mathrm{Int}_{X^2}(j_*(x),y) = \mathrm{Int}_{X^2}((t-1)z_x, y) = \mathrm{Int}_{X^2}(z_x, (t-1)y) = 0.$$

Thus, the intersection form Int_{X^2} is represented by a block sum of the intersection matrices on A_{-1} and A_1 and the zero matrix on A_0. Since the signature of the intersection matrix on A_1 is equal to $\mathrm{sign}(X)$, we have

$$\begin{aligned} |\overline{\tau}_{-1}(L_1) - \overline{\tau}_{-1}(L_0)|/2 &= |\mathrm{sign}(X^2) - \mathrm{sign}(X)|/2 \\ &\le (\dim_{\mathbf{Q}} A_{-1})/2 \le (\dim_{\mathbf{Q}} B_{-1})/2 = 1. \end{aligned}$$

This completes the proof.

3. Applications

The *τ-unknotting number* of a link L which we denote by $u^\tau(L)$ is the number $d^\tau(L,O)$ where O is a trivial link with $\#O = \#L$. Using d^κ in place of d^τ, we have the usual unknotting number $u(L)$ of L. Since we have $e_S(O) = \#O - 1$ for any S and $\tau_a(O) = \lambda(O) = 0$, we have the following corollary:

COROLLARY 3.1. *For every link L, every multiplicative set S of Λ, and every $a \in [-1,1]$, we have*

$$u(L) \ge u^\tau(L) \ge \max(|e_S(L) - \#L + 1|, |\overline{\tau}_a(L)|/2).$$

For example, let K_n be the n-fold connected sum of a twist knot (= a twisted double of a trivial knot) with Alexander polynomial $\Delta_m(t) = mt^2 + (1-2m)t + m$ for a non-zero integer m, where we note that the twist knots (up to the mirror images) are uniquely determined by the Alexander polynomials. Then $H_1(\widetilde{E}(K_n))$ is an n-fold direct sum of the Λ-cyclic module $\Lambda/(\Delta_m(t))$. Since $e_S(K_n) \ge e_{\bar{S}}(K_n) = n$ for the set S of units of Λ and $u(K_n) \le n$, we have

$$u(K_n) = u^\tau(K_n) = n.$$

As another example, let L be any link with non-zero one-variable Alexander polynomial $\Delta_L(t)$. Then we see

$$u(L) \ge u^\tau(L) \ge \#L - 1$$

by taking $S = \Lambda - \{0\}$ to have $\Lambda_S = Q(\Lambda)$.

The *τ-fibering number* of a link L which we denote by $f^\tau(L)$ is the minimum of $d^\tau(L, L^f)$ for all fibered links L^f with $\#L^f = \#L$. Using $d^\times$ in place of d^τ, we have the *fibering number* $f(L)$ of L. Let S^f be the multiplicative set of Λ which consists of Laurent polynomials such that the coefficients of the highest and lowest terms are ± 1. Since the one-variable Alexander polynomial $\Delta_{L^f}(t)$ of a fibered link L^f is in S^f, we have $e_{S^f}(L^f) = 0$, for any square Λ-presentation matrix of $H_1(\widetilde{E}(L^f))$ is an *invertible* Λ_{S^f}-presentation matrix of $H_1(\widetilde{E}(L^f))_{S^f}$. Hence we obtain the following corollary:

COROLLARY 3.2. *For any link L, we have*

$$f(L) \geq f^\tau(L) \geq e_{S^f}(L).$$

For example, let K_n be the n-fold connected sum of the twist knot with Alexander polynomial $\Delta_m(t) = mt^2 + (1 - 2m)t + m$ for an integer m with $|m| \geq 2$. Then $H_1(\widetilde{E}(K_n))_{S^f}$ is an n-fold direct sum of the Λ_{S^f}-cyclic module $\Lambda_{S^f}/(\Delta_m(t))$. Since $\Lambda_{\bar{S}^f}$ is a principal ideal domain and $\Lambda_{\bar{S}^f}/(\Delta_m(t))$ is non-zero, it follows that $e_{S^f}(K_n) \geq e_{\bar{S}^f}(K_n) = n$. Using that the trivial knot is fibered, we have

$$f(K_n) = f^\tau(K_n) = e_{S^f}(K_n) = n.$$

For a trivial link O, we have the following result by taking the τ-distance to the $(\#O - 1)$-fold connected sum of a Hopf link:

$$f(O) = f^\tau(O) = e_{S^f}(O) = \#O - 1.$$

We denote the mirror image or the inversed mirror image of a link L by L^* or $-L^*$. A link L is *amphicheiral* if L has the same type as L^* or $-L^*$. The *τ-chiral number* of a link L which we denote by $c^\tau(L)$ is the minimum of $d^\tau(L, L^a)$ for all amphicheiral links L^a with $\#L^a = \#L$. The *τ-self-chiral number* of a link L, denoted by $sc^\tau(L)$ is the number $\min(d^\tau(L, L^*), d^\tau(L, -L^*))$. Using $d^\times$ in place of d^τ, we have the *chiral number* $c(L)$ and the *self-chiral number* $sc(L)$ of L. By the definitions, we have

$$sc^\tau(L) \leq 2c^\tau(L) \quad \text{and} \quad sc(L) \leq 2c(L)$$

for all links L. Since $\tau_a(\pm L^*) = -\tau_a(L)$ and $\lambda(\pm L^*) = -\lambda(L)$, we have the following corollary:

COROLLARY 3.3. *For every link L and every $a \in [-1, 1]$, we have*

$$c(L) \geq c^\tau(L) \geq |\overline{\tau}_a(L)|/2,$$
$$sc(L) \geq sc^\tau(L) \geq |\overline{\tau}_a(L)|.$$

For example, let K' and K'' be the twist knots with Alexander polynomials $\Delta_{m'}(t) = m't^2+(1-2m')t+m'$ and $\Delta_{m''}(t) = m''t^2+(1-2m'')t+m''$, respectively, where m' and m'' are integers with $0 < m' < m''$ and with the same signature which must be ± 2. Let K_n be the n-fold connected sum of the connected sum $-(K')^*\#K''$. Then we see easily that the usual knot signature $\overline{\tau}_{-1}(K_n) = 0$, but $\overline{\tau}_a(K_n) = \pm 2n$ for any a with $(0 <)\ (2m'-1)/2m' < a < (2m''-1)/2m''$ (< 1), so that $c(K_n) \geq c^{\tau}(K_n) \geq n$, for $\lambda(K_n) = 0$. Since K'' moves to K' by one zero-linking twist and $-(K')^*\#K'$ is amphicheiral, we see that

$$c^{\tau}(K_n) = n \quad \text{and} \quad sc^{\tau}(K_n) = 2n.$$

For $m'' = m' + 1$, we also have $c(K_n) = n$ and $sc(K_n) = 2n$, but for $m'' > m' + 1, c(K_n)$ and $sc(L)$ are not yet determined.

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