ENUMERATING 3-MANIFOLDS BY A CANONICAL ORDER

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A well-order was introduced in the set of links by A. Kawauchi [2]. This well-order also naturally induces a well-order in the set of prime link exteriors and eventually induces a well-order in the set of closed connected orientable 3-manifolds. By this order, we enumerated the prime links with lengths up to 10 and the prime link exteriors with lengths up to 9. Our present plan is to enumerate the 3−manifolds on the enumeration basis of the prime link exteriors. In this paper, we show our latest result in this plan and as an application, give a new proof of the identification of Perko's pair.

Keywords: Lattice point, Length, Link, Exterior, 3-manifold, Table.

1. Introduction

In [2] we suggested a method of enumerating the links, link exteriors and closed connected orientable 3-manifolds. The idea is to introduce a well-order in the set of links by embedding it into a well-ordered set of lattice points. This well-order also naturally induces a well-order in the set of prime link exteriors and eventually induces a well-order in the set of closed connected orientable 3-manifolds.

By using this method, the first 28, 26 and 26 lattice points of lengths up to 7 corresponding to the prime links, prime link exteriors and closed connected orientable 3-manifolds are respectively tabulated without any computer aid in [2]. We enlarged the table of the first 28 lattice points of length up to 7 corresponding to the prime links into that of the first 443 lattice points of length up to 10 in [4] and enlarged the table of the first 26 lattice points of length up to 7 corresponding to the prime link exteriors into that of the first 142 lattice points of lengths up to 9. Our goal of this study is to enumerate the lattice points of lengths up to 10 corresponding to the 3-manifolds. In this paper, we show our latest result on making a table of 3-manifolds and as an application, give a new proof of the identification of Perko's pair by using elementary transformations on the lattice points.

2. Definition of a well-order in the set of links

Let $\mathbf{Z}$ be the set of integers, and $\mathbf{Z}^n$ the product of n copies of $\mathbf{Z}$. We put

$$\mathbf{X} = \coprod_{n=1}^{\infty} \mathbf{Z}^n = \{(x_1, x_2, \ldots, x_n) \,|\, x_i \in \mathbf{Z},\ n = 1, 2, \ldots\}.$$

We call elements of $\mathbf{X}$ *lattice points*. For a lattice point $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{X}$, we put $\ell(\mathbf{x}) = n$ and call it the *length* of $\mathbf{x}$. Let $|\mathbf{x}|$ and $|\mathbf{x}|_N$ be the lattice points determined from $\mathbf{x}$ by the following formulas:

$|\mathbf{x}| = (|x_1|, |x_2|, \ldots, |x_n|)$ and $|\mathbf{x}|_N = (|x_{j_1}|, |x_{j_2}|, \ldots, |x_{j_n}|)$ where $|x_{j_1}| \leqq |x_{j_2}| \leqq \cdots \leqq |x_{j_n}|$ and $\{j_1, j_2, \ldots, j_n\} = \{1, 2, \ldots, n\}$.

We define a well-order (called a *canonical order* [2]) in $\mathbf{X}$ as follows :

Definition 2.1. We define a well-order in $\mathbf{Z}$ by $0 < 1 < -1 < 2 < -2 < 3 < -3 \cdots$, and for $\mathbf{x}, \mathbf{y} \in \mathbf{X}$ we define $\mathbf{x} < \mathbf{y}$ if we have one of the following conditions (1)-(4):

(1) $\ell(\mathbf{x}) < \ell(\mathbf{y})$.
(2) $\ell(\mathbf{x}) = \ell(\mathbf{y})$ and $|\mathbf{x}|_N < |\mathbf{y}|_N$ by the lexicographic order.
(3) $|\mathbf{x}|_N = |\mathbf{y}|_N$ and $|\mathbf{x}| < |\mathbf{y}|$ by the lexicographic order.
(4) $|\mathbf{x}| = |\mathbf{y}|$ and $\mathbf{x} < \mathbf{y}$ by the lexicographic order on the well-order of $\mathbf{Z}$ defined above.

