# On the Alexander polynomials of knots with Gordian distance one

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## Abstract

We consider a condition on a pair of the Alexander polynomials of knots which are realizable by a pair of knots with Gordian distance one. We show that there are infinitely many mutually disjoint infinite subsets in the set of the Alexander polynomials of knots such that every pair of distinct elements in each subset is not realizable by any pair of knots with Gordian distance one. As one of the subsets, we have an infinite set containing the Alexander polynomials of the trefoil knot and the figure eight knot. We also show that every pair of distinct Alexander polynomials such that one is the Alexander polynomial of a slice knot is realizable by a pair of knots of Gordian distance one, so that every pair of distinct elements in the infinite subset consisting of the Alexander polynomials of slice knots is realizable by a pair of knots with Gordian distance one. These results solve problems given by Y. Nakanishi and by I. Jong.

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## 1. Introduction

The Gordian distance $d_G(J,K)$ from a knot $J$ to a knot $K$ is the least number of cross-changes needed to obtain $K$ from $J$. Then $d_G(J,K) = d_G(K,J)$ and the Gordian distance $u(K) = d_G(O,K)$ for the trivial knot $O$ is the unknotting number of $K$. Let $\Lambda = Z[t,t^{-1}]$ be the one-variable integral Laurent polynomial ring. For an element $a \in \Lambda$, an element $\bar{a} \in \Lambda$ is defined by the identity $\bar{a}(t) = a(t^{-1})$. By an *Alexander polynomial*, we mean an element $a \in \Lambda$ such that $a = \bar{a}$ and $a(1) = 1$. The

Laurent polynomial degree of an Alexander polynomial is simply called the *degree*. Our problem in this paper is to know when a pair of Alexander polynomials $a, b$ are realizable by a pair of knots $J$, $K$ with $d_G(J, K) = 1$. Let $(a_1, a_2, \ldots, a_r)$ be the $\Lambda$-ideal generated by the elements $a_i$ $(i = 1, 2, \ldots, r)$. Elements $a_i \in \Lambda$ $(i = 1, 2, \ldots, r)$ $(r > 1)$ are said to be *strongly coprime* if the $\Lambda$-ideal $(a_1, a_2, \ldots, a_r)$ coincides with $\Lambda$ itself, and *coprime* if there are only unit common divisors of the elements $a_i$ $(i = 1, 2, \ldots, r)$. Strongly coprime elements in $\Lambda$ are coprime. The *determinant ring* of a pair of elements $a, b \in \Lambda$ is the quotient ring $D(a, b) = \Lambda/(a, b)$, which is a finite $\Lambda$-module if $a$ and $b$ are coprime (see 2.1 later). For a finite $\Lambda$-module $D$, let $\hom(D, Q/Z)$ be the $\Lambda$-module consisting of all abelian homomorphisms $f : D \to Q/Z$ with the $t$-action defined by the identity $(t \cdot f)(x) = f(tx)$ for $f : D \to Q/Z$ and $x \in D$. For a finitely generated $\Lambda$-module $H$, we denote by $H^{op}$ the same $\Lambda$-module as $H$ but with the $t$-action by the identity $t \cdot x = t^{-1}x$ for every $x \in H$. The *dual* $\Lambda$*-module* $D^\#$ of a finite $\Lambda$-module $D$ is the same $\Lambda$-module as $\hom(D, Q/Z)$ but with the $t$-action by the identity $(t \cdot f)(x) = f(t^{-1}x)$ for $f : D \to Q/Z$ and $x \in D$. Namely, we have $D^\# = \hom(D, Q/Z)^{op}$. We shall show the following theorem:

**Theorem 1.1.** Let $a$, $a'$ be a pair of coprime Alexander polynomials which is realizable by a pair of knots $K$, $K'$ with $d_G(K, K') = 1$. Then there is a $\Lambda$-submodule $R$ of the determinant ring $D(a, a')$ admitting a short exact sequence

$$0 \to R \to D(a, a') \to R^\# \to 0.$$

We shall give this finite $\Lambda$-module $R$ a meaning, called "the residue module of the twist family"in §2, on the cross-change of a pair of knots of Gordian distance one with the Alexander polynomials $a, a'$. The proof of Theorem 1.1 is done with this meaning of $R$ in §3 by using the three dualities on an infinite cyclic covering in [4]. An application of Theorem 1.1 is done for the Alexander polynomials of degree 2 in §4, where we find infinitely many mutually disjoint infinite sets of Alexander polynomials such that every pair of distinct elements in each set is not realizable by any pair of knots with Gordian distance one, solving problems given by I. Jong [1, 2, 3]. In particular, we show that as such a member, there is an infinite set containing the Alexander polynomials $A_1 = -1 + (t + t^{-1})$ and $A_{-1} = 3 - (t + t^{-1})$ of the trefoil knot $3_1$ and the figure eight knot $4_1$ are not realizable by any pair of knots with Gordian distance one, solving an earlier problem given by Y. Nakanishi in [11]. This problem was originally caused from H. Murakami's result in [9] that any knot with Alexander polynomial $A_{-1} = 3 - (t + t^{-1})$ cannot be transformed into $3_1$ by one cross-change and then from Nakanishi's result in [11] that any knot with Alexander polynomial $A_1 = -1 + (t + t^{-1})$ cannot be transformed into $4_1$ by one cross-change. We also observe how Theorem 1.1 relates to Nakanishi's work in [12, 13], from which we can derive the following theorem:

**Theorem 1.2.** If $a$ and $a'$ are the Alexander polynomials of knots $K$ and $K'$, respectively, such that $d_G(O,K) = d_G(K,K') = 1$, then there is an element $c \in \Lambda$ such that $a' \equiv \pm c\bar{c} \pmod{a}$. Conversely, for any distinct Alexander polynomials $a, a'$ such that $a' \equiv \pm c\bar{c} \pmod{a}$ for an element $c \in \Lambda$, there are knots $K$, $K'$ with $d_G(O,K) = d_G(K,K') = 1$ such that $a$ and $a'$ are the Alexander polynomials of $K$ and $K'$, respectively.

In Theorem 1.2, we assume that $a$ and $a'$ are coprime. Then the determinant ring $D(a,a')$ is finite and equal to $\Lambda/(a, c\bar{c})$, which induces a short exact sequence

$$0 \to D(a,c) \to D(a,b) \to D(a,\bar{c}) \to 0$$

and $D(a,c)^{\#} = D(a,\bar{c})$. Thus, Theorem 1.1 is a generalization of this property to every pair of coprime Alexander polynomials which are realizable by a pair of knots with Gordian distance one. In [13], Nakanishi has characterized the Alexander polynomials of the knots obtained from the trefoil knot and the figure eight knot by one cross-change in terms of special values of the Alexander polynomials. In §5, we explain how Theorem 1.2 is derived from [12, 13]. As a corollary to Theorem 1.2, we shall also show in §5 that every pair of distinct Alexander polynomials $a_0, a$ such that $a_0$ is of slice type, i.e., $a_0 = c\bar{c}$ for some $c \in \Lambda$ is realizable by a pair of knots $K_0, K$ with $d_G(K_0, K) = 1$. This answers positively Nakanishi's recent question asking whether there is an Alexander polynomial $a' \neq 1$ such that every pair of distinct Alexander polynomials $a', a$ is realizable by a pair of knots $K', K$ with $d_G(K', K) = 1$. In [17], Y. Uchida and M. Hirasawa constructed a set of $n+1$ knots $K_i$ $(i = 0, 1, 2, \dots, n)$ for every $n \geq 2$ such that every pair of distinct elements in the set has the Gordian distance one. By taking the set of Alexander polynomials of slice type, we also answers positively a relating Jong's question asking whether there is an infinite set of Alexander polynomials such that every pair of distinct elements in the set is realizable by a pair of knots with Gordian distance one. In §6, we investigate the effect by one cross-change in the classes of the 2-bridge knots of genus 2 and the 3-strand pretzel knots of genus 2, realizing all the Alexander polynomials of degree 2. We show in Example 6.2 that the set of Alexander polynomials with degree 2 of slice type is directly shown to be an infinite set of Alexander polynomials such that every pair of distinct elements in the set is realizable by a pair of knots with Gordian distance one. In §7, Appendix A on the Mayer-Vietoris exact sequence for a certain triplet is given, which we use for the proof of Theorem 1.1.