For $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbf{X}$, we put

$$\min|\mathbf{x}| = \min_{1 \leqq i \leqq n} |x_i| \quad \text{and} \quad \max|\mathbf{x}| = \max_{1 \leqq i \leqq n} |x_i|.$$

Let $\beta(\mathbf{x})$ be the $(\max|\mathbf{x}|+1)$-string braid determined from $\mathbf{x}$ by the identity

$$\beta(\mathbf{x}) = \sigma_{|x_1|}^{\mathrm{sign}(x_1)} \sigma_{|x_2|}^{\mathrm{sign}(x_2)} \cdots \sigma_{|x_n|}^{\mathrm{sign}(x_n)},$$

where we define $\sigma_{|0|}^{\mathrm{sign}(0)} = 1$. Let $\mathrm{cl}\beta(\mathbf{x})$ be the closure of the braid $\beta(\mathbf{x})$. Let $\mathbf{L}$ be the set of links. Then we have a map

$$\mathrm{cl}\beta : \mathbf{X} \to \mathbf{L}$$

sending $\mathbf{x}$ to $\mathrm{cl}\beta(\mathbf{x})$. By Alexander's braiding theorem, the map $\mathrm{cl}\beta$ is surjective. For $L \in \mathbf{L}$, we define a map

$$\sigma : \mathbf{L} \to \mathbf{X}$$

by $\sigma(L) = \min\{\mathbf{x} \in \mathbf{X} \,|\, \mathrm{cl}\beta(\mathbf{x}) = L\}$. Then σ is a right inverse of $\mathrm{cl}\beta$ and hence is injective. Now we have a well-order in $\mathbf{L}$ by the following definition:

Definition 2.2. For $L, L' \in \mathbf{L}$, we define $L < L'$ if $\sigma(L) < \sigma(L')$.

For a link $L \in \mathbf{L}$, we call $\ell(\sigma(L))$ the *length* of L.

3. A method of a tabulation of prime links

Let $\mathbf{L}^{\mathrm{p}}$ be the set of prime links. For $k \in \mathbf{Z}$, let k^n and $-k^n$ be the lattice points determined by

$$k^n = (\underbrace{k, k, \ldots, k}_{n}) \quad \text{and} \quad -k^n = (-k)^n,$$

respectively.

For $\mathbf{x} = (x_1, x_2, \ldots, x_n),\ \mathbf{y} = (y_1, y_2, \ldots, y_m) \in \mathbf{X}$, let $\mathbf{x}^T$, $-\mathbf{x}$, $(\mathbf{x},\ \mathbf{y})$ and $\delta(\mathbf{x})$ be the lattice points determined by the following formulas:

$$\begin{aligned}
\mathbf{x}^T &= (x_n, \ldots, x_2, x_1), \\
-\mathbf{x} &= (-x_1, -x_2, \ldots, -x_n), \\
(\mathbf{x},\ \mathbf{y}) &= (x_1, \ldots, x_n, y_1, \ldots, y_m), \\
\delta(\mathbf{x}) &= (x_1', x_2', \ldots, x_n'), \\
\text{where} \quad x_i' &= \begin{cases} \mathrm{sign}(x_i)(\max|\mathbf{x}| + 1 - |x_i|) & (x_i \neq 0) \\ 0 & (x_i = 0). \end{cases}
\end{aligned}$$

Definition 3.1. Let $\mathbf{x},\ \mathbf{y},\ \mathbf{z},\ \mathbf{w} \in \mathbf{X},\ k,\ l,\ n \in \mathbf{Z}$ with $n > 0$ and $\varepsilon = \pm 1$. An *elementary transformation* on lattice points is one of the following operations (1)-(12) and their inverses $(1)^-$-$(12)^-$.