## 2. Some concepts on the knot module

For a finitely generated $\Lambda$-module $H$, let $TH$ be the $\Lambda$-torsion part of $H$, and $BH = H/TH$ the $\Lambda$-torsion-free part of $H$. Let

$$DH = \{x \in H \mid \exists \text{coprime } \ a_1, a_2, \dots, a_r \in \Lambda (r > 1) \ \text{ with } \ a_i x = 0 (\forall i)\},$$

which is known to be the unique maximal finite $\Lambda$-submodule of $H$. Let $T_D H = TH/DH$. These concepts are indispensable in computing the extension $\Lambda$-modules $E^q H = Ext^q_\Lambda(H;\Lambda)$ $(q \geq 0)$ needed to prove Theorem 1.1. Some properties on the extension $\Lambda$-modules are stated as follows (cf. [4, §3], [8]):

**2.1 Properties on the extension $\Lambda$-modules.** We have the following properties (1)-(5) of the type $q$th extension $\Lambda$-module $E^q H = Ext^q_\Lambda(H;\Lambda)$ of a finitely generated $\Lambda$-module $H$.

(1) There is a natural $\Lambda$-isomorphism $E^0 H = \hom_\Lambda(H,\Lambda) \cong \hom_\Lambda(BH,\Lambda)$ which is necessarily a free $\Lambda$-module.

(2) There is a natural exact sequence

$$\hom_\Lambda(H, Q(\Lambda)) \to \hom_\Lambda(H, Q(\Lambda)/\Lambda) \to E^1 H \to 0$$

for the quotient field $Q(\Lambda)$ of $\Lambda$, and the short exact sequence $0 \to TH \to H \to BH \to 0$ induces a natural short exact sequence

$$0 \to E^1 BH \to E^1 H \to E^1 TH \to 0.$$

In this short exact sequence, we have $E^1 BH = DE^1 H$ and a natural $\Lambda$-isomorphism $E^1 E^1 H \cong T_D H$. Further, $E^1 BH = 0$ if and only if $BH$ is a free $\Lambda$-module.

(3) There are natural $\Lambda$-isomorphisms

$$E^2 H \cong E^2 DH \cong \hom(DH, Q/Z) \quad \text{and} \quad E^2 E^2 H \cong DH.$$

(4) There is a natural short exact sequence

$$0 \to BH \to E^0 E^0 BH \to E^2 E^1 BH \to 0.$$

(5) $E^q H = 0$ $(q \geqq 3)$.

For example, we have $E^1(\Lambda/(a)) \cong \Lambda/(a)$ and $E^q(\Lambda/(a)) = 0$ $(q \neq 1)$ for a non-zero element $a \in \Lambda$. In particular, for the ring $\Lambda_n = Z_n[t, t^{-1}] = \Lambda/(n)$ with $n$ a non-zero integer, we have $E^1 \Lambda_n \cong \Lambda_n$ and $E^q \Lambda_n = 0$ for every $q \neq 1$. If $a, b \in \Lambda$ are coprime, then we have $E^2 D(a,b) \cong D(a,b)$ and $E^q D(a,b) = 0$ for $q \neq 2$. For a finitely generated $\Lambda$-module $H$, we denote by $H^{op}$ the same $\Lambda$-module as $H$ but with the action of $t$ by the identity $t{\cdot}x = t^{-1}x$ for every $x \in H$. Let $E = E(K)$ be the compact exterior of a knot $K$ in the 3-sphere $S^3$. Let $\tilde{E} \to E$ be the infinite cyclic connected covering associated with $H_1(E) \cong Z$.[1] The $\Lambda$-module $M(K) = H_1(\tilde{E})$ is called the *module* of

[1] Throughout this paper, homologies will mean homologies with integer coefficients unless otherwise stated.

the knot $K$. Let $X = X(K)$ be the 0-surgery of $S^3$ along $K$. Taking the infinite cyclic connected covering $\tilde{X} \to X$ associated with $H_1(X) \cong H_1(E) = Z$, we have a natural isomorphism $H_1(\tilde{E}) \cong H_1(\tilde{X})$, and thus the $\Lambda$-module $H_1(\tilde{X})$ is also regarded as the module $M(K)$. We also consider the $\Lambda_Q$-module $M(K;Q) = M(K) \otimes Q$ of a knot $K$ for $\Lambda_Q = Q[t, t^{-1}]$. A $\Lambda_Q$-*cyclic knot* is a knot $K$ whose $\Lambda_Q$-module $M(K;Q)$ is a $\Lambda_Q$-cyclic module $\Lambda_Q/(a)$ where we can take as $a$ the Alexander polynomial of the knot $K$ in $\Lambda$. A $\Lambda$-*cyclic knot* is a knot $K$ whose $\Lambda$-module $M(K)$ is a $\Lambda$-cyclic module $\Lambda/(a)$. A *cross-change loop* of a knot $K$ in the 3-sphere $S^3$ is a loop $\gamma \subset E$ which is the boundary of a disk $\Delta$ in $S^3$ meeting $K$ transversely in two interior points with intersection number $\mathrm{Int}(\Delta, K) = 0$. The *twist family* of a knot $K$ along a cross-change loop $\gamma$ is the family $\mathcal{F}(K;\gamma)$ of the knot $K^n$ obtained from $K$ by the $n$-full twist along $\gamma$ for any $n \in Z$, where we take $K^0 = K$. We can regard the loop $\gamma$ as a cross-change loop of every knot $K^n \in \mathcal{F}(K;\gamma)$. Then we see that $d_G(K^n, K^{n+1}) \leqq 1$ for every $n$ and every pair of knots with Gordian distance one occurs as a pair of consecutive knots $K^n, K^{n+1} \in \mathcal{F}(K;\gamma)$ for a knot $K$ and a cross-change loop $\gamma$. Let $\tilde{\gamma}$ be the preimage of a cross-change loop $\gamma$ of a knot $K$ under the covering $\tilde{E} \to E$. The *cross-change module* of a knot $K$ on a cross-change loop $\gamma$ is a $\Lambda$-submodule $M(\gamma;K) \subset M(K)$ which is the image of the natural homomorphism $H_1(\tilde{\gamma}) \to H_1(\tilde{E}) = M(K)$. The quotient module $R = R(K;\gamma) = M(K)/M(\gamma;K)$ is called the *residue module* of a knot $K$ on a cross-change loop $\gamma$. We have a short exact sequence

$$0 \to M(\gamma;K) \to M(K) \to R \to 0$$

determined uniquely by the cross-change loop $\gamma$ of the knot $K$ which we call the *cross-change short exact sequence*. We note that *the cross-change module $M(\gamma;K)$ is $\Lambda$-cyclic, i.e.,*

$$M(\gamma;K) \cong \Lambda/(c)$$

*for an element $c \in \Lambda$ with $c(1) = \pm 1$.* In fact, the kernel of the natural homomorphism $H_1(\tilde{\gamma}) \to M(K)$ must be a free $\Lambda$-module of rank one because $H_1(\tilde{\gamma}) \cong \Lambda$ and every $\Lambda$-submodule of $M(K)$ is a torsion $\Lambda$-module of projective dimension $\leqq 1$. Hence $M(\gamma;K) \cong \Lambda/(c)$ for an element $c \in \Lambda$. Since $t-1$ induces an automorphism of every quotient $\Lambda$-module of $M(K)$, we see that $t-1$ is an automorphism of every $\Lambda$-submodule of $M(K)$ and hence , in particular, of $M(\gamma;K) = \Lambda/(c)$. Thus, we have $c(1) = \pm 1$.

The following lemma shows that the residue module $R(K;\gamma)$ is independent of a choice of a knot $K^n \in \mathcal{F}(K;\gamma)$.

**Lemma 2.2.** The residue module $R(K;\gamma)$ is $\Lambda$-isomorphic to the residue module $R(K^n;\gamma)$ of every knot $K^n \in \mathcal{F}(K;\gamma)$.

**Proof.** Since $H_0(\tilde{\gamma}) \cong \Lambda$, we have

$$H_1(\tilde{E}, \tilde{\gamma}) \cong TH_1(\tilde{E}, \tilde{\gamma}) \oplus \Lambda$$

by the homology exact sequence of the pair $(\tilde{E}(K), \tilde{\gamma})$ and hence the residue module $R(K;\gamma)$ is $\Lambda$-isomorphic to $TH_1(\tilde{E}, \tilde{\gamma})$. Using that there is an excision $\Lambda$-isomorphism $H_1(\tilde{E}(K^n), \tilde{\gamma}) \cong H_1(\tilde{E}(K), \tilde{\gamma})$, we see that the residue module $R(K^n;\gamma)$ coincides with the residue module $R(K;\gamma)$. This completes the proof.