(1) $(\mathbf{x},\ k,\ -k,\ \mathbf{y}) \to (\mathbf{x},\ \mathbf{y})$
(2) $(\mathbf{x},\ k,\ \mathbf{y}) \to (\mathbf{x},\ \mathbf{y})$, where $|k| > \max|\mathbf{x}|,\ \max|\mathbf{y}|$.
(3) $(\mathbf{x},\ k,\ l,\ \mathbf{y}) \to (\mathbf{x},\ l,\ k,\ \mathbf{y})$, where $|k| > |l| + 1$ or $|l| > |k| + 1$.
(4) $(\mathbf{x},\ \varepsilon k^n,\ k+1,\ k,\ \mathbf{y}) \to (\mathbf{x},\ k+1,\ k,\ \varepsilon(k+1)^n,\ \mathbf{y})$, where $k(k+1) \neq 0$.
(5) $(\mathbf{x},\ k,\ \varepsilon(k+1)^n,\ -k,\ \mathbf{y}) \to (\mathbf{x},\ -(k+1),\ \varepsilon k^n,\ k+1,\ \mathbf{y})$, where $k(k+1) \neq 0$.
(6) $(\mathbf{x},\ \mathbf{y}) \to (\mathbf{y},\ \mathbf{x})$
(7) $\mathbf{x} \to \mathbf{x}^T$
(8) $\mathbf{x} \to -\mathbf{x}$
(9) $\mathbf{x} \to \delta(\mathbf{x})$

(10) $(1^n,\ \mathbf{x},\ \varepsilon,\ \mathbf{y}) \to (1^n,\ \mathbf{y},\ \varepsilon,\ \mathbf{x})$, where $\min|\mathbf{x}| \geqq 2$ and $\min|\mathbf{y}| \geqq 2$.
(11) $(k^2,\ \mathbf{x},\ \mathbf{y},\ -k^2,\ \mathbf{z},\ \mathbf{w}) \to (-k^2,\ \mathbf{x},\ \mathbf{w}^T,\ k^2,\ \mathbf{z},\ \mathbf{y}^T)$, where $\max|\mathbf{x}| < k < \min|\mathbf{y}|$, $\max|\mathbf{z}| < k < \min|\mathbf{w}|$ and $\mathbf{x}, \mathbf{y}, \mathbf{z}$ or $\mathbf{w}$ may be empty.
(12) $(\mathbf{x},\ k,\ (k+1)^2,\ k,\ \mathbf{y}) \to (\mathbf{x},\ -k,\ -(k+1)^2,\ -k,\ \mathbf{y}^T)$, where $\max|\mathbf{x}| < k < \min|\mathbf{y}|$ and $\mathbf{x}$ or $\mathbf{y}$ may be empty.

A meaning of Definition 3.1 is given by the following lemma (See [2,4]):

Lemma 3.2. If a lattice point $\mathbf{x}$ is transformed into a lattice point $\mathbf{y}$ by an elementary transformation, then we have $\mathrm{cl}\beta(\mathbf{x}) = \mathrm{cl}\beta(\mathbf{y})$ (modulo a split union of a trivial link for (9)).

The outline of a tabulation of prime links is the following (See [2,4] for the details): Let Δ be the subset of $\mathbf{X}$ consisting of 0, 1^m for $m \geqq 2$ and $(x_1, x_2, \ldots, x_n)$, where $n \geqq 4$, $x_1 = 1$, $1 \leqq |x_i| \leqq \frac{n}{2}$, $|x_n| \geqq 2$ and $\{|x_1|, |x_2|, \ldots, |x_n|\} = \{1, 2, \ldots, \max|\mathbf{x}|\}$. Then we have $\sharp\{\mathbf{y} \in \Delta | \mathbf{y} < \mathbf{x}\} < \infty$ for every $\mathbf{x} \in \Delta$ and have $\sigma(\mathbf{L}^{\mathrm{p}}) \subset \Delta$. First, we enumerate the lattice points of Δ under the canonical order and then we omit $x \in \Delta$ from the sequence if $\mathrm{cl}\beta(x)$ is the non-prime link or the link which has already appeared in the table of prime links. By using Lemma 3.2, we can remove most of the links which have already appeared. We show a table of prime links with length up to 9 below:

$O < 2_1^2 < 3_1 < 4_1^2 < 4_1 < 5_1 < 5_1^2 < 6_1^2 < 5_2 < 6_2 < 6_3^3 < 6_1^3 < 6_3 < 6_2^3 < 6_3^2 < 7_1 < 6_2^2 < 7_1^2 < 7_7^2 < 7_8^2 < 7_4^2 < 7_2^2 < 7_5^2 < 7_6^2 < 6_1 < 7_6 < 7_7 < 7_1^3 < 8_1^2 < 7_3 < 8_2 < 8_7^3 < 8_8^3 < 8_1^3 < 8_{19} < 8_{20} < 8_5 < 7_5 < 8_7 < 8_{21} < 8_{10} < 8_9^3 < 8_5^3 < 8_{16} < 8_9 < 8_2^3 < 8_{17} < 8_6^3 < 8_{10}^3 < 8_4^3 < 8_{18} < 7_3^2 < 8_5^2 < 8_{16}^2 < 8_{15}^2 < 8_9^2 < 8_8^2 < 8_{12}^2 < 8_{13}^2 < 8_7^2 < 8_{10}^2 < 8_{11}^2 < 8_3^4 < 8_2^4 < 8_1^4 < 8_{14}^2 < 8_{12} < 9_1 < 8_2^2 < 9_1^2 < 9_{43}^2 < 9_{44}^2 < 9_{13}^2 < 9_{49}^2 < 9_{51}^2 < 9_{19}^2 < 9_{50}^2 < 8_3^2 < 9_2^2 < 9_{52}^2 < 9_{20}^2 < 9_{55}^2 < 9_{31}^2 < 9_{53}^2 < 9_{54}^2 < 8_4^2 < 9_{23}^2 < 9_{57}^2 < 9_{35}^2 < 9_{40}^2 < 9_5^2 < 9_{14}^2 < 9_{21}^2 < 9_{34}^2 < 9_{37}^2 < 9_{59}^2 < 9_{29}^2 < 9_{39}^2 < 9_{61}^2 < 9_{41}^2 < 9_{42}^2 < 8_6 < 9_{11} < 9_{43} < 9_{44} < 9_{36} < 9_{42} < 7_2 < 8_{14} < 9_{26} < 8_4 < 8_3^3 < 9_6^3 < 9_{13}^3 < 9_{14}^3 < 9_2^3 < 9_{19}^3 < 9_{18}^3 < 9_8^3 < 9_{45} < 9_{32} < 9_{11}^3 < 8_8 < 9_{20} < 9_1^3 < 7_4 < 8_{11} < 9_{27} < 8_{13} < 8_{15} < 9_{24} < 9_{30} < 9_{17}^3 < 9_{16}^3 < 9_{15}^3 < 9_4^3 < 9_{10}^3 < 9_{20}^3 < 9_{12}^3 < 9_{21}^3 < 9_{33} < 9_{46} < 9_{34} < 9_{47} < 9_{31} < 9_{28} < 9_{40} < 9_{11}^2 < 9_{17} < 9_{22} < 9_5^3 < 9_9^3 < 9_{29} < 9_{12}^2 < 8_6^2 < 9_{25}^2$

Table 1

As an application, we show that the knots of Perko's pair are equivalent to each other, by using Lemma 3.2. We describe each knot as a closed braid, which induces a lattice point. For example 10_{161} and 10_{162} are corresponding to $(1^3, 2^2, 1, -2, 1, 2^2)$ and $(4, -3^2, -4, -3, 2, -1, -2, -3, -2, -3, 4, -3, -4, -1, -2)$, respectively. We transform each of them into the smallest one by using Lemma 3.3. we see that they are the same lattice point as follows and we have $10_{161} = 10_{162}$.