By Lemma 2.2, we may call $R = R(K;\gamma) = R(K^n;\gamma)$ the *residue module* of the twist family $\mathcal{F}(K;\gamma)$. For the Alexander polynomial $a$ of a knot $K$, we denote by $a^n$ the Alexander polynomial of the knot $K^n \in \mathcal{F}(K;\gamma)$. We state the finite condition of the residue module $R$ of the twist family $\mathcal{F}(K;\gamma)$ in terms of the cross-change module and the residue module as follows:

**Lemma 2.3.** The following conditions (1)-(3) on the twist family $\mathcal{F}(K;\gamma)$ are mutually equivalent:

(1) The residue module $R$ of the twist family $\mathcal{F}(K;\gamma)$ is a finite $\Lambda$-module.

(2) The cross-change module $M(\gamma;K^n)$ of every knot $K^n \in \mathcal{F}(K;\gamma)$ is $\Lambda$-isomorphic to the $\Lambda$-cyclic module $\Lambda/(a^n)$.

(3) The cross-change module $M(\gamma;K^n)$ of a knot $K^n \in \mathcal{F}(K;\gamma)$ is $\Lambda$-isomorphic to the $\Lambda$-cyclic module $\Lambda/(a^n)$.

**Proof.** To see that (1) $\Rightarrow$ (2), assume that the residue module $R$ of every knot $K^n \in \mathcal{F}(K;\gamma)$ is finite. Since the cross-change module $M(\gamma;K^n)$ is a $\Lambda$-cyclic module $\Lambda/(c)$ for an element $c \in \Lambda$ with $c(1) = \pm 1$ and induces a $\Lambda_Q$-isomorphism $M(\gamma;K^n) \otimes Q \cong M(K^n;Q)$, we have $c = ra^n$ for a rational number $r$ and the Alexander polynomial $a^n$ of $K^n$. Using that $a^n(1) = 1$, we see that $r = \pm 1$ and $M(\gamma;K^n) \cong \Lambda/(a^n)$, showing (2). There is nothing to prove that (2) $\Rightarrow$ (3). We show that (3) $\Rightarrow$ (1). Since the natural $\Lambda_Q$-homomorphism $M(\gamma;K^n) \otimes Q \to M(K^n;Q)$ is a monomorphism and

$$\dim_Q M(\gamma;K^n) \otimes Q = \dim_Q M(K^n;Q) = \text{ the degree of } \ a^n,$$

we have a natural $\Lambda$-isomorphism $M(\gamma;K^n) \otimes Q \cong M(K^n;Q)$ and hence we see from the cross-change short exact sequence that $R \otimes Q = 0$, so that $R$ is a $Z$-torsion $\Lambda$-module. Using that $R$ is finitely generated over $\Lambda$ and admits $t-1$ as an automorphism, we can see that $R$ is finite (cf. see, for example, [4, §3]), implying that (3) $\Rightarrow$ (1). This completes the proof.

## 3. Proof of Theorem 1.1

We first investigate the Alexander polynomials $a^0, a^1$ of the knots $K^0, K^1$ in the twist family $\mathcal{F}(K;\gamma)$. Let $X^0 = \chi(K^0, 0)$ and $X^1 = \chi(K^1, 0)$ be the 0-surgery manifolds of $K^0$ and $K^1$, respectively. We obtain the pair $(S^3, K^1)$ from the pair $(S^3, K^0)$ by a $+1$-surgery of $S^3$ along the cross-change loop $\gamma$. This enables us to construct a compact oriented 4-manifold $Y$ with $\partial Y = (-X^0) \cup X^1$ obtained from $X^0 \times [-1,1]$ by attaching a 2-handle $h^2 = \mathbf{D}^2 \times \mathbf{D}^2$ so that $\dot{h}^2 = \partial(\mathbf{D}^2) \times \mathbf{D}^2$ is identified with a tubular neighborhood $N(\gamma) \times 1$ of $\gamma \times 1$ in $X^0 \times 1$ by the specified framing $+1$. Then we have natural isomorphisms on infinite cyclic groups:

$$H_1(X^0) \xrightarrow{\cong} H_1(Y) \xleftarrow{\cong} H_1(X^1).$$

Let $(\tilde{Y}; \tilde{X}^0, \tilde{X}^1)$ be the infinite cyclic connected covering of the cobordism $(Y; X^0, X^1)$. The following computations are used to prove Theorem 1.1.

**Lemma 3.1.** (1) If $a^0, a^1$ are coprime, then the residue module $R$ of the twist family $\mathcal{F}(K;\gamma)$ is a finite $\Lambda$-module.

(2) We have $H_2(\tilde{Y}) \cong \Lambda \oplus \Lambda/(t-1)$, $H_2(\tilde{Y}, \tilde{X}^\varepsilon) \cong \Lambda$ and $H_k(\tilde{Y}, \tilde{X}^\varepsilon) = 0$ for $\varepsilon = 0, 1$ and $k \neq 2$.

(3) The residue module $R$ of the twist family $\mathcal{F}(K;\gamma)$ is $\Lambda$-isomorphic to $H_1(\tilde{Y})$.

(4) If $H_1(\tilde{Y})$ is a finite $\Lambda$-module, then the cokernel of the natural $\Lambda$-homomorphism $i_*^\varepsilon : H_2(\tilde{Y}) \to H_2(\tilde{Y}, \tilde{X}^\varepsilon)$ is $\Lambda$-isomorphic to $\Lambda/(a^\varepsilon)$ for every $\varepsilon = 0, 1$.

(5) We have $H_3(\tilde{Y}, \partial\tilde{Y}) \cong \Lambda/(t-1)$. If $H_1(\tilde{Y})$ is a finite $\Lambda$-module, then $H_2(\tilde{Y}, \partial\tilde{Y})$ is a $\Lambda$-torsion-free module and there is a $\Lambda$-isomorphism $E^1 H_2(\tilde{Y}, \partial\tilde{Y}) \cong H_1(\tilde{Y})^{op}$.

**Proof.** Since $a^\varepsilon M(K^\varepsilon) = 0$ for every $\varepsilon = 0, 1$, we see that $a^\varepsilon R = 0$ for every $\varepsilon = 0, 1$. If $R$ is generated by $s$ elements over $\Lambda$, we have a $\Lambda$-epimorphism from the $s$-fold direct sum $D(a^0, a^1)^s$ of $D(a^0, a^1)$ onto $R$. If $a^0, a^1$ are coprime, then the determinant ring $D(a^0, a^1)$ is finite and hence $R$ is finite, showing (1). By the exact sequence of the pair $(\tilde{Y}, \tilde{X}^0)$, we have an exact sequence

$$H_2(\tilde{Y}) \to H_2(\tilde{Y}, \tilde{X}^0) \xrightarrow{\partial} H_1(\tilde{X}^0) \to H_1(\tilde{Y}) \to H_1(\tilde{Y}, \tilde{X}^0).$$

Since $H_k(\tilde{Y}, \tilde{X}^0) \cong H_k(\tilde{h}^2, \tilde{\dot{h}}^2)$ for the lift $(\tilde{h}^2, \tilde{\dot{h}}^2)$ of $(h^2, \dot{h}^2)$ to $\tilde{Y}$ by an excision isomorphism, we see that $H_k(\tilde{Y}, \tilde{X}^0) = 0$ for $k \neq 2$ and $H_2(\tilde{Y}, \tilde{X}^0) \cong \Lambda$ with a generator represented by a lifting of a handle core $\mathbf{D}^2 \times 0$ of $h^2$. By considering the dual cobordism $(-Y; X^1, X^0)$, we see also that $H_k(\tilde{Y}, \tilde{X}^1) = 0$ for $k \neq 2$ and $H_2(\tilde{Y}, \tilde{X}^1) \cong \Lambda$. Next, we apply the duality theorem in [4] for the triplet $(\tilde{Y}; \tilde{X}^0, \tilde{X}^1)$ to see that $H_2(\tilde{Y}) \cong \Lambda \oplus \Lambda/(t-1)$. Since the natural homomorphism $H_1(\tilde{X}^0) \to H_1(\tilde{Y})$ is onto, we have

$$T_D H_1(\tilde{Y}, \partial\tilde{Y}) = H_1(\tilde{Y}, \partial\tilde{Y}) \cong \tilde{H}_0(\partial\tilde{Y}) = \Lambda/(t-1).$$

Hence, by the first duality, $T_D H_2(\tilde{Y}) \cong E^1 T_D H_1(\tilde{Y}, \partial\tilde{Y})^\# \cong \Lambda/(t-1)$. Using that $BH_1(\tilde{Y}, \partial\tilde{Y}) = 0$ and $DH_0(\tilde{Y}, \partial\tilde{Y}) = 0$, we have $DH_2(\tilde{Y}) = 0$ by the second duality, showing that $TH_2(\tilde{Y}) = T_D H_2(\tilde{Y}) \cong \Lambda/(t-1)$. Since $DH_1(\tilde{Y}, \partial\tilde{Y}) = 0$, the second duality implies $E^1 BH_2(\tilde{Y}) = 0$. This means that $BH_2(\tilde{Y})$ is $\Lambda$-free. Using that the natural homomorphism $i^0 : H_2(\tilde{Y}) \to H_2(\tilde{Y}, \tilde{X}^0) \cong \Lambda$ is injective whose cokernel is a torsion $\Lambda$-module, we see that $BH_2(\tilde{Y}) \cong \Lambda$, so that $H_2(\tilde{Y}) \cong \Lambda \oplus \Lambda/(t-1)$, showing (2). The exact sequence of the pair $(\tilde{Y}, \tilde{X}^0)$ now induces an exact sequence

$$H_2(\tilde{Y}, \tilde{X}^0) \stackrel{\partial}{\to} H_1(\tilde{X}^0) \to H_1(\tilde{Y}) \to 0.$$

Here $H_1(\tilde{X}^0) = M(K^0)$ and the image of the boundary map $\partial : H_2(\tilde{Y}, \tilde{X}^0) \to H_1(\tilde{X}^0)$ is equal to the cross-change module $M(\gamma; K^0)$, we see that the residue module $R$ of $K^0$ on $\gamma$ and hence of the twist family $\mathcal{F}(K;\gamma)$ by Lemma 2.1 is $\Lambda$-isomorphic to $H_1(\tilde{Y})$, showing (3). The exact sequence of the pair $(\tilde{Y}, \tilde{X}^0)$ implies the following exact sequence:

$$\Lambda \stackrel{i^0_*}{\to} \Lambda \to H_1(\tilde{X}^0) \to H_1(\tilde{Y}) \to 0,$$

which induces a short exact sequence $0 \to \Lambda/(c) \to H_1(\tilde{X}^0) \to H_1(\tilde{Y}) \to 0$, where $c = i^0_*(1)$. By (3), we have $M(\gamma; K^0) = \Lambda/(c)$ and $R = H_1(\tilde{Y})$. If $R$ is finite, then we see from Lemma 2.3 that $c = \pm a^0$, showing (4). By the zeroth duality, $BH_1(\tilde{Y}) = 0$ implies $BH_3(\tilde{Y}, \partial\tilde{Y}) = 0$. By the first duality, $T_D H_3(\tilde{Y}, \partial\tilde{Y}) \cong E^1 H_0(\tilde{Y})^{op} \cong \Lambda/(t-1)$. Since $BH_0(\tilde{Y}) = DH_{-1}(\tilde{Y}) = 0$, we have $DH_3(\tilde{Y}, \partial\tilde{Y}) = 0$ by the second duality, implying that $H_3(\tilde{Y}, \partial\tilde{Y}) \cong \Lambda/(t-1)$. We note that $T_D H_2(\tilde{Y}, \partial\tilde{Y}) \cong E^1 T_D H_1(\tilde{Y})^{op}$ by the first duality. Assume that $H_1(\tilde{Y})$ is a finite $\Lambda$-module. Then we see that $T_D H_1(\tilde{Y}) = 0$, so that $T_D H_2(\tilde{Y}, \partial\tilde{Y}) = 0$. Since $BH_1(\tilde{Y}) = DH_0(\tilde{Y}) = 0$, we have $DH_2(\tilde{Y}, \partial\tilde{Y}) = 0$ by the second duality. This shows that $H_2(\tilde{Y}, \partial\tilde{Y})$ is $\Lambda$-torsion-free. Then, by the second duality, we have $H_1(\tilde{Y})^{op} \cong E^1 H_2(\tilde{Y}, \partial\tilde{Y})$, showing (5). This completes the proof of Lemma 3.1.

The following lemma means that the coprimeness of the Alexander polynomials $a^n, a^m$ $(n \neq m)$ is equivalent to the finiteness of the residue module $R$.

**Lemma 3.2.** The following conditions (1)-(3) on the twist family $\mathcal{F}(K;\gamma)$ are mutually equivalent:

(1) The residue module $R$ of the twist family $\mathcal{F}(K;\gamma)$ is a finite $\Lambda$-module.

(2) The Alexander polynomials $a^m$, $a^n$ of any pair $K^m, K^n \in \mathcal{F}(K;\gamma)$ with $m \neq n$ are coprime.

(3) The Alexander polynomials $a^m$, $a^n$ of a pair $K^m, K^n \in \mathcal{F}(K;\gamma)$ with $m \neq n$ are coprime.

**Proof.** To show that (1)$\Rightarrow$ (2), we first show that $a^0, a^1$ is coprime. Assume that the residue module $R$ is finite, meaning that $H_1(\tilde{Y})$ is finite by Lemma 3.1(3). We consider the following Mayer-Vietoris exact sequence for the triplet $(\tilde{Y}; \tilde{X}^0, \tilde{X}^1)$ which is explained as a general setting in Appendix A of this paper.

$$\begin{aligned} H_3(\tilde{Y},\partial\tilde{Y}) &\to H_2(\tilde{Y}) \xrightarrow{i^0_*-i^1_*} H_2(\tilde{Y},\tilde{X}^0)\oplus H_2(\tilde{Y},\tilde{X}^1) \\ &\xrightarrow{j_*} H_2(\tilde{Y},\partial\tilde{Y}) \to H_1(\tilde{Y}) \to H_1(\tilde{Y},\tilde{X}^0)\oplus H_1(\tilde{Y},\tilde{X}^1) = 0. \end{aligned}$$

Let $J$ be the image of $j_*$. Using that $H_3(\tilde{Y},\partial\tilde{Y}) = \Lambda/(t-1)$ by Lemma 3.1 (5), we have the following two short exact sequences (MV1) and (MV2):

(MV1) $\quad 0 \to BH_2(\tilde{Y}) \xrightarrow{i^0_*-i^1_*} H_2(\tilde{Y},\tilde{X}^0)\oplus H_2(\tilde{Y},\tilde{X}^1) \to J \to 0.$

(MV2) $\quad 0 \to J \to H_2(\tilde{Y},\partial\tilde{Y}) \to H_1(\tilde{Y}) \to 0.$

The short exact sequence (MV1) induces an exact sequence

$$0 \to E^0J \to E^0H_2(\tilde{Y},\tilde{X}^0)\oplus E^0H_2(\tilde{Y},\tilde{X}^1) \xrightarrow{(i^0_*)'-(i^1_*)'} E^0BH_2(\tilde{Y}) \to E^1J \to 0,$$

where the map $(i^0_*)'-(i^1_*)'$ denotes the induced map of $i^0_*-i^1_*$. The image of this map $(i^0_*)'-(i^1_*)'$ is just the $\Lambda$-ideal $(a^0,a^1)$ in $E^0BH_2(\tilde{Y}) = E^0\Lambda = \Lambda$ and we have

$$E^1J \cong \Lambda/(a^0,a^1),$$

which is finite by 2.1(2) because $J$ is $\Lambda$-torsion-free by Lemma 3.1(5). This means that $a^0,a^1$ must be coprime. To see that $a^m,a^n$ are coprime for $m\neq n$, we compute the Conway polynomial $\nabla(K^n;z)$ of $K^n$ for all $n$ (cf. [6]). By Conway's identity, we have the identity

$$\nabla(K^n;z) = \nabla(K^0;z) + nz\nabla(L;z)$$

for the Conway polynomial $\nabla(L;z)$ of the 2-component link $L$ from $K^1$ by splicing the crossing point of $K^1$ caused by twisting $K^0$ along $\gamma$. Note that

$$a^n = \nabla(K^n;t^{-\frac{1}{2}}-t^{\frac{1}{2}}).$$

Let $b = (t^{-\frac{1}{2}}-t^{\frac{1}{2}})\nabla(L;t^{-\frac{1}{2}}-t^{\frac{1}{2}}) \in \Lambda$. Using that $a^0,a^1$ are coprime, we can see that $a^0$ is coprime to $b$, so that $a^m = a^0+mb, a^n = a^0+nb$ $(m\neq n)$ are coprime, showing that (1) $\Rightarrow$ (2). There is nothing to prove that (2) $\Rightarrow$ (3). The assertion that (3) $\Rightarrow$ (1) follows from Lemma 3.1 (1) by considering $a^0$ and $a^1$. This completes the proof of Lemma 3.2.