$$(1^3, 2^2, 1, -2, 1, 2^2) \to (1^3, 2, -1, 2, 1^2, 2^2) \quad \text{by (4)}$$

$$\begin{aligned}
&(4, -3^2, -4, -3, 2, -1, -2, -3, -2, -3, 4, -3, -4, -1, -2)\\
\to &(-3^2, -4, -3, 2, -1, -2, -3, -2, -3, 4, -3, -4, -1, -2, 4) \quad \text{by (6)}\\
\to &(-3^2, -4, -3, 2, -1, -2, -3, -2, -3, 4, -3, -4, 4, -1, -2) \quad \text{by (3)}\\
\to &(-3^2, -4, -3, 2, -1, -2, -3, -2, -3, 4, -3, -1, -2) \quad \text{by (1)}\\
\to &(-3, -1, -2, -3^2, -4, -3, 2, -1, -2, -3, -2, -3, 4) \quad \text{by (6)}\\
\to &(-1, -3, -2, -3^2, -4, -3, 2, -1, -2, -3, -2, -3, 4) \quad \text{by (3)}\\
\to &(-1, -2^2, -3, -2, -4, -3, 2, -1, -2, -3, -2, -3, 4) \quad \text{by (4)}\\
\to &(4 - 1, -2^2, -3, -2, -4, -3, 2, -1, -2, -3, -2, -3) \quad \text{by (6)}\\
\to &(-1, -2^2, 4, -3, -4, -2, -3, 2, -1, -2, -3, -2, -3) \quad \text{by (3)}\\
\to &(-1, -2^2, -3, -4, 3, -2, -3, 2, -1, -2, -3, -2, -3) \quad \text{by (4)}\\
\to &(-1, -2^2, -3, 3, -2, -3, 2, -1, -2, -3, -2, -3) \quad \text{by (2)}\\
\to &(-1, -2^3, -3, 2, -1, -2, -3, -2, -3) \quad \text{by (1)}\\
\to &(-1, -2^3, -3, -1, -2, 1, -2, -3, -2) \quad \text{by (4)}\\
\to &(-2, -1, -2^3, -3, -1, -2, 1, -2, -3) \quad \text{by (6)}\\
\to &(-1^3, -2, -1, -3, -1, -2, 1, -2, -3) \quad \text{by (4)}\\
\to &(-3, -1^3, -2, -1, -3, -1, -2, 1, -2) \quad \text{by (6)}\\
\to &(-1^3, -3 - 2, -3, -1^2, -2, 1, -2) \quad \text{by (3)}\\
\to &(-1^3, -2 - 3, -2, -1^2, -2, 1, -2) \quad \text{by (4)}\\
\to &(-1^3, -2^2, -1^2, -2, 1, -2) \quad \text{by (2)}\\
\to &(-2^2, -1^2, -2, 1, -2, -1^3) \quad \text{by (6)}\\
\to &(1^3, 2, -1, 2, 1^2, 2^2) \quad \text{by (8), (9)}
\end{aligned}$$

4. A method of a tabulation of prime link exteriors

Since a knot is determined by its exterior by the Gordon-Luecke Theorem, we classify the exteriors of two or more component links.

We obtain a table of prime link exteriors, by omitting $7^2_7, 7^2_8, 8^3_7, 8^3_8, 8^2_{16}, 8^2_{15}, 9^2_{43}, 9^2_{44}, 9^2_{49}, 9^3_{13}, 9^3_{14}, 9^3_{19}, 9^3_{18}, 9^3_{17}$ from Table 1 and replacing the rest of the links with their exteriors because the exteriors of the above 14 links have already appeared (See [5]). So we have the following table of link exteriors:

$$E(O) < E(2^2_1) < E(3_1) < E(4^2_1) < E(4_1) < E(5_1) < E(5^2_1) < E(6^2_1) < E(5_2) < E(6_2) < E(6^3_3) < E(6^3_1) < E(6_3) < E(6^3_2) < E(6^2_3) < E(7_1) < E(6^2_2) < E(7^2_1) < E(7^2_4) < E(7^2_2) < E(7^2_5) < E(7^2_6) < E(6_1) < E(7_6) < E(7_7) < E(7^3_1) < E(8^2_1) < E(7_3) < E(8_2) < E(8^3_1) < E(8_{19}) < E(8_{20}) < E(8_5) < E(7_5) < E(8_7) < E(8_{21}) < E(8_{10}) < E(8^3_9) < E(8^3_5) < E(8_{16}) < E(8_9) < E(8^3_2) < E(8_{17}) < E(8^3_6) < E(8^3_{10}) < E(8^3_4) < E(8_{18}) < E(7^2_3) < E(8^2_5) < E(8^2_9) < E(8^2_8) < E(8^2_{12}) < E(8^2_{13}) < E(8^2_7) < E(8^2_{10}) < E(8^2_{11}) < E(8^4_3) < E(8^4_2) < E(8^4_1) < E(8^2_{14}) < E(8_{12}) < E(9_1) < E(8^2_2) < E(9^2_1) < E(9^2_{13}) < E(9^2_{51}) < E(9^2_{19}) < E(9^2_{50}) < E(8^2_3) < E(9^2_2) < E(9^2_{52}) < E(9^2_{20}) < E(9^2_{55}) < E(9^2_{31}) < E(9^2_{53}) < E(9^2_{54}) < E(8^2_4) < E(9^2_{23}) < E(9^2_{57}) < E(9^2_{35}) < E(9^2_{40}) < E(9^2_5) < E(9^2_{14}) < E(9^2_{21}) < E(9^2_{34}) < E(9^2_{37}) < E(9^2_{59}) < E(9^2_{29}) < E(9^2_{39}) < E(9^2_{61}) < E(9^2_{41}) < E(9^2_{42}) < E(8_6) < E(9_{11}) < E(9_{43}) < E(9_{44}) < E(9_{36}) < E(9_{42}) < E(7_2) < E(8_{14}) < E(9_{26}) < E(8_4) < E(8^3_3) < E(9^3_6) < E(9^3_2) < E(9^3_8) < E(9_{45}) < E(9_{32}) < E(9^3_{11}) < E(8_8) < E(9_{20}) < E(9^3_1) < E(7_4) < E(8_{11}) < E(9_{27}) < E(8_{13}) < E(8_{15}) < E(9_{24}) < E(9_{30}) < E(9^3_{16}) < E(9^3_{15}) < E(9^3_4) < E(9^3_{10}) < E(9^3_{20}) < E(9^3_{12}) < E(9^3_{21}) < E(9_{33}) < E(9_{46}) < E(9_{34}) < E(9_{47}) < E(9_{31}) < E(9_{28}) < E(9_{40}) < E(9^2_{11}) < E(9_{17}) < E(9_{22}) < E(9^3_5) < E(9^3_9) < E(9_{29}) < E(9^2_{12}) < E(8^2_6) < E(9^2_{25}).$$

Table 2

5. A method of a tabulation of 3-manifolds

We make a list of closed connected orientable 3-manifolds by constructing a sequence of 3-manifolds obtained from 0 surgery along the links in Table 2 and removing the manifolds which have already appeared (See [2]). Let $\chi(L, 0)$ denote the manifold obtained from 0 surgery along a link L. We classify $\chi(L, 0)$ for L in Table 2 according to the first homology group $H_1(\chi(L, 0))$. There are 10 types of groups 0, $\mathbf{Z}$, $\mathbf{Z} \oplus \mathbf{Z}$, $\mathbf{Z} \oplus \mathbf{Z} \oplus \mathbf{Z}$, $\mathbf{Z} \oplus$

$\mathbf{Z}_2 \oplus \mathbf{Z}_2,\ \mathbf{Z}_2,\ \mathbf{Z}_2 \oplus \mathbf{Z}_2,\ \mathbf{Z}_3 \oplus \mathbf{Z}_3,\ \mathbf{Z}_4,\ \mathbf{Z}_4 \oplus \mathbf{Z}_4$ and we have respectively 16, 62, 16, 4, 5, 7, 15, 7, 5, 5 links with these types of groups. In this paper we enumerate the manifolds with $H_1(\chi(L,0)) \cong \mathbf{Z}$. The links with this condition are the following:

$O < 3_1 < 4_1 < 5_1 < 5_2 < 6_2 < 6_3 < 7_1 < 6_1 < 7_6 < 7_7 < 7_3 < 8_2 < 8_{19} < 8_{20} < 8_5 < 7_5 < 8_7 < 8_{21} < 8_{10} < 8^3_5 < 8_{16} < 8_9 < 8_{17} < 8^3_6 < 8_{18} < 8_{12} < 9_1 < 8_6 < 9_{11} < 9_{43} < 9_{44} < 9_{36} < 9_{42} < 7_2 < 8_{14} < 9_{26} < 8_4 < 9^3_2 < 9_{45} < 9_{32} < 8_8 < 9_{20} < 9^3_1 < 7_4 < 8_{11} < 9_{27} < 8_{13} < 8_{15} < 9_{24} < 9_{30} < 9^3_{10} < 9_{33} < 9_{46} < 9_{34} < 9_{47} < 9_{31} < 9_{28} < 9_{40} < 9_{17} < 9_{22} < 9_{29}$.