The following corollary is direct from the calculation of $a^n-a^0$ done in the proof of Lemma 3.2.

**Corollary 3.3.** The determinant ring $D(a^n,a^{n+1})$ for the Alexander polynomials $a^n$ of the knots $K^n\in\mathcal{F}(K;\gamma)$ is independent of a choice of the integer $n$.

We are ready to prove Theorem 1.1.

**3.4 Proof of Theorem 1.1.** We take $K^0 = K$, $K^1 = K'$ for a cross-change loop $\gamma$. Then we take $a^0 = a$, and $a^1 = a'$. Since $a, a'$ are coprime, the residue module $R$ of the twist family $\mathcal{F}(K;\gamma)$ is finite and $R = H_1(\tilde{Y})$ by Lemma 3.1. Then we have $E^1H_1(\tilde{Y}) = 0$ by 2.1 (2). The short exact sequence (MV2) in the proof of Lemma 3.2 induces a short exact sequence

$$0 \to E^1H_2(\tilde{Y}, \partial\tilde{Y}) \to E^1J \to E^2H_1(\tilde{Y}) \to 0.$$

We have already shown that $E^1J = \Lambda/(a^0, a^1) = D(a^0, a^1)$ in the proof of Lemma 3.2. By 2.1(3), we have

$$E^2H_1(\tilde{Y}) \cong \hom(R, Q/Z).$$

By Lemma 3.1 (5), we have $E^1H_2(\tilde{Y}, \partial\tilde{Y}) \cong R^{op}$. Thus, we have an exact sequence

$$0 \to R^{op} \to D(a^0, a^1) \to \hom(R, Q/Z) \to 0.$$

Since $D(a^0, a^1)^{op} \cong D(a^0, a^1)$, this exact sequence is changed into an exact sequence

$$0 \to R \to D(a^0, a^1) \to R^{\#} \to 0.$$

This completes the proof of Theorem 1.1.

If $K$ is $\Lambda$-cyclic, then Theorem 1.1 may be stated as follows:

**Corollary 3.5.** Let $a$, $a'$ be a pair of coprime Alexander polynomials which is realizable by a pair of knots $K$, $K'$ with $d_G(K, K') = 1$. If $K$ is $\Lambda$-cyclic, then there is an element $c \in \Lambda$ with $c(1)$ any previously given integer such that $D(a, a') = D(a, c\bar{c})$.

**Proof.** Since $a$, $a'$ are coprime and $K$ is $\Lambda$-cyclic, we see from Lemmas 2.3, 3.2 that the cross-change short exact sequence is given by

$$0 \to \Lambda/(a) \xrightarrow{e} \Lambda/(a) \to R \to 0$$

for the finite residue module $R$. Then we have $R = D(a, c)$ for any element $c \in \Lambda$ representing the image of the class of $1 \in \Lambda$ in $\Lambda/(a)$ under the monomorphism $e$. Then $c$ is coprime to $a$. For any integer $m$, we can replace $c$ by an element $c' \in \Lambda$ with $c'(1) = m$, because the element $c' = c - (c(1) - m)a \in \Lambda$ has $D(a, c') = D(a, c)$ and $c'(1) = m$ for $a(1) = 1$. The short exact sequence in Theorem 1.1 implies the following short exact sequence (over $\Lambda$)

$$0 \to D(a, c) \xrightarrow{f} D(a, a') \xrightarrow{g} D(a, \bar{c}) \to 0.$$

We show that the class $[c\bar{c}] \in D(a,a')$ represented by the element $c\bar{c} \in \Lambda$ is zero. In fact, since $g([\bar{c}]) = \bar{c}g([1]) = 0 \in D(a,\bar{c})$, there is an element $x \in D(a,c)$ with $f(x) = [\bar{c}] \in D(a,a')$. Then $cx = 0 \in D(a,c)$ and $[c\bar{c}] = cf(x) = f(cx) = 0 \in D(a,a')$, as desired. Thus, there is a natural $\Lambda$-epimorphism $D(a,c\bar{c}) \to D(a,a')$. Because we have also a natural short exact sequence

$$0 \to D(a,c) \to D(a,c\bar{c}) \to D(a,\bar{c}) \to 0,$$

the abelian group orders of $D(a,c\bar{c})$ and $D(a,a')$ are equal, so that $D(a,c\bar{c}) = D(a,a')$. This completes the proof.

The following lemma explains when the coprime condition and the strongly coprime condition break down.

**Lemma 3.6.** For the pair $a, a'$ of Alexander polynomials of a pair of knots $K$, $K'$ with $d_G(K,K') = 1$, we have the following (1) and (2).

(1) If the knot $K$ or $K'$ is not $\Lambda_Q$-cyclic, then the Alexander polynomials $a, a'$ are not coprime. If $K$ is $\Lambda_Q$-cyclic but $K'$ is not $\Lambda_Q$-cyclic, then the $\Lambda_Q$-module $M(K';Q)$ is a direct sum of two $\Lambda_Q$-cyclic modules.

(2) If the knot $K$ or $K'$ is not $\Lambda$-cyclic, then the Alexander polynomials $a, a'$ are not strongly coprime.

**Proof.** We take $K = K^0$, $K^1 = K'$, $a^0 = a$, and $a^1 = a'$. For (1), suppose that $a^0$ and $a^1$ are coprime. Then the residue module $R$ is finite by Lemma 3.2. Since $M(\gamma; K^n) \cong \Lambda/(a^n)$ and $R \otimes Q = 0$, we see from the cross-change short exact sequence that $M(K^n;Q) \cong \Lambda_Q(a^n)$, contradicting that $K^0$ or $K^1$ is not $\Lambda_Q$-cyclic. Hence $a^0, a^1$ are not coprime. If $K^0$ is $\Lambda_Q$-cyclic, then the $\Lambda_Q$-module $R \otimes Q$ is also $\Lambda_Q$-cyclic by the cross-change short exact sequence. Using that the cross-change-module is always $\Lambda$-cyclic, we see from the cross-change short exact sequence that if the knot $K^1$ is not $\Lambda_Q$-cyclic, then the $\Lambda_Q$-module $M(K^1;Q)$ is a direct sum of two $\Lambda_Q$-cyclic modules (since $\Lambda_Q$ is PID), showing (1). For (2), suppose that $a^0$ and $a^1$ are strongly coprime. Then the determinant ring $D(a^0,a^1) = 0$, so that $R = 0$ by Theorem 1.1. Hence we have $M(K^n) \cong M(\gamma; K^n) = \Lambda/(a^n)$, contradicting that $K^0$ or $K^1$ is not $\Lambda$-cyclic, completing the proof of Lemma 3.6.

Every Alexander polynomial $a$ is realized by an unknotting number one knot $K$ (see [7, 15] ). Since $a$ is strongly coprime to $1 \in \Lambda$, we have $M(K) \cong \Lambda/(a)$ by Lemma 3.6 taking $b = 1$ (cf. [5, 10]).

## 4. The Alexander polynomial distance

*For the remainder of this paper, we often use the notation $K_a$ for a knot $K$ with Alexander polynomial $a$.* The *Alexander polynomial distance* between Alexander polynomials $a, b$ is defined by the identity:

$$\rho(a,b) = \min_{K_a, K_b} d_G(K_a, K_b).$$

Since there is an unknotting number one knot $K_a$ for any Alexander polynomial $a$, we have $\rho(a,1) \leqq 1$, so that the inequality $\rho(a,b) \leqq 2$ holds for all Alexander polynomials $a, b$, because we have

$$\rho(a,b) \leqq \rho(a,1) + \rho(1,b) \leqq 1 + 1 = 2.$$

Thus, the following problem by Jong comes to mind (cf. [1, 2, 3]):

**Jong's Problem.** Determine a pair of Alexander polynomials $a, b$ such that $\rho(a,b) = 2$.

We solve this problem partially in this section. This problem was caused from a study on an asymptotic behavior on the Alexander polynomials of alternating knots (cf. [1, 2, 3]). To determine a pair with the Alexander polynomial distance 2, we split the short exact sequence $0 \to R \to D \to R^{\#} \to 0$ with $D = D(a,b)$ in Theorem 1.1 into the short exact sequences

$$0 \to R_p \to D_p \to R_p^{\#} \to 0$$

of the $p$-primary components $R_p$ and $D_p$ of $R$ and $D$, respectively, for all primes $p$. The following lemma is useful in applying Theorem 1.1.

**Lemma 4.1.** If the $p$-primary component $D_p$ is given in the form $\Lambda/(p^m, c)$ where $m$ is a positive integer and $c \in \Lambda$ such that the $p$-reduction $c_p \in \Lambda_p$ of $c$ is a non-unit irreducible element, then $m$ is even and the element $\bar{c}_p \in \Lambda_p$ is equal to $c_p$ up to units of $\Lambda_p$.

**Proof.** Let $p^k$ be the highest order of the elements of $R_p$. Since the $\Lambda$-module $R_{p,i} = \{x \in R_p \,|\, p^i x = 0\}$ for every $i$ $(1 \leqq i \leqq k)$ is a non-trivial $\Lambda$-submodule of $p^{m-i}\Lambda/(p^m, c) \cong \Lambda/(p^i, c)$, we see that the $\Lambda_p$-module $R_{p,i}/R_{p,i-1}$ $(1 \leqq i \leqq k)$ is a non-trivial $\Lambda_p$-submodule of $\Lambda_p/(c_p)$, where we take $R_{p,0} = 0$. Since $c_p \in \Lambda_p$ is irreducible, we see that $R_{p,i}/R_{p,i-1}$ is $\Lambda$-isomorphic to $\Lambda_p/(c_p)$ for every $i$ $(1 \leqq i \leqq k)$. Thus, the abelian group order of $R_p$ is $p^{dk}$ where $d$ denotes the Laurent polynomial degree($\geqq 1$) of $c_p \in \Lambda$. Using that $R_p$ and $R_p^{\#}$ are isomorphic to each other as abelian groups, we see that the group order of $R_p^{\#}$ is also equal to $p^{dk}$, so that the group order $p^{md}$ of $D_p$ is equal to $p^{2dk}$. Thus, we have $m = 2k$. Since $D \cong D^{op}$ as $\Lambda$-modules, we have $D_p = \Lambda/(p^m, c) \cong D_p^{op} = \Lambda/(p^m, \bar{c})$ as $\Lambda$-modules, so that $\Lambda_p/(c_p) \cong \Lambda_p/(\bar{c}_p)$, implying that $\bar{c}_p$ is equal to $c_p$ up to units of $\Lambda_p$. This completes the proof of Lemma 4.1.

We observe that the $p$-primary determinant ring $D_p$ is calculable in principle as follows: Since $a, b$ are coprime, there is a positive integer $n$ such that $aa' + bb' = n$ for some $a', b' \in \Lambda$. Taking the prime decomposition $n = p_1^{n_1} p_2^{n_2} \dots p_s^{n_s}$, we have

$$D = \Lambda/(n, a, b) \cong \oplus_{i=1}^{s} \Lambda/(p_i^{n_i}, a, b).$$

and hence $D_{p_i} = \Lambda/(p_i^{n_i}, a, b)$. We consider the Alexander polynomials

$$A_n = (1 - 2n) + n(t + t^{-1}) \quad (n \in Z).$$

To consider the Alexander polynomial distance $d(A_m, A_n)$, let $d(\geqq 1)$ be the greatest common divisor of $m$ and $n$, and $m = dm'$, $n = dn'$. We have the identity

$$n'A_m - m'A_n = n' - m'$$

showing that the elements $A_m, A_n \in \Lambda$ are coprime when $m \neq n$. This coprimeness is also confirmed by Lemma 3.2, because, for the Whitehead link $L = O_1 \cup O_2$, the polynomials $A_n (n \in Z)$ are the Alexander polynomials of the knots in the twist family of $O_1$ along the cross-change loop $\gamma = O_2$ whose residue module $R$ is zero. Let $\delta' = |n' - m'|$ and $\delta = |n - m| = d\delta'$. Let $\delta^* = \delta' d'$ for any factor $d'$ of $d$. Since $m = n \pm d\delta'$, we have $m \equiv n \pmod{\delta^*}$. Hence the determinant ring $D(A_m, A_n)$ is given by

$$\Lambda/(\delta^*, A_m, A_n) = \Lambda/(\delta^*, A_n).$$

We note that there is a $\Lambda$-isomorphism

$$\Lambda/(\delta^*, A_n) \to \Lambda/(\delta', A_n).$$

In particular, we have

$$\Lambda/(\delta, A_n) \cong \Lambda/(\delta', A_n).$$

This is because there is an exact sequence

$$0 \to d'\Lambda/(\delta^*, A_n) \xrightarrow{\subset} \Lambda/(\delta^*, A_n) \to \Lambda/(d', A_n) \to 0$$

and we have $\Lambda/(\delta', A_n) = d'\Lambda/(\delta^*, A_n)$ and $\Lambda/(d', A_n) = 0$. The following corollary is a result from Theorem 1.1 and Lemma 4.1 and solves in part Jong's problem.

**Corollary 4.2.** Let $p$ be any prime number, and $n, \ell$ integers coprime to $p$. If $p$ is odd prime, then assume that $p$ is coprime to $1 - 4n$ with the Legendre symbol $\left(\dfrac{1-4n}{p}\right) = -1$. By fixing $p$, $n$ and $\ell$, let $\mathcal{S}(n + \ell p^*)$ be the infinite set consisting of the Alexander polynomials $A_n$ and $A_{n+\ell p^{2s+1}}$ for all $s = 0, 1, 2, \dots$. Then we have $\rho(A_{n'}, A_{n''}) = 2$ for any distinct elements $A_{n'}, A_{n''} \in \mathcal{S}(n + \ell p^*)$.

**Proof.** First, consider the case that $n' = n$ and $n'' = n + \ell p^{2s+1}$. The $p$-primary determinant ring $D_p = D(A_n, A_{n+\ell p^{2s+1}})_p$ is given by $\Lambda/(p^{2s+1}, A_n)$, for $\ell$ is coprime to $p$. When $p = 2$ and $n$ is odd, the $Z_2$-polynomial $A_n = 1 + t + t^{-1} \in \Lambda_2$ is irreducible. By Theorem 1.1 and Lemma 4.1, we have $\rho(A_n, A_{n+\ell 2^{2s+1}}) = 2$. When $p$ is an odd prime number coprime to $n$, then the elements $2, t^{-1}n, n \in \Lambda_p$ are units and we have

$$\frac{A_n}{t^{-1}n} = t^2 - \frac{2n-1}{n}t + 1 = (t - \frac{2n-1}{2n})^2 - \frac{1-4n}{4n^2}.$$

The Alexander polynomial $A_n$ is irreducible in $\Lambda_p$ if and only if the integer $1-4n$ is a quadratic non-residue modulo $p$, that is, $1-4n$ is coprime to $p$ and $\left(\dfrac{1-4n}{p}\right) = -1$. In this case, we have $\rho(A_n, A_{n+\ell p^{2s+1}}) = 2$ by Theorem 1.1 and Lemma 3.1. Next, consider the case that $n' = n + \ell p^{2s+1}$ and $n'' = n + \ell p^{2s'+1}$ for $s' > s$. Then we note that $n'' = n' + \ell' p^{2s+1}$ with $\ell' = \ell(p^{2(s'-s)} - 1)$ coprime to $p$. Further, $n'$ is coprime to $p$, and if $p$ is odd prime, then $1 - 4n'$ is coprime to $p$ and $\left(\dfrac{1-4n'}{p}\right) = -1$. By the first half argument, we have $\rho(A_{n'}, A_{n'+\ell' p^{2s+1}}) = \rho(A_{n'}, A_{n''}) = 2$. This completes the proof.