We see that $\chi(9^3_2,0) \cong \chi(6_3,0)$, $\chi(9^3_1,0) \cong \chi(6_2,0)$ and $\chi(9_{46},0) \cong \chi(8^3_5,0)$. So we omit $\chi(9^3_2,0)$, $\chi(9^3_1,0)$ and $\chi(9_{46},0)$ from the sequence. For the rest of the links, we calculate their Alexander polynomials or Alexander modules and we see that the manifolds are different from each other except the following two cases:

$$\Delta(\chi(O,0)) = \Delta(\chi(9^3_{10},0)),\ \Delta(\chi(9_{28},0)) = \Delta(\chi(9_{29},0)).$$

However, we have

$$\chi(O,0) \not\cong \chi(9^3_{10},0),\ \chi(9_{28},0) \not\cong \chi(9_{29},0)$$

by the following discussion. For the first case, we transform the framed link 9^3_{10} with coefficient 0 into a framed knot K with coefficient 0 by the Kirby calculus on handle slides. We see that K is a non-trivial knot by computing the $p_0(\ell)$-polynomial of the HOMFLY polynomial $P(K;\ell,m)$ and we have $\chi(9^3_{10},0) \cong \chi(K,0) \not\cong \chi(O,0)$ by Gabai's property R [1]. For the second case, we substitute the fifth roots of unity for the Jones polynomial of 9_{28} and 9_{29} and we have

$$J_{9_{28}}(\omega) = -5 - 10\omega + 3\omega^2 - 12\omega^3$$

$$J_{9_{29}}(\omega) = -3 + 9\omega - 6\omega^2 + 6\omega^3,$$

where ω is any one of the fifth roots of unity. We see that $J_{9_{28}}(\omega) \neq J_{9_{29}}(\omega')$ for any of the fifth roots of unity ω, ω' and we have $\chi(9_{28},0) \not\cong \chi(9_{29},0)$ by Kirby and Melvin's theorem [6,p.530].

Therefore we have:

$\chi(O,0) < \chi(3_1,0) < \chi(4_1,0) < \chi(5_1,0) < \chi(5_2,0) < \chi(6_2,0) < \chi(6_3,0) < \chi(7_1,0) < \chi(6_1,0) < \chi(7_6,0) < \chi(7_7,0) < \chi(7_3,0) < \chi(8_2,0) < \chi(8_{19},0) < \chi(8_{20},0) < \chi(8_5,0) < \chi(7_5,0) < \chi(8_7,0) < \chi(8_{21},0) <$

$\chi(8_{10},0) < \chi(8^3_5,0) < \chi(8_{16},0) < \chi(8_9,0) < \chi(8_{17},0) < \chi(8^3_6,0) < \chi(8_{18},0) < \chi(8_{12},0) < \chi(9_1,0) < \chi(8_6,0) < \chi(9_{11},0) < \chi(9_{43},0) < \chi(9_{44},0) < \chi(9_{36},0) < \chi(9_{42},0) < \chi(7_2,0) < \chi(8_{14},0) < \chi(9_{26},0) < \chi(8_4,0) < \chi(9_{45},0) < \chi(9_{32},0) < \chi(8_8,0) < \chi(9_{20},0) < \chi(7_4,0) < \chi(8_{11},0) < \chi(9_{27},0) < \chi(8_{13},0) < \chi(8_{15},0) < \chi(9_{24},0) < \chi(9_{30},0) < \chi(9^3_{10},0) < \chi(9_{33},0) < \chi(9_{34},0) < \chi(9_{47},0) < \chi(9_{31},0) < \chi(9_{28},0) < \chi(9_{40},0) < \chi(9_{17},0) < \chi(9_{22},0) < \chi(9_{29},0).$

Table 3

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