For example, we obtain $\rho(A_1, A_{-1}) = 2$ because $A_1, A_{-1} \in \mathcal{S}(1 + (-1)2^*)$, solving Nakanishi's problem. For $p = 3, 5, 7$, we see that $\left(\dfrac{1-4n}{p}\right) = -1$ if and only if $n \equiv -1 \pmod 3$, $n \equiv 1, 2 \pmod 5$, $n \equiv -1, \pm 3 \pmod 7$, respectively. For this check, we use Euler's criterion

$$\left(\frac{m}{p}\right) \equiv m^{\frac{p-1}{2}} \pmod p$$

for an integer $m$ coprime to an odd prime $p$.

## 5. Proof of Theorem 1.2 and its consequence

We prove Theorem 1.2 here by using Proposition 6 in Nakanishi's paper [13].

**5.1 Proof of Theorem 1.2.** Let $a$ be the Alexander polynomial of a knot $K$ with $d_G(O, K) = 1$, and $a'$ the Alexander polynomial of a knot $K'$ a knot with $d_G(K, K') = 1$. Nakanishi observed in [13, Proposition 6] that there are elements $b, c \in \Lambda$ with $\bar{b} = b$, $b(1) = 1$, and $c(1) = 0$ such that the matrix $\begin{pmatrix} \varepsilon a & c \\ \bar{c} & \varepsilon' b \end{pmatrix}$ for some $\varepsilon, \varepsilon' = \pm 1$ is an Alexander matrix of $K'$, whose determinant $\varepsilon\varepsilon' ab - c\bar{c}$ is equal to

$\varepsilon\varepsilon' a'$. This means that $a' \equiv \pm c\bar{c} \pmod{a}$ for an element $c \in \Lambda$. Conversely, assume that $a' \equiv \pm c\bar{c} \pmod{a}$ for distinct Alexander polynomials $a, a'$ and an element $c \in \Lambda$. Let $c' = c - c(1)a \in \Lambda$. Then $\bar{c}' = \bar{c} - c(1)a$, $c'(1) = c(1) - c(1)a(1) = 0$ and

$$c'\bar{c}' = c\bar{c} - (c + \bar{c})c(1)a + c(1)^2a^2 \equiv c\bar{c} \equiv \pm a' \pmod{a}.$$

Thus, we have $a' \equiv \pm c'\bar{c}' \pmod{a}$ and $c'(1) = 0$. By [7, 15], $a$ is the Alexander polynomial of a knot $K$ with $d_G(O, K) = 1$. Let $\varepsilon a' = \varepsilon ab - c'\bar{c}'$ for an element $b \in \Lambda$ and an $\varepsilon = \pm 1$. Then $b$ is an Alexander polynomial, i.e., $b(1) = 1$ and $\bar{b} = b$. Nakanishi observed in [13, Proposition 6] that the matrix $\begin{pmatrix} \varepsilon a & c' \\ \bar{c}' & b \end{pmatrix}$ is an Alexander matrix of a knot $K'$ with $d_G(K, K') = 1$. Since the determinant of this matrix is $\varepsilon ab - c'\bar{c}' = \varepsilon a'$, we see that $a'$ is the Alexander polynomial of the knot $K'$. This completes the proof of Theorem 1.2.

The following corollary is obtained from Theorem 1.2, where the latter half also answers Nakanishi's question asking whether there is an Alexander polynomial $a'$ except 1 such that $\rho(a', a) = 1$ for every Alexander polynomial $a$ distinct from $a'$.

**Corollary 5.2.** If $a_0$ is of slice type, i.e., $a_0 = c\bar{c}$ for an element $c \in \Lambda$, then we have $\rho(a_0, a) = 1$ for every Alexander polynomial $a$ distinct from $a_0$.

It is pointed out by Y. Nakanishi [14] that the knot $K = 8_9$ is a slice knot, but $d_G(K, K_{A_1}) \geqq 2$ for any knot $K_{A_1}$ with Alexander polynomial $A_1 = -1 + (t + t^{-1})$. By Corollary 5.2, we have $\rho(a, b) = 1$ for any distinct elements $a, b$ in the set of the Alexander polynomials of slice type (which is an infinite set), answering a question by Jong asking whether there is an infinite set of Alexander polynomials such that we have $\rho(a, b) = 1$ for any distinct elements $a, b$ in the set. We reconfirm this result directly for the set of the Alexander polynomials with degree 2 of slice type in Example 6.2.

## 6. Some examples on the Alexander polynomials of degree two

In the following example, we consider the Alexander polynomials of 2-bridge knots of genus one.

**Example 6.1.** Let $K(k, \ell)$ be the 2-bridge knot given by the Conway notation $C(2k, 2\ell)$ (see Fig. 1), whose Alexander polynomial is $A_{-k\ell} = (1 + 2k\ell) - k\ell(t + t^{-1})$. Since the polynomial $A_n$ is realized by the knot $K(1, -n)$ which is made a trivial knot by one cross-change, we have $\rho(A_n, A_0) = 1$ for all non-zero integers $n$. We note that the knot $K(k, \ell)$ is a trivial knot if $k\ell = 0$, the figure eight knot $4_1$ if $k\ell = 1$ and the

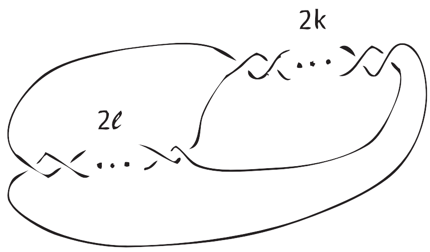


Figure 1: Two-bridge knots of genus one

(positive or negative) trefoil knot $3_1$ if $k\ell = -1$. Since the knot $K(k, \ell-1)$ is obtained from the knot $K(k, \ell)$ by one cross-change, we have the Alexander polynomial distance

$$\rho(A_{-k(\ell-1)}, A_{-k\ell}) = 1$$

for all integers $k, \ell$ with $k \neq 0$. By $\delta' = 1$, the elements $A_{-k(\ell-1)}, A_{-k\ell} \in \Lambda$ are strongly coprime and the determinant ring $D(A_{-k(\ell-1)}, A_{-k\ell}) = 0$. By Lemma 3.6, any knots $K_{A_{-k(\ell-1)}}$ and $K_{A_{-k\ell}}$ with $d_G(K_{A_{-k(\ell-1)}}, K_{A_{-k\ell}}) = 1$ are $\Lambda$-cyclic.

In the following example, we consider the Alexander polynomials of 3-strand pretzel knots of genus one.

**Example 6.2.** We consider the pretzel knot $K(k, \ell, m)$ of the three 2-string braids of $2k+1, 2\ell+1, 2m+1$-half-twists (see Fig. 2)[2], whose Alexander polynomial is given by

$$A_{[k,\ell,m]} = (1 - 2[k, \ell, m]) + [k, \ell, m](t + t^{-1})$$

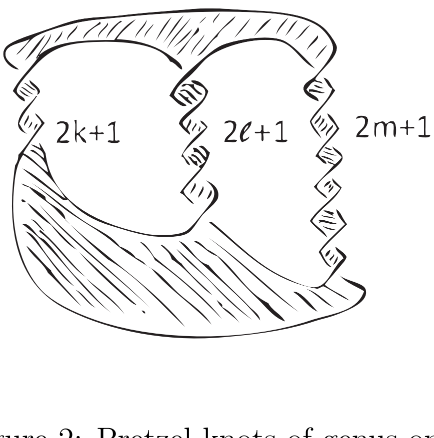


Figure 2: Pretzel knots of genus one

[2]A canonical Seifert surface of genus one is illustrated by a shadow in the figure.

with $[k,\ell,m] = (k+1)(\ell+1)(m+1) - k\ell m$. Let $m = -\ell + i$. Then we have

$$\begin{aligned} m' &= [k,\ell,-\ell+i] = k(i+1) - (\ell+1)(\ell-i-1) \quad \text{and} \\ m'' &= [k,\ell,-\ell+i-1] = ki - (\ell+1)(\ell-i). \end{aligned}$$

Since $K(k,\ell,m-1)$ is obtained from the knot $K(k,\ell,m)$ by one cross-change, we see that $\rho(A_{m'}, A_{m''}) = 1$ or 0 according to whether $\delta > 0$ or $\delta = 0$, where we note that $\delta = |m'' - m'| = |\ell + k + 1|$. Let $\delta > 0$. The determinant ring $D = D(A_{m'}, A_{m''})$ is given by

$$D = \Lambda/(\delta, A_{m'}) = \Lambda/\,(\delta, c \cdot \bar{c}) \quad \text{for} \quad c = kt - k - 1,$$

because

$$A_{m'} \equiv k^2 + (k+1)^2 - k(k+1)(t + t^{-1}) = -c\bar{c} \pmod{\delta}.$$

This determinant ring $D$ admits a short exact sequence

$$0 \to R \to D \to R^{\#} \to 0$$

with a finite $\Lambda$-module $R = \Lambda/(\delta, c)$. We can check that $R$ is in fact a common residue module of the knots $K(k,\ell,m-1)$ and $K(k,\ell,m)$ (which are $\Lambda_Q$-cyclic by Lemma 3.6) as follows. The following matrix $V$ is a Seifert matrix $V$ associated with the canonical Seifert surface $F$ of genus one in Fig. 2.

$$V = \begin{pmatrix} k+\ell+1 & -\ell \\ -\ell-1 & \ell+m+1 \end{pmatrix}.$$

The matrix $tV^T - V$ is a matrix of a $\Lambda$-homomorphism

$$tj_*^+ - j_*^- : H_1(F) \otimes \Lambda \to H_1(S^3 \backslash F) \otimes \Lambda$$

with respect to $\Lambda$-bases given by a $Z$-basis $e_1, e_2$ for $H_1(F)$ and a $Z$-basis $e_1'$, $e_2'$ for $H_1(S^3 \backslash F)$ with the linking number $\mathrm{Link}(e_i', e_j) = \delta_{i,j}$ whose cokernel is the $\Lambda$-module $H_1(\tilde{E}) \cong M(K(k,\ell,m))$ of the knot $K(k,\ell,m)$ (see [6, p.69]). Since our cross-change corresponds to a $\pm 1$-twist along a loop $\gamma$ representing $e_2'$, the quotient module of $M(K(k,\ell,m))$ by the $\Lambda$-cyclic submodule generated by the image of $e_2'$ is the $\Lambda$-module

$$R' = \Lambda/\,((k+\ell+1)(t-1), -(\ell+1)t + \ell)\,.$$

Using that $t-1$ is an automorphism of this $\Lambda$-module $R'$, we see that $\delta = |k+\ell+1| = 0$ in $R'$. Thus, $R'$ is $\Lambda$-isomorphic to the $\Lambda$-module

$$R = \Lambda/(\delta, -(\ell+1)t + \ell) = \Lambda/(\delta, kt - k - 1) = \Lambda/(\delta, c),$$

showing that $R$ is the residue module of the $\Lambda_Q$-cyclic knot $K(k,\ell,m)$ on the cross-change loop $\gamma$. By the same argument or by Lemma 2.2, the $\Lambda$-module $R$ is the residue module of the $\Lambda_Q$-cyclic knot $K(k,\ell,m-1)$ on $\gamma$. Taking $i=0$, we have $m' = k-(\ell+1)(\ell-1)$ and $m'' = -(\ell+1)\ell$. This means that

$$\rho(A_n, A_{-\ell(\ell+1)}) = 1$$

for all integers $\ell$ and $n$ with $n \neq -\ell(\ell+1)$ where we take $k = n+(\ell-1)(\ell+1)$. In particular, we have $\rho(A_n, A_{-2}) = 1$ for every integer $n \neq -2$. The set of the Alexander polynomials $A_{-\ell(\ell+1)}$ $(\ell \in Z)$ is easily seen to coincide with the set of Alexander polynomials with degree 2 of slice type and gives a concrete infinite set of Alexander polynomials such that any two distinct elements in the set have the Alexander polynomial distance one. For some small values of $m'$ and $m''$, we have the following calculations:

(1) $[-5,1,-3]=1,\ [-5,1,-4]=4,\ \rho(A_1,A_4)=1.$

(2) $[-8,2,-4]=-1,\ [-8,2,-5]=4,\ \rho(A_{-1},A_4)=1.$

(3) $[-12,3,-5]=-4,\ [-12,3,-6]=4,\ \rho(A_{-4},A_4)=1.$

(4) $[3,3,-3]=-5,\ [3,3,-2]=2,\ \rho(A_{-5},A_2)=1.$

Using this result and Corollaries 4.3, 5.2 and Examples 6.1, 6.2, we can make table 1 of the Alexander polynomial distances between the Alexander polynomials $A_n$ with $|n| \leqq 5$. Corollary 5.2 is sometimes useful to know the Alexander polynomial distance one by applying it to the difference $A_m - A_n$. For example, we can see from this method that $\rho(A_1, A_{-3}) = \rho(A_{-1}, A_3) = 1$.

**Table 1**

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_5$ | 2 | 1 | 2 | 1 | 2 | 1 | 1 | 2 | 2 | 1 | 0 |
| $A_4$ | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 |
| $A_3$ | 2 | 2 | 2 | 1 | 1 | 1 | 2 | 1 | 0 | 1 | 2 |
| $A_2$ | 1 | 2 | 2 | 1 | 2 | 1 | 1 | 0 | 1 | 1 | 2 |
| $A_1$ | 2 | 2 | 1 | 1 | 2 | 1 | 0 | 1 | 2 | 1 | 1 |
| $A_0$ | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 |
| $A_{-1}$ | 1 | 2 | 2 | 1 | 0 | 1 | 2 | 2 | 1 | 1 | 2 |
| $A_{-2}$ | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $A_{-3}$ | 2 | 1 | 0 | 1 | 2 | 1 | 1 | 2 | 2 | 2 | 2 |
| $A_{-4}$ | 1 | 0 | 1 | 1 | 2 | 1 | 2 | 2 | 2 | 1 | 1 |
| $A_{-5}$ | 0 | 1 | 2 | 1 | 1 | 1 | 2 | 1 | 2 | 1 | 2 |
| | $A_{-5}$ | $A_{-4}$ | $A_{-3}$ | $A_{-2}$ | $A_{-1}$ | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ |

## 7. Appendix: The Mayer-Vietoris exact sequence for a certain triplet

Let $(U; V^0, V^1)$ be a triplet consisting of a (not necessarily compact) polyhedron $U$ and (not necessarily compact) subpolyhedra $V^\varepsilon$ $(\varepsilon = 0, 1)$ of $U$ such that $V^0 \cap V^1 = \emptyset$. For $\varepsilon = 0, 1$, let $V^\varepsilon * v^\varepsilon$ be the cone over $V^\varepsilon$ with $v^\varepsilon$ as the vertex, and $U^\varepsilon = U \cup V^\varepsilon * v^\varepsilon$. We consider that $U^0 \cap U^1 = U$. Then we have the following Mayer-Vietoris exact sequence for the singular homology (cf. E. H. Spanier [16]):

$$\cdots \stackrel{j_*}{\to} H_{q+1}(U^0 \cup U^1) \stackrel{\partial_*}{\to} H_q(U) \xrightarrow{i_*^0 - i_*^1} H_q(U^0) \oplus H_q(U^1) \stackrel{j_*}{\to} H_q(U^0 \cup U^1) \stackrel{\partial_*}{\to} \cdots.$$

We have the following natural isomorphisms by using excision isomorphisms in [16]:

$$H_q(U^\varepsilon) \cong H_q(U^\varepsilon, \{v^\varepsilon\}) \cong H_q(U^\varepsilon, V^\varepsilon * v^\varepsilon) \cong H_q(U, V^\varepsilon) \text{ for } q > 0,\ \varepsilon = 0, 1 \text{ and}$$

$$\begin{aligned} H_q(U^0 \cup U^1) \cong H_q(U^0 \cup U^1, \{v^0, v^1\}) &\cong H_q(U^0 \cup U^1, V^0 * v^0 \cup V^1 * v^1) \\ &\cong H_q(U, V^0 \cup V^1) \quad \text{for } q > 1. \end{aligned}$$

Further, we have a natural monomorphism

$$H_1(U^0 \cup U^1) \to H_1(U^0 \cup U^1, \{v^0, v^1\}) \cong H_1(U, V^0 \cup V^1).$$

Thus, we have the following Mayer-Vietoris exact sequence:

$$\begin{aligned} \cdots \stackrel{\partial_*}{\to} H_2(U) \xrightarrow{i_*^0 - i_*^1} H_2(U, V^0) \oplus H_2(U, V^1) &\stackrel{j_*}{\to} H_2(U, V^0 \cup V^1) \\ \stackrel{\partial_*}{\to} H_1(U) \xrightarrow{i_*^0 - i_*^1} H_1(U, V^0) \oplus H_1(U, V^1) &\stackrel{j_*}{\to} H_1(U, V^0 \cup V^1). \end{aligned}$$

